Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 14, No. 4, 2022 Article ID IJIM-1530, 10 pages
DOI: http://dx.doi.org/10.30495/ijim.2022.20272
Research Article

On the Modified Block-Pulse Function for Volterra Integral Equation of The First Kind

M. Mohammadi [*], A. R. Vahidi [†‡], T. Damercheli [§], S. Khezerloo [¶], M. Nouri [‖]

Received Date: 2021-06-26 Revised Date: 2021-09-21 Accepted Date: 2021-10-26

Abstract

In this work, we present a simple efficient modified Block-Pulse functions (MBPFs) for numerical solution of class of linear Volterra integral equation of first kind. The peresent method is based on converting Volterra integral equation of the first kind into Volterra integral equation of the second kind. Some theorems are included to show the convergence and advantage of this method. Numerical results show that the approximate solutions have a good degree of accuracy.

Keywords : Volterra integral equation of the first kind; Block-pulse functions; Operational matrix; Function expansion; Convergence.

1 Introduction

Integral equations are widely used for solving many problems in engineering, physics and mathematics [1, 2, 3, 4, 5]. These equations are generally ill-posed. That is, small changes in the problem's data can make very large changes in the obtained answers [6, 7]. So, obtaining numerical solutions is very difficult. Consider the following Voletrra integral equation of the first kind as the form

$$\int_0^t k(t,s)x(s)ds = f(t), \quad 0 \le t < 1, \qquad (1.1)$$

where f and k are known functions, x is the unknown function. Moreover, $k(t,s) \in L^2([0,1) \times [0,1))$ and $x,\ f \in L^2([0,1))$.

In recent years, many different basic functions have been used to estimate the solution of Voletrra integral equations of the first kind. Maleknejad et al. [8] use a numerical technique based on Bernstein's approximation method to find the solution of Voletrra integral equations of the first kind. Bahmanpour and Fariborzi Aragi [9] used wavelet bases method to solve Voletrra integral equations of the first kind. Isaac and Unanam [10] used the trapezoidal quadrature

*Department of Mathematics, South Tehran Branch, Islamic Azad university, Tehran, Iran.
†Corresponding author. alrevahidi@yahoo.com, Tel:+98(912)3599569.
‡Department of Mathematics, Yadegar-e-Imam Khomeini (RAH) Shahre Rey Branch, Islamic Azad University, Tehran, Iran.
§Department of Mathematics, Yadegar-e-Imam Khomeini (RAH) Shahre Rey Branch, Islamic Azad University, Tehran, Iran.
¶Department of Mathematics, South Tehran Branch, Islamic Azad university, Tehran, Iran.
‖Department of Mathematics, Islamic Azad university South Tehran Branch, Tehran, Iran.

method for solving Voletrra integral equations of the first kind. Khan et al. [11] applied optimal homotopy asymptotic method for solving a Voletrra integral equation of the first kind.
In this paper, firstly Voletrra integral equation of the first kind is reduced to a Volterra integral equation of the second kind by differetiating and applying Leibnitz rule. Secondly, the approximate solution of Volterra integral equation of the second kind is obtained by extending MBPFs.
This paper is divided into the following sections. In Section 2, we review the MBPFs and operational matrix. Our proposed method is studied in Section 3. In Section 4, an error analysis for the proposed method is presented and it is proved that the convergence rate is $O(\frac{h}{k})$. Then, numerical results are given in Section 5. Finally, conclusion will be in Section 6.

2 Modified Block-Pulse Function

Definition 2.1. *An $(n+1)$–set of MBPFs $\phi_i(t)$, $i = 0, \ldots, n$ on the interval $[0, T)$ are defined as [13]:*

$$\phi_0(t) = \begin{cases} 1, & t \in \left[0, \frac{T}{n} - \varepsilon\right) = I_0, \\ 0, & o.w, \end{cases}$$
$$\phi_n(t) = \begin{cases} 1, & t \in [T-\varepsilon, T) = I_n, \\ 0, & o.w, \end{cases} \tag{2.2}$$
$$\phi_i(t) = \begin{cases} 1, & t \in \left[\frac{iT}{n} - \varepsilon, \frac{(i+1)T}{n} - \varepsilon\right) = I_i, \quad 0 < i < n, \\ 0, & o.w. \end{cases}$$

There are some properties for MBPFs as follows: MBPFs are disjoint and orthogonal as

$$\phi_i(t)\phi_j(t) = \begin{cases} \phi_i(t), & i = j, \\ 0, & i \neq j \end{cases}, i, j = 0, \ldots, n, \tag{2.3}$$

$$\int_0^1 \phi_i(t)\phi_j(t)dt = h\delta_{ij}, \quad 0 < i, j < n, \tag{2.4}$$

$$\int_0^1 \phi_0(t)\phi_0(t)dt = h - \varepsilon, \tag{2.5}$$
$$\int_0^1 \phi_n(t)\phi_n(t)dt = \varepsilon, \tag{2.6}$$

and MBPFs like BPFs are complete [13]:

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\phi_i(t)\|^2, \tag{2.7}$$

using notation $\Phi_n(t) = [\phi_0(t), \ldots, \phi_n(t)]^T$, the following properties are achieved:

$$\Phi_{n+1}(t)\Phi_{n+1}^T(t) = \begin{bmatrix} \phi_0(t) & 0 & 0 & \ldots & 0 \\ 0 & \phi_1(t) & 0 & \ldots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & \ldots & 0 & \ldots & \phi_n(t) \end{bmatrix} \tag{2.8}$$

$$\Phi_{n+1}^T(t)\Phi_{n+1}(t) = 1, \tag{2.9}$$
$$\Phi_{n+1}(t)\Phi_{n+1}^T(t)V = \tilde{V}\Phi_{n+1}(t), \tag{2.10}$$
$$\Phi_{n+1}^T(t)B\Phi_{n+1}(t) = \hat{B}^T\Phi_{n+1}(t), \tag{2.11}$$

where V is an $(n+1)$-vector and $\tilde{V}$ =diag(V), and also B is an $(n+1)\times(n+1)$ matrix where $\hat{B}$ is an $(n+1)$-vector with elements equal to the diagonal entries of matrix B. If $h = \frac{T}{n}$, the operational matrix of MBPFs is defined as follows [13]:

$$P_{(n+1)\times(n+1)} = \begin{bmatrix} \frac{h-\varepsilon}{2} & h-\varepsilon & h-\varepsilon & \ldots & h-\varepsilon & h-\varepsilon \\ 0 & \frac{h}{2} & h & \ldots & h & h \\ 0 & 0 & \frac{h}{2} & \ldots & h & h \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{h}{2} & h \\ 0 & 0 & 0 & \ldots & 0 & \frac{\varepsilon}{2} \end{bmatrix}. \tag{2.12}$$

Definition 2.2. *[13] MBPFs expansion of continuous function $f \in L^2([0, 1))$ with respect to ϕ_i, $i = 0, \ldots, n$ is defined as:*

$$f(t) \simeq \hat{f}_{n+1} = \sum_{i=0}^{n} f_i\phi_i(t), \tag{2.13}$$

where

$$f_i = \frac{1}{\Delta(I_i)} \int_0^1 f(t)\phi_i(t)dt, \tag{2.14}$$

and $\Delta(I_i)$ is the length of interval I_i defined in Equations (2.2).

3 Extension of MBPFs for Volterra integral equation of second kind

Consider Volterra integral Equation (1.1). Now, by differentiating both sides with respect to t and using Leibnitz rule, we have

$$f'(t) = k(t,t)x(t) + \int_0^t k_t(t,s)x(s)ds, \quad (3.15)$$

by solving $x(t)$, if $k(t,t) \neq 0$, we obtain Volterra integral equation of the second kind given by [16]

$$x(t) = \frac{f'(t)}{k(t,t)} - \int_0^t \frac{1}{k(t,t)} k_t(t,s)x(s)ds. \quad (3.16)$$

Assuming $\mathcal{F}(t) = \frac{f'(t)}{k(t,t)}$ and $g(t,s) = \frac{-1}{k(t,t)} k_t(t,s)$, then:

$$x(t) = \mathcal{F}(t) + \int_0^t g(t,s)x(s)ds. \quad (3.17)$$

Maleknejad in [13] introduces MBPFs to solve Volterra integral equation of the first kind. Here, we extend this method to solve Volterra integral equation of the second kind. To this end, approximating functions $\mathcal{F}$, x and g with respect to MBPFs gives:

$$\begin{aligned} g(t,s) &\simeq \Phi^T(t)G\Phi(s), \\ \mathcal{F}(t) &\simeq F^t\Phi(t) = \Phi^T(t)F, \\ x(t) &\simeq X^T\Phi(t) = \Phi^T(t)X. \end{aligned} \quad (3.18)$$

Here, m–vectors F, X, and $m \times m$ matrix G are MBPFs coefficients of $\mathcal{F}$, x and g, respectively. Note that X in Equations (3.18) is the unknown vector and should be obtained. Therefore, substituting (3.18) into (3.17) gives

$$\begin{aligned} X^T\Phi(t) &\simeq F^T\Phi(t) + \int_0^t \Phi^T(t)G\Phi(s)\Phi^T(s)Xds \\ &= F^T\Phi(t) + \Phi^T(t)G\int_0^t \Phi(s)\Phi^T(s)Xds, \end{aligned} \quad (3.19)$$

by using (2.10) and operational matrix P in (2.12), we have

$$X^T\Phi(t) \simeq F^T\Phi(t) + \Phi^T(t)G\tilde{X}P\Phi(t), \quad (3.20)$$

here, $G\tilde{X}P$ is an $(n+1)\times(n+1)$ matrix. Thus, if ε equals to 0, just n BPFs would exist and the dimension of vectors is reduced to n.

By Equation (2.11), we can write:

$$X^T\Phi(t) \simeq F^T\Phi(t) + \hat{X}^T\Phi(t), \quad (3.21)$$

where $\hat{X}$ is an $(n+1)$–vector with components equal to the diagonal entries of matrix $G\tilde{X}P$. Finally,

$$X - \hat{X} \simeq F.$$

So, the vector $\hat{X}$ can be writen as follows: Now, substituting respect Equation (3.22) into Equation (3.21) would give: Now, by replacing $\simeq$ with $=$, Equation (3.21) is reduced to a linear lower triangular system as:

$$Q = F. \quad (3.24)$$

Consequently, unknown cofficients X_j, $j = 0, 1, \ldots, n$ are calculated by solving this linear equation system.

Remark

[16] Before, it was stated that if $k(t,t) = 0$ in Equation (3.16), then the conversion of the first kind to the second kind fails. Therefore, by differentiating Equation (1.1) with respect to t twice and assuming that $k'_t(t,s)|_{s=t} \neq 0$, we obtain Volterra integral equation of the second kind:

$$\begin{aligned} x(t) = &\frac{f''(t)}{k'_t(t,s)|_{s=t}} \\ &- \int_0^t \frac{1}{k'_t(t,s)|_{s=t}} k''_{tt}(t,s)x(s)ds, \end{aligned} \quad (3.25)$$

If $k'_t(t,t) = 0$, we can again apply differentiation, and so on. If the first $m-2$ partial derivatives of the kernel with respect to t are identically zero and the $(m-1)$ st derivative is nonzero, then the m–fold differentiation of the original equation gives the following Volterra integral equation of the second kind:

$$\begin{aligned} x(t) = &\frac{f^{(m)}(t)}{k_t^{(m-1)}(t,s)|_{s=t}} \\ &- \int_0^t \frac{1}{k_t^{(m-1)}(t,s)|_{s=t}} k_t^{(m)}(t,s)x(s)ds. \end{aligned} \quad (3.26)$$

$$\hat{X} = \begin{bmatrix} \frac{h-\varepsilon}{2} g_{0,0} x_0 \\ (h-\varepsilon) g_{1,0} x_0 + \frac{h}{2} g_{1,1} x_1 \\ (h-\varepsilon) g_{2,0} x_0 + h g_{2,1} x_1 + \frac{h}{2} g_{2,2} x_2 \\ \vdots \\ (h-\varepsilon) g_{n,0} x_0 + h g_{n,1} x_1 + \cdots + h g_{n,(n-1)} x_{n-1} + \frac{\varepsilon}{2} g_{n,n} x_n \end{bmatrix}. \tag{3.22}$$

$$Q = \begin{bmatrix} 1-\left(\frac{h-\varepsilon}{2}\right) g_{0,0} & 0 & 0 & \ldots & 0 \\ -(h-\varepsilon) g_{1,0} & 1-\left(\frac{h}{2}\right) g_{1,1} & 0 & \ldots & 0 \\ -(h-\varepsilon) g_{2,0} & -h g_{2,1} & 1-\left(\frac{h}{2}\right) g_{2,2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(h-\varepsilon) g_{n,0} & -h g_{n,1} & -(h) g_{n,2} & \ldots & 1-\left(\frac{\varepsilon}{2}\right) g_{n,n} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{3.23}$$

4 Error Analysis

Maleknejad and Rahimi [13] considered the error analysis of Volterra integral equations of the first kind based on modified Block-pulse functions. In this section, an error analysis on modified Block-pulse functions of Volterra integral equations of the second kind is presented. In the following theorems, for simplicity, we assume $T = 1$ and $h = \frac{1}{n}$.

Theorem 4.1. *Let* $f_{n,\varepsilon}(t) = \sum_{i=0}^{n} f_i \phi_i(t)$ *and* $f_i = \frac{1}{\Delta(I_i)} \int_0^1 f(t)\phi_i(t)dt$, $i = 0, \ldots, n$. *Then the following equation* $\int_0^1 (f(t) - f_{n,\varepsilon}(t))^2 dt$ *achieves its minimum value and also we have*

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\phi_i\|^2. \tag{4.27}$$

Proof. We refer the reader to the theorem which was proved in [15]. □

Theorem 4.2. *Assume $f(t)$ is continuous and differentiable in $[-h, 1+h]$, with bounded derivative; that is, $|f'(t)| < M$. And $f_{n,\varepsilon_i}(t), \varepsilon_i = \frac{ih}{k}$, for $i = 0, 1, \ldots, k-1$, are correspondingly MBPFs, $\frac{h}{k}$ MBPFs, $\ldots$, $\frac{(k-1)h}{k}$ MBPFs expansions of $f(t)$ based on $(n+1)$ MBPFs over internal $[0, 1)$ and*

$$\bar{f}_{n,k}(t) = \frac{1}{k} \sum_{i=0}^{k-1} f_{n,\varepsilon_i}(t), \tag{4.28}$$

then for sufficient larg n, we have

$$\|f - \bar{f}\|_\infty \leq \frac{M}{2kn}, \tag{4.29}$$

therefore,

$$\|f - \bar{f}\|_\infty = O(\frac{1}{kn}).$$

Proof. See [13]. □

Theorem 4.3. *Assume $f(x, y)$ is continuous and differentiable over district $[-h, 1+h] \times [-h, 1+h]$, and $f_{n,\varepsilon_i}(x, y), \varepsilon_i = \frac{ih}{k}$, for $i = 0, 1, \ldots, k-1$, are correspondingly 2D-MBPFs(ε_0), 2D-MBPFs(ε_1), $\ldots$, 2D-MBPFs(ε_{k-1}) expansions of $f(x, y)$ based on $(n+1)$2D-MBPFs over district $[0, 1) \times [0, 1)$ and*

$$\bar{f}_{n,k}(x, y) = \frac{1}{k} \sum_{i=0}^{k-1} f_{n,\varepsilon_i}(x, y), \tag{4.30}$$

then for sufficient larg n, we have

$$\|f - \bar{f}_{m,k}\|_\infty \leq \frac{\sqrt{2}M}{kn}, \tag{4.31}$$

therefore,

$$\|f - \bar{f}_{m,k}\|_\infty = O(\frac{1}{kn}),$$

where M is bounded of $\|Df(x, y)\|_\infty$ and n shows number of 2D-MBPFs.

Proof. See [17]. □

Theorem 4.4. *Suppose $x(t)$ be the exact solution of Equation (3.17) and $\bar{x}_{n,k}(t)$ be the MBPFs approximate solution of Equation (3.17). Also assume that*

(i) $\| x \|_2 \leq C_1, \quad t \in [0, 1)$

(ii) $\parallel g \parallel_2 \leq C_2, \quad (s,t) \in [0,1) \times [0,1).$

Then

$$\|x - \bar{x}_{n,k}\|_2 = O(\frac{1}{kn}), \tag{4.32}$$

and

$$\lim_{n\to\infty} \bar{x}_{n,k}(t) = x(t). \tag{4.33}$$

Proof. From Eq. (3.17), we get

$$\begin{aligned} x(t) - \bar{x}_{n,k}(t) &= \mathcal{F}(t) - \bar{\mathcal{F}}_{n,k}(t) \\ &+ \int_0^t (g(s,t)x(t) - \bar{g}_{n,k}(s,t)\bar{x}_{n,k}(s))ds, \end{aligned} \tag{4.34}$$

then by mean value theorem, we have

$$\begin{aligned} \parallel x - \bar{x}_{n,k} \parallel_2 \leq &\parallel \mathcal{F} - \bar{\mathcal{F}}_{n,k} \parallel_2 \\ &+ \parallel gx - \bar{g}_{n,k}\bar{x}_{n,k} \parallel_2 . \end{aligned} \tag{4.35}$$

By using Cauchy-Schwartz inequality and Theorems 4.2 and 4.3, we obtain

$$\begin{aligned} \parallel gx - \bar{g}_{n,k}\bar{x}_{n,k} \parallel_2 &\leq \parallel g - \bar{g}_{n,k} \parallel_2 \parallel x \parallel_2 \\ &+ \parallel \bar{g}_{n,k} \parallel_2 \parallel x - \bar{x}_{n,k} \parallel_2 \\ &\leq \frac{\sqrt{2}M}{kn} C_1 + C_2 \parallel x - \bar{x}_{n,k} \parallel_2, \end{aligned} \tag{4.36}$$

and

$$\parallel \mathcal{F} - \bar{\mathcal{F}}_{n,k} \parallel_2 \leq \frac{M}{2kn}. \tag{4.37}$$

Substituting Equations (4.36), (4.37) in Equation (4.35), we can write

$$\begin{aligned} \parallel x - \bar{x}_{n,k} \parallel_2 \leq &\frac{M}{2kn} + \frac{\sqrt{2}M}{kn} C_1 \\ &+ C_2 \parallel x - \bar{x}_{n,k} \parallel_2, \end{aligned}$$

therefore,

$$\parallel x - \bar{x}_{n,k} \parallel_2 \leq \frac{M}{2kn} \left(\frac{1 + 2\sqrt{2}C_1}{1 - C_2} \right), \tag{4.38}$$

hence,

$$\parallel x - \bar{x}_{n,k} \parallel_2 = O(\frac{1}{kn}),$$

and

$$\lim_{n\to\infty} \bar{x}_{n,k}(t) = x(t).$$

□

5 Numerical examples

In this section, we consider three examples of Volterra integral equations of the first kind. All three examples are converted to Volterra integral equation of the second kind to obtain the approximate solution by using the proposed method in Section 3. Note that if $k(t,s) = k(\mid t - s \mid)$ and $k(t,t) \to 0$, then the system of linear equations obtained in [12] and [13] will have a large condition number and the methods will be consequently inefficient. Furthermore, the proposed method in [14] reduces recurrence relation with a singular matrix. Matlab 2017a software package was used on a laptop with core i5-3210M CPU. Intel processor and 4GB RAM for making all computations.

Example 5.1. *Consider following integral equation [3, 18]:*

$$\int_0^t e^{t-s} x(s)ds = \sin(t), \quad 0 \leq t < 1, \tag{5.39}$$

with the exact solution $x(t) = \cos(t) - \sin(t)$. By using differentiation, we have:

$$x(t) = \cos t - \int_0^t e^{t-s} x(s)ds. \tag{5.40}$$

Table 1 reports the numerical results. Moreover, Figure 1 shows the results of the exact and the approximate solution by the present method for $n = 128$ and $k = 3$.

Example 5.2. *This example consists of following Volterra integral equation of the first kind [16]:*

$$t \sinh t = 2 \int_0^t \sinh(s-t)x(s)ds, \tag{5.41}$$

with the exact solution $x(t) = \cosh t$. By using differentiation, we obtain:

$$t \cosh t + \sinh t = 2 \int_0^t \cosh(s-t)x(s)ds, \tag{5.42}$$

which is still a Volterra integral equation of the first kind. According to the Remark on Page 6,

Table 1: Numerical results for Example 5.1.

nodes s	Exact solution	Approximate solution for present method $n=32$ $k=1$	$n=64$ $k=3$	$n=128$ $k=3$
0	1.000000	0.996068	0.995602	0.997802
0.1	0.895171	0.902121	0.897763	0.891323
0.2	0.781397	0.796176	0.773113	0.784623
0.3	0.659816	0.644209	0.658857	0.661286
0.4	0.531643	0.523503	0.538815	0.531134
0.5	0.398157	0.398199	0.392861	0.395508
0.6	0.260693	0.269399	0.263954	0.255806
0.7	0.120625	0.138234	0.110713	0.124477
0.8	−0.020649	−0.038339	−0.021760	−0.018994
0.9	−0.161717	−0.170526	−0.154041	−0.162268

Table 2: Numerical results for Example 5.2.

nodes s	Exact solution	Approximate solution for present method $n=32$ $k=1$	$n=64$ $k=3$	$n=128$ $k=3$
0	1.000000	1.000081	1.000046	1.000011
0.1	1.005004	1.004601	1.004823	1.005375
0.2	1.020067	1.017833	1.021558	1.019533
0.3	1.045339	1.049429	1.045627	1.044995
0.4	1.081072	1.083862	1.078892	1.081245
0.5	1.127626	1.127825	1.129720	1.128658
0.6	1.185465	1.181706	1.184025	1.187723
0.7	1.255169	1.245979	1.260582	1.253112
0.8	1.337435	1.348832	1.338177	1.336407
0.9	1.433086	1.439717	1.427541	1.433499

Table 3: Numerical results for Example 5.2.

nodes s	Exact solution	Approximate solution for present method $n=32$ $k=1$	$n=64$ $k=3$	$n=128$ $k=3$
0	1.000000	1.000081	1.000046	1.000011
0.1	1.005004	1.004601	1.004823	1.005375
0.2	1.020067	1.017833	1.021558	1.019533
0.3	1.045339	1.049429	1.045627	1.044995
0.4	1.081072	1.083862	1.078892	1.081245
0.5	1.127626	1.127825	1.129720	1.128658
0.6	1.185465	1.181706	1.184025	1.187723
0.7	1.255169	1.245979	1.260582	1.253112
0.8	1.337435	1.348832	1.338177	1.336407
0.9	1.433086	1.439717	1.427541	1.433499

Table 4: Numerical results for Example 5.3.

nodes s	Exact solution	Approximate solution for present method $n=32$ $k=1$	$n=64$ $k=3$	$n=128$ $k=3$
0	1.000000	1.007813	1.005859	1.002930
0.1	1.105171	1.098158	1.102552	1.109055
0.2	1.221403	1.206096	1.229987	1.218059
0.3	1.349859	1.366692	1.350878	1.348269
0.4	1.491825	1.501020	1.483649	1.492398
0.5	1.648721	1.648549	1.655131	1.651934
0.6	1.822119	1.810576	1.817805	1.828524
0.7	2.013753	1.988527	2.027908	2.008241
0.8	2.225541	2.253291	2.227220	2.222919
0.9	2.459603	2.474750	2.446121	2.460547

Table 5: Bound of error.

Example	Bound of error $\|f(t)-\bar{f}(t)\|_\infty$ $n=128,\ k=3$ for VIE1	$n=128,\ k=3$ present method
1	1.2×10^{-2}	4.9×10^{-3}
2	Inf	2.3×10^{-3}
3	Inf	6.4×10^{-3}

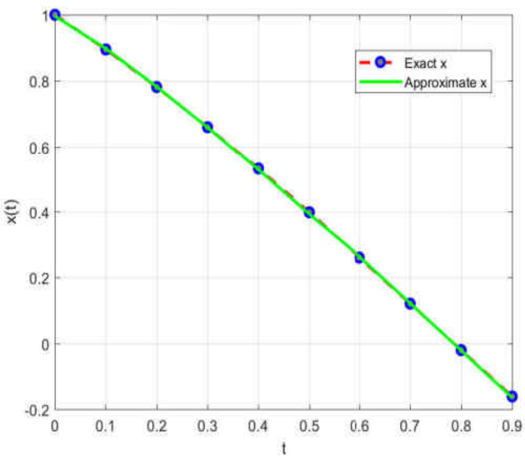

Figure 1: The soluion results of the proposed scheme for $n=128$, $k=3$ (Example 5.1).

because $k_t(t,t)\neq 0$, we differentiate again to obtain Volterra integral equation of the second kind as:

$$x(t)=\cosh t+\frac{1}{2}t\sinh t-\int_0^t \sinh(s-t)x(s)ds. \tag{5.43}$$

Now, we apply the MBPFs method for solving Equation (5.43). Table 3 presents the numerical results. Moreover, Figure 2 shows the results of the exact and the approximate solutions by the present method for $n=128$, $k=3$.

Example 5.3. *Consider the following Volterra integral equation of the first kind [16]:*

$$e^t-\sin t-\cos t=2\int_0^t \sin(s-t)x(s)ds, \tag{5.44}$$

with the exact solution $x(t)=e^t$. By using differentiation, we have:

$$e^t-\cos t+\sin t=2\int_0^t \cos(s-t)x(s)ds, \tag{5.45}$$

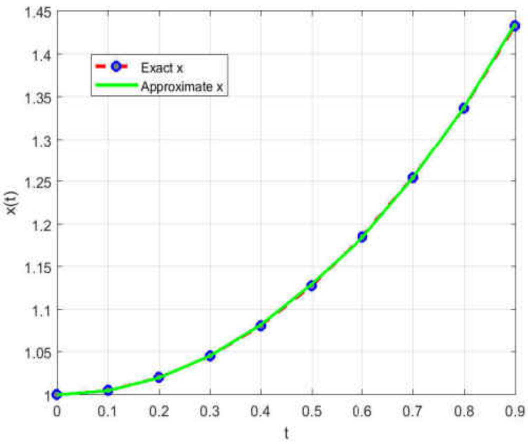

Figure 2: The soluion results of the proposed scheme for $n = 128$, $k = 3$(Example 5.2).

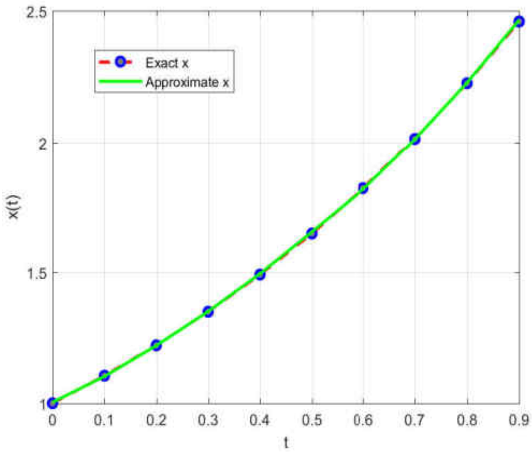

Figure 3: The soluion results of the proposed scheme for $n = 128$, $k = 3$(Example 5.3).

which is still a Volterra integral equation of the first kind. According to the Remark on Page 6*, because $k_t(t,t) \neq 0$, we differentiate again to obtain Volterra integral equation of the second kind as:*

$$x(t) = \frac{1}{2}(e^t + \sin t + \cos t) - \int_0^t \sin(s-t)x(s)ds. \tag{5.46}$$

Now, we apply the MBPFs method for solving Equation (5.46). Table 4 presents the numerical results. Moreover, Figure 3 depicts the results of the exact and the approximate solutions by the present method for $n = 128$, $k = 3$.

6 Conclusion

In this paper, a numerical method based on MBPFs and its operational matrix to solve Volterra integral equation of the second kind arisen from Volterra integral equation of the first kind are presented. Since Volterra integral equations of the first kind with kernels like $k(t,s) = k(\mid t-s \mid)$ and $k(t,t) \to 0$, that were converted to a system of linear equations have a large condition number, in fact, ill position has been eliminated by converting Volterra integral equations of the first kind to Volterra integral equations of the second kind. Furthermore, by proving the convergence of the proposed method, the convergence rate of the method is calculated. Numerical examples confirm the reliability and the accuracy of the above mentioned points.

References

[1] S. Hatamzadeh-Varmazyar, M. Naser-Moghadasi, Z. Masouri, A moment method simulation of electromagnetic scattering from coducting bodis, *Progress in Electromagnetics Research* 81 (2008) 99-119.

[2] S. Hatamzadeh-Varmazyar, M. Naser-Moghadasi, E. Babolian, Z. Masouri, Numerical approach to survery the problem of electromagnetic scattering from resistive strips based on using a set of orthogonal basis functions, *Progress in Electromagnetics Research* 81 (2008) 393-412.

[3] S. Aggarwal, R. Chauhan, N. Sharma, Application of elzaki transform for solving linear Volterra integral equation first kind, *International Journul of Research in Advent Technology* 6 (2018) 3687-3692.

[4] A. R. Vahidi, T. Damercheli, A modified ADM for solving systems of linear Fredholm integral equations of the second kind, *Applied Mathematical Sciences* 6 (2012) 1267-1273.

[5] A. Shoja, A. R. Vahidi, E. Babolian, A spectral iterative method for solving nonlinear singular Volterra integral equations of Abel type, *Applied Mathematical Sciences* 112 (2017) 79-90.

[6] M. A. Alzhrani, H. O. Bakodah, M. Al-mazmumy, A 3/8 Simpson's numerical scheme for the classes of Volterra integral equations of first kind, *Nonlinear Analysis and Differential Equations* 7 (2019) 99-113. http://dx.doi.org/10.12988/nade.2019.9812/

[7] K. Maleknejad, R. Mollapourasl, M. Alizadeh, Numerical solution of Volterra type integral equation of the first kind with wavelet basis,*Appl. Math. Comput.* 194 (2007) 400-405.

[8] K. Maleknejad, E. Hashemizadeh, R. Ezzati, A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation, *Commun. Nonlin. Sci. Numer. Silmulat.* 16 (2011) 647-655.

[9] M. Bahmanpour, M. A. Fariborzi Araghi, Numerical solution of Fredholm and Volterra integral equations of the first kind using wavelets bases, *The Journul of Mathematics and Computer Science* 5 (2012) 337-345.

[10] I. O. Isaac, A. O. Unanam, Quadrature method for one-dimensional Volterra integral equations of the first kind, *Journul of Modern Mathematics and Statistics* 4 (2010) 109-114.

[11] N. Khan, M. S. Hashmi, S. Iqbal, T. Mahmood, Optimal homotopy asympotic method for solving Volterra integral equations of first kind, *Alexandria Engineering Journal* 5 (2014) 751-755.

[12] E. Babolian, Z. Masouri, Direct method to solve Volterra integral equation of the first kind using operational matrix with Block-Pulse functions, *Journal of Computational and Applied Mathematics* 220 (2008) 51-57.

[13] K. Maleknejad, B. Rahimi, Modification of block–pulse functions and their application to solve numbering Volterra integral equation of the first kind, *Commuh. Nonlin. SCi. Numer. Simulat.* 16 (2011) 2469-2477.

[14] Z. Masouri, E. Babolian, S. H. Varmazyar, An expansion-iterative method for numeically solving Volterra integral equation of the first kind, *Comput. Math. APPl.* 59 (2010) 1491-149.

[15] ZH. Jiang, W. Schaufelberger, Block-pulse functions and their applications in control systems, *Springer-Verlag, Berlin*, (1992).

[16] A. M. Wazwaz, Linear and Nonlinear Integral Equations, Methods and Applications, *Springer Heidelberg Dordrecht London NewYork*, ISBN 978-3-642-21448-6.

[17] F. Mirzaee, E. Haddadiyan, Approximate solutions for mixed nonlinear Volterra-Fredholm type integral equations via modified block-pulse functions, *Journul of the Association of Arab Universities for Basic and Applied Sciences* 12 (2012) 65-73.

[18] S. Aggarwal, N. Sharma, Laplace transform for the solution of first kind linear Volterra integral equation, *Journul of Advanced Researchin Applied Mathematics and Statistics* 4 (2019) 16-23.

Mahnaz Mohammadi is a Ph.D student of Applied Mathematics (Numerical Analysis) in Islamic Azad University South Tehran Branch, Tehran, Iran. Her research interests include: Differential equations, linear and nonlinear integral equations.

Alireza Vahidi is Associate Professor of Applied Mathematics at the Department of mathematic, Islamic Azad University. His main researches interest is including numerical solution of integral equations and differential equations.

Tayebeh Damercheli is Assistance Professor of Applied Mathematics at the Department of mathematic, Islamic Azad University. His main research interest is including numerical solution of deterministic and stochastic integral equations, differential equations and dynamic systems arise from optimal control problems.

Saeid Khezerloo is Ph.D of Applied Mathematics. He is member of Department of Mathematics, South Tehran Branch, Islamic Azad University, Tehran, Iran. Interested in fuzzy mathematics, numerical solution of fuzzy differential equations, numerical analysis fuzzy linear algebra and fuzzy ranking.

Mostafa Nouri is Ph.D. of applied mathematics. He is member of mathematics department, Islamic Azad University, South Tehran Branch, Tehran, Iran. Interested in Integral Equations, Finance Mathematics, Stochastic Equations.