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Research Article

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A Fast Numerical Method Based on Hybrid Taylor and Block-Pulse Functions for Solving Delay Differential Equations

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Abstract

In this article, a fast numerical approach is proposed for finding the solution of nonlinear delay differential equations by using hybrid Taylor and Block-pulse Functions (HTBPFs). Firstly, some features of hybrid functions which are a combination of Block-Pulse functions and Taylor polynomials on the interval are introduced $[0, 1)$. In this spectral approach, the operational matrices of stretch, derivation and coefficient matrices are utilized. Based on these piecewise functions, we transfer delay differential equations (DDEs) into a system of linear or nonlinear algebraic equations. Also, in this numerical approach, it is shown that these operational matrices are sparse which is an effective advantage of the fast implementation of numerical computation. Then, error analysis is done. Finally, three examples are solved to show that the new proposed approach is comparable with other methods of high accuracy and efficiency.

Keywords : Delay differential equations; Hybrid functions; Operation matrix; Taylor function; Block-Pulse function; Coefficient matrix.

1 Preliminaries and problem formulation

IN recent years, extensive scientific research has been done on DDEs. These equations play an important role in the modeling and analyzing of many problems that arise in the fields of population dynamics, physiological, infections and chemical kinetics [1, 4, 5, 28]. The first delay models in engineering were introduced by Von Schlippe and Dietrich for modelling wheel Shimmy [24], and Minorsky for ship stabilization [20]. Special calculation and theorems for DDEs are quite well-developed by publishing articles, text books, useful software packages. Some of them are shown in [6, 22, 26]. Time delay is also a key element of the population of machine tool chatter [8]. Most of all sophisticated models have been appeared for turning and milling applications in the last decade [2]. Therefore, the application and performance of these equations in different fields of science and engineering like

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transmission lines, communication networks, biological models and population dynamics [18, 27] have caused many authors and researchers to be interested in solving DDEs with various methods, such as the process of solutions the material, energy balances and electrodynamics are presented by a delay dynamical system including delayed states [10, 17].

Since DDEs are used in different fields and many scientists are interested in them, the notion of analytical and numerical method has been used to solve these equations. For example, authors of [14] presented a computer algebra system for solving DDEs. In [3], M. Behroozifar and S. A. Yosefi solved DDEs by using operational matrices of hybrid Block-Pulse function and Bernstein. Musa Cakir et al. [7] have applied Adomian decomposition method and the differential transform method for solving multi-pantograph DDEs. Also, researchers of [31, 32] employed a direct method for solving time-varying delay systems and linear delay differential equations. Also, M.S. Hafshejani et al. in [13] obtained numerical solution of DDE using Legendre wavelet method. The main purpose of this paper is to express a new spectral approach based on HTBPFs for solving DDE of the from

$$\begin{cases} \frac{d}{dt}y(t) = p(t)\,y(\frac{t}{\lambda}) + q(t)y(t),\ 0 < t \leq t_f \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where p, q are known continuous functions on $I(t_f) := [0, t_f]$ and $\lambda > 1$ is stretched argument that plays an important role in modeling with DDEs.

This paper is organized as follows:

In Section 2, some preliminaries and basic definitions of the HTBPFs are given. Section 3 is devoted to introduce the operational matrices such as stretch, product and derivative of HTBPFs. In Section 4, a numerical approach is proposed to solve problem (1.1) using the results obtained for the first time. Section 5 has been estimated the error analysis of our proposed technique. In Section 6, the scheme is applied on three examples. Also some graphs and tables are presented to show the applicability and accuracy of our technique. Finally, in Section 7 the conclusion of this paper is presented.

2 The HTBPFs and Their Properties

In this section, we describe the important definition and attributes that are required for the computation and implementation of the new numerical method.

Definition 2.1. *A set $\{b_i(t) : i = 1, 2, \ldots, m\}$ of Block-Pulse functions are defined on $[0, 1)$ by [9].*

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m} \leq t < \frac{i}{m}, \\ 0, & otherwise. \end{cases} \tag{2.2}$$

Also, these piecewise functions are disjoint and orthogonal. Also, we have $b_i(t)\,b_j(t) = \delta_{ij} b_i(t)$. By considering Taylor polynomials $T_m(t) = t^m$ on the interval $[0, 1]$, the HTBPFs can be presented as follows.

Definition 2.2. *For $m = 0, 1, \ldots, M-1$ and $n = 1, 2, ..., N$, the orthogonal set of HTBPFs $b(n, m, t)$ are defined on $[0, 1)$ as*

$$b(n,m,t) = \begin{cases} T_m(Nt-(n-1)), & \frac{n-1}{N} \leq t < \frac{n}{N}, \\ 0, & otherwise. \end{cases} \tag{2.3}$$

On the other hand, this Hybrid functions can be expressed in the following way. The piecewise function for fixed $a \in [0, 1)$ is defined as follows

$$U_a(t) = \begin{cases} T_m(Nt-(n-1)), & t \geq a, \\ 0, & t < a, \end{cases} \tag{2.4}$$

where $t \in [0, 1)$. From Equation (2.4), one can write

$$U_{\frac{n-1}{N}}(t) = \begin{cases} T_m(Nt-(n-1)), & t \geq \frac{n-1}{N}, \\ 0, & t < \frac{n-1}{N}, \end{cases} \tag{2.5}$$

and

$$U_{\frac{n}{N}}(t) = \begin{cases} T_m(Nt-(n-1)), & t \geq \frac{n}{N}, \\ 0, & t < \frac{n}{N}. \end{cases} \tag{2.6}$$

Using functions $U_{\frac{n-1}{N}}(t)$, $U_{\frac{n}{N}}(t)$ yields

$$b(n,m,t) = U_{\frac{n-1}{N}}(t) - U_{\frac{n}{N}}(t)\,. \tag{2.7}$$

2.1 Function approximation

In this section, an arbitrary function $f \in C^M[0,1]$ may be approximated in terms of HTBPFs as follows

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} b_{nm}(t). \tag{2.8}$$

Here, coefficient c_{nm} can be calculated in two ways

$$c_{nm} = \frac{\langle f, b_{nm} \rangle}{\langle b_{nm}, b_{nm} \rangle}, \tag{2.9}$$

where $m = 0, 1, \ldots, M-1, n = 1, 2, \ldots, N,$ and $\langle ., . \rangle$ denotes the inner product. Coefficient c_{nm} can be obtained as

$$c_{nm} = \frac{1}{N^m m!} \left[\frac{d^m f(t)}{dt^m} \right]_{t=\left(\frac{n-1}{N}\right)}. \tag{2.10}$$

Therefore, the function f can be estimated by truncating the infinite series in (2.8) as [19]

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n\,m} b_{n\,m}(t) = C^T B(t), \tag{2.11}$$

where

$$C = [c_{10}, \ldots, c_{1(M-1)}, c_{20}, c_{21}, \ldots,$$
$$c_{2(M-1)}, \ldots, c_{N0}, c_{N1}, \ldots, c_{N(M-1)}]^T, \tag{2.12}$$

and

$$B(t) = [b_{10}(t), \ldots, b_{1(M-1)}(t), b_{20}(t), \ldots,$$
$$b_{2(M-1)}(t), \ldots, b_{N0}(t), \ldots, b_{N(M-1)}(t)]^T. \tag{2.13}$$

3 Operational matrices of HTBPFs

In this part of the study, operational matrices stretch and derivative that play an important role in simplifying types of DDEs and the implementation of the proposed framework are introduced.
The operational matrix of derivative D is expressed by

$$\frac{d}{dt} B(t) \simeq DB(t), \tag{3.14}$$

where $\frac{d}{dt} b_{nm}(t)$ is approximated by $\sum_{i=1}^{N} \sum_{j=0}^{M-1} c_{ij}{}^{(n,m)} b_{ij}(t)$ and,

$$c_{ij}{}^{(n,m)} = \frac{1}{N^j j!} \left[\frac{d^j}{dt^j} \left(\frac{d}{dt} b_{nm}(t) \right) \right]_{t=\frac{i-1}{N}}, \tag{3.15}$$

where $i, n = 1, ..., N$ and $j, m = 0, ..., M-1$. For example, for $N = 2$, $M = 3$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 0 \end{pmatrix}.$$

Also, stretch operational matrix of HTBPFs is defined as matrix S satisfying in the relation

$$B\left(\frac{t}{\lambda}\right) \simeq SB(t), \tag{3.16}$$

here, vector $B(\frac{t}{\lambda})$ can be written as

$$B\left(\frac{t}{\lambda}\right) = [b_{10}(\tfrac{t}{\lambda}), \ldots, b_{1(M-1)}(\tfrac{t}{\lambda}), b_{20}(\tfrac{t}{\lambda}), \ldots,$$
$$b_{2(M-1)}(\tfrac{t}{\lambda}), \ldots, b_{N0}(\tfrac{t}{\lambda}), \ldots, b_{N(M-1)}(\tfrac{t}{\lambda})]^T. \tag{3.17}$$

Then, vector elements $B(\frac{t}{\lambda})$ in equation (3.16) can be approximated by using HTBPFs as

$$\sum_{i=1}^{N} \sum_{j=0}^{M-1} r_{nm,ij} b_{ij}(t) = R^T{}_{nm} B(t), \tag{3.18}$$

where

$$r_{nm,ij} = \frac{1}{N^j j!} \left[\frac{d^j}{dt^j} b_{nm} \left(\frac{t}{\lambda} \right) \right]_{t=\left(\frac{i-1}{N}\right)}, \tag{3.19}$$

where $n, i = 1, \ldots, N$ and $m, j = 0, 1, \ldots, M-1$. Hence,

$$S = [R^T_{10}, \ldots, R^T_{1M-1}, \ldots, R^T_{N0}, \ldots, R^T_{NM-1}].$$

For $\lambda = 2, N = 2, M = 3$, we have

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{3}{4} & \frac{1}{2} & 0 \\ \frac{1}{16} & \frac{1}{4} & \frac{1}{4} & \frac{9}{16} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Table 1: Comparison of maximum absolute errors between our proposed method for $N = 1$, $M = 25$, $\lambda = 2$ and other methods for Example 6.1.

t	e_{SM} $h = 0.001$[25]	$e_{VIM}, e_{ADM}, e_{HPM}$	e_{SM} $h = 0.001$[11]	e_{LWM} [16]	$e_{(1,25)}$
0.0	0.00	0.00	0.00	0.00	0.00
0.2	1.37E-11	0.00	3.10E-15	1.00E-15	2.00E-15
0.4	3.27E-11	1.00E-15	7.54E-15	0.00	2.88658E-15
0.6	5.86E-11	2.19E-13	1.39E-14	2.00E-15	3.77476E-16
0.8	9.54E-11	9.36E-12	2.13E-14	5.00E-15	3.55271E-15
0.9	1.43E-10	1.72E-10	3.19E-14	3.00E-15	2.7182E-15

Table 2: The maximum absolute errors for $N = 1$ and different values of M with $\lambda = 2$, for Example 6.2

t	$e_{(1,9)}$	$e_{(1,13)}$
0.0	0.00000000	0.00000000
0.2	6.99828E-8	1.75415E-13
0.4	1.01575E-7	1.76303E-13
0.6	8.95865E-8	2.20046E-13
0.8	1.26004E-8	2.50244E-13
0.9	2.13E-13	1.42035E-14

Table 3: The approximate solutions values of $N = 1$ and different values M with $\lambda = 1.25$ for Example 6.3.

t	$y_{(1,10)}$	$y_{(1,12)}$	$y_{(1,15)}$	$y_{(1,20)}$	$y_{(1,25)}$
0.0	1	1	1	1	1
0.08	0.859908	0.854663	0.851031	0.851092	0.851031
0.16	0.732028	0.725705	0.722282	0.722319	0.722283
0.24	0.617757	0.613116	0.611232	0.611256	0.611232
0.32	0.518054	0.516257	0.515648	0.515673	0.515648
0.4	0.433181	0.433811	0.433561	0.433583	0.433561

Table 4: The approximate solutions values of $N = 2$ and different values M with $\lambda = 1.25$ for Example 6.3.

t	$y_{(2,10)}$	$y_{(2,12)}$	$y_{(2,15)}$	$y_{(2,20)}$	$y_{(2,25)}$
0.0	1	1	1	1	1
0.08	0.855136	0.852321	0.851263	0.851042	0.851032
0.16	0.723588	0.722575	0.722358	0.722292	0.722283
0.24	0.610688	0.611564	0.611387	0.611239	0.611233
0.32	0.518009	0.516583	0.515702	0.515655	0.515649
0.4	0.437755	0.433125	0.433666	0.433562	0.433561

4 Description of spectral method for solving DDEs

The main objective of this stage of the paper is to implement a very effective numerical approach to solve DDE (1.1), with initial conditions $y(0) = y_0$ by employing HTBPFs. Using equation (2.11), our unknown function y is approximated as

$$y(t) \simeq C^T B(t), \tag{4.20}$$

where C and $B(t)$ are given in equations (2.12) and (2.13). Considering equations (3.14) and (2.11), we obtain

$$\frac{d}{dt} y(t) \simeq C^T B'(t) \simeq C^T D B(t), \tag{4.21}$$

Table 5: The approximate solutions absolute errors in solution of $N = 1$ and different values M with $\lambda = 1.25$ for Example 6.3.

t	$\|y_{(1,9)} - y_{(1,10)}\|_\infty$	$\|y_{(1,10)} - y_{(1,11)}\|_\infty$	$\|y_{(1,24)} - y_{(1,25)}\|_\infty$
0.0	0.00000000	0.00000000	0.000000000
0.2	0.00882114	0.00846740	4.318120E-5
0.4	0.00001281	0.00035625	3.622701E-5
0.6	0.00205286	0.00200203	1.166110E-5
0.8	0.00126533	0.00114213	1.167641E-5
0.9	0.00208259	0.00221691	7.534360E-7

Table 6: The approximate solutions absolute errors in solution of $N = 2$ and different values M with $\lambda = 1.25$ for Example 6.3.

t	$\|y_{(2,10)} - y_{(2,12)}\|_\infty$	$\|y_{(2,12)} - y_{(2,15)}\|_\infty$	$\|y_{(2,15)} - y_{(2,20)}\|_\infty$	$\|y_{(2,20)} - y_{(2,25)}\|_\infty$
0.0	0.00000000	0.000000000	0.000000000	0.00000000
0.2	0.00245903	0.000012336	0.000098801	8.69526E-6
0.4	0.00463037	0.000541443	0.000103834	9.79565E-7
0.6	0.00052784	0.001049940	0.000182796	1.04518E-5
0.8	0.03337610	0.013465800	0.004002501	1.05292E-3
0.9	0.11131107	0.061628956	0.014736464	0.009576403

also, we use

$$B\left(\frac{t}{\lambda}\right) \simeq SB\left(t\right), \tag{4.22}$$

where D and S are defined in equations (3.14) and (3.15).
Substituting equations (4.20), (4.21) and (4.22) into the main problem (1.1) and replacing $\simeq$ by $=$ give

$$C^T DB\left(t\right) = P\left(t\right) SB\left(t\right) + q\left(t\right) C^T B\left(t\right). \tag{4.23}$$

Furthermore, the initial condition in equation (1.1) has to be used i.e.

$$C^T B\left(0\right) \simeq y_0. \tag{4.24}$$

Finally, equations (4.23) and (4.24) give a system of linear equations. $NM-1$ Newton-Cotes points are applied for finding C as

$$x_p = \frac{2p-1}{2NM}, \; p = 1, 2, \ldots, NM-1, \tag{4.25}$$

to obtain,

$$C^T DB(x_p) = P(x_p)S\, B(x_p) + q(x_p)\, C^T B\left(x_p\right), \tag{4.26}$$

where $p = 1, 2, \ldots, NM-1$. Now, with combining equations (4.24) and (4.26), a system of $N \times M$ linear equations is obtained that can be solved easily. Then vector C is used for finding the approximate solution as

$$y\left(t\right) \simeq C^T B\left(t\right), t \in [0, 1). \tag{4.27}$$

5 Error Analysis

In this section, error analysis is performed for our numerical approach, so an upper norm for the error is found. Consider the following DDE

$$\begin{cases} y'(t) = ay(t) + by(rt), & 0 \leq t \leq 1, \\ y(0) = y_0, \end{cases} \tag{5.28}$$

where $y \in PC^1[0, 1)$, with $PC^1[0, 1)$ as the set of piecewise functions having continuous First Derivative, $0 < r < 1$ and $a, b \in R$.
Consider arbitrary function $f \in C^M[0, 1]$. Taylor polynomial of degree $M-1$ is utilized to approximate f.

$$\begin{aligned} e_n\left(t\right) &= f\left(t\right) - \sum_{i=0}^{M-1} \frac{f^{(i)}\left(a\right)}{M!}(t-a)^i \\ &= \frac{(t-a)^M}{M!} f^{(M)}\left(\xi\right), \end{aligned} \tag{5.29}$$

where

$$\|e_n\|_\infty \le \frac{(b-a)^M}{M!}\left\|f^{(M)}\right\|_\infty. \quad (5.30)$$

Now, if the HTBPFs on the interval $[0,1)$ are considered, then for ith sub interval $\left[\frac{i-1}{N}, \frac{i}{N}\right)$, the truncation error satisfies the following inequality (see [23]).

$$\begin{aligned}\|e_n\|_\infty &\le \frac{\left(\frac{i}{N}-\frac{i-1}{N}\right)^M}{M!}\left\|f^{(M)}\right\|_\infty \\ &= \frac{1}{M!\,N^M}\left\|f^{(M)}\right\|_\infty,\end{aligned} \quad (5.31)$$

and

$$\|e_n\|_\infty \le \frac{1}{M!\,N^M}\left\|f^{(M)}\right\|_\infty. \quad (5.32)$$

Integrating both sides of DDE (5.28) from 0 to t yields

$$\begin{aligned}\int_0^t y'(s)\,ds = \int_0^t ay(s)\,ds \\ + \int_0^t by(rs)\,ds,\ 0 \le t \le 1.\end{aligned} \quad (5.33)$$

Therefore,

$$\begin{aligned}y(t) - y(0) = a\int_0^t y(s)\,ds \\ + b\int_0^t y(rs)\,ds,\end{aligned} \quad (5.34)$$

or

$$\begin{aligned}0 = y(t) - y(0) - a\int_0^t y(s)\,ds \\ - b\int_0^t y(rs)\,ds.\end{aligned} \quad (5.35)$$

On the other hand, using (2.8), (2.10) and (2.11), results

$$\left\|y - \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm}b_{nm}\right\|_\infty \le \frac{1}{M!\,N^M}\left\|y^{(M)}\right\|_\infty. \quad (5.36)$$

Therefore, if the function approximation error with $e_J(t) = \left|y(t) - \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm}b_{nm}(t)\right|$ is shown, then (5.36) can be rewritten as follows

$$\|e_J\|_\infty \le \frac{1}{M!\,N^M}\left\|y^{(M)}\right\|_\infty. \quad (5.37)$$

On the other hand, we have

$$y(t) - y(0) \simeq y_J(t) - y(0). \quad (5.38)$$

So, using (5.35) yields

$$\begin{aligned}e(t) = y_J(t) - y_0 - a\int_0^t y_J(s)\,ds \\ - b\int_0^t y_J(rs)\,ds,\ 0 \le t \le 1,\end{aligned} \quad (5.39)$$

where, y_J is an approximation function of y. Therefore, applying equations (2.11) and (5.39), gives

$$\begin{aligned}e(t) = c^T B(t) - y(0) - a\int_0^t c^T B(s)\,ds \\ - b\int_0^t c^T sB(s)\,ds,\ 0 \le t \le 1.\end{aligned} \quad (5.40)$$

Now, we subtract equation (5.40) from equation (5.35) to obtain

$$\begin{aligned}e(t) - 0 = c^T B(t) - y(t) \\ - a\int_0^t c^T B(s)\,ds + a\int_0^t y(s)\,ds \\ - b\int_0^t c^T sB(s)\,ds + b\int_0^t y(rs)\,ds.\end{aligned} \quad (5.41)$$

Then

$$\begin{aligned}e(t) = c^T B(t) - y(t) \\ - a\left(\int_0^t \left(c^T B(s) - y(s)\right)ds\right) \\ - b\left(\int_0^t \left(c^T sB(s) - y(rs)\right)ds,\right.\end{aligned} \quad (5.42)$$

and

$$e(t) = e_J(t) - a\int_0^t e_J(s)\,ds \qquad (5.43)$$
$$- b\int_0^t e_J(rs)\,ds.$$

Thus, according to the property of the absolute value function, (5.43) is obtained as

$$|e(t)| \le \left|e_J(t)\right| + |a|\int_0^t |e_J(s)|ds \qquad (5.44)$$
$$+ |b|\int_0^t |e_J(rs)|ds.$$

Finally, following is obtained

$$\|e\|_\infty \le \frac{(1+|a|+|b|)}{M!\,N^M}\left\|y^{(M)}\right\|_\infty. \qquad (5.45)$$

6 Illustrative examples

In this section, three different examples are solved which illustrate the accuracy, applicability and efficiency of the scheme. The error of the new numerical approach based on HTBPFs is

$$e(N,M) = \left\|y - y_{(N,M)}\right\|_\infty \qquad (6.46)$$
$$= max_{0\le t\le 1}\left|y(t) - y_{(N,M)}(t)\right|,$$

where y is the exact solution and $y_{(N,M)}$ is the approximate solution obtained by the proposed method, with N, M defined in Definition 2.2. In our implementation, the calculations are done on a personal computer with core-i5 processor, 2.67 GHZ frequency, and 4GB memory. In numerical solution, all examples of this paper are used for computations of the Mathematica 11 software.

Example 6.1. *As the first example, the following DDE [11, 13, 15, 16, 25] is considered*

$$\begin{cases} \frac{d}{dt}y(t) = \frac{1}{2}e^{\frac{t}{2}}y\left(\frac{t}{2}\right) + \frac{1}{2}y(t), & 0 < t \le 1 \\ y(0) = 1, \end{cases} \qquad (6.47)$$

with the exact solution $y(t) = e^t$.

First, this equation with different values of N and M was solved. The operational matrices of stretch and derivative for $N = 1,\ M = 3,\ \lambda = 2$ are obtained in the following forms

$$S = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{16} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix},$$

and,

$$c_{10} = 1, c_{11} = 0.513467, c_{11} = 0.568481,$$

give

$$y_{(1,3)}(t) = \begin{cases} 1 + 0.513467t + 0.568481t^2, & t \in (0,1], \\ 0, & t \notin (0,1]. \end{cases} \qquad (6.48)$$

Figure 1, compares the exact solution with the approximate solution obtained by proposed method for $N = 1,\ M = 25$. The absolute errors for $N = 1,\ M = 25$ are shown in Figure 2 where extremely high accuracy can be seen. In Table

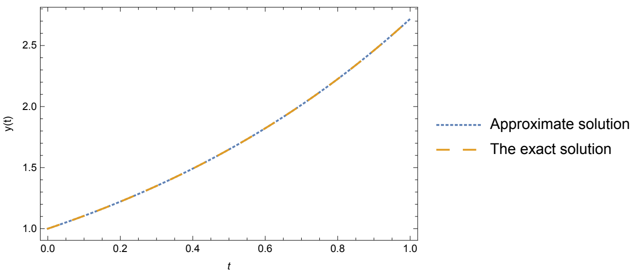

Figure 1: *Comparison of the computed and exact solutions forN = 1, M = 25, with λ = 2 for Example 6.1.*

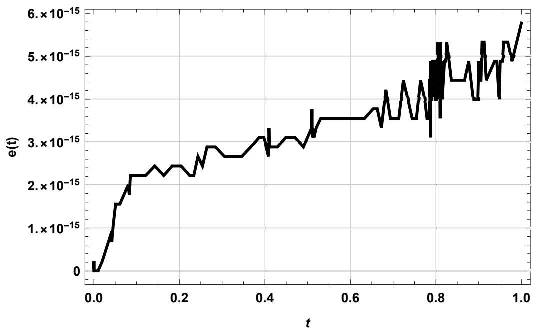

Figure 2: *Plot of the absolute error for $N = 1,\ M =$* 25, *with $\lambda = 2$ for Example 6.1.*

1, our results for $N = 1,\ M = 25,\ \lambda = 2$ are compared with spline methods [11, 25], Adomian decomposition method [30], homotopy perturbation method [29] and Legendre wavelet method [16].

Example 6.2. *Next the pantograph equation is discussed as follows [4, 15, 16].*

$$\begin{cases} \frac{d}{dt}y(t) - y(\frac{t}{2}) = 0, \quad 0 < t \leq 1, \\ y(0) = 1, \end{cases} \tag{6.49}$$

with the exact solution

$$y(t) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^{\frac{1}{2}k(k-1)}}{k!} t^k.$$

Table 2 shows the maximum errors for Example 6.2.

Example 6.3. *Here the third example proposed in [21] is considered and also is extensively studied by Fox et al. [12]. Since the exact solution of this problem is not available, many authors have tried to solve it.*
Consider the following DDE of the form

$$\begin{cases} \frac{dy(t)}{dt} = ay(t/\lambda) + by(t), \quad 0 < t \leq 1, \\ y(0) = y_0. \end{cases} \tag{6.50}$$

The numerical solution with the new spectral approach for this equation is under the following conditions

$$a = -1, b = -1, \lambda = 1.25 \quad and \quad y_0 = 1.$$

7 Conclusion

In this article a spectral method based on HTBPFs for solving DDEs is presented. The error analysis of the spectral approach has been done. The computing time of implementing this technique compared with other known methods are very low. Three numerical examples are provided to confirm the applicability and accuracy of the scheme.

References

[1] N. Ahmadi, A. R. Vahidi, T. Allahviranloo, An efficient approach based on radial basis functions for solving stochastic fractional differential equations,*Mathematical Sciences* 11 (2017) 113-118.

[2] Y. Altintas, S. Engin, E. Budak, Analytical stability prediction and design of variable pitch cutters, *J. Manuf. Sci. Eng.* 121 (1999) 173-178.

[3] M. Behroozifar, S. A. Yousefi, Numerical solution of delay differential equations via operational matrices of hybrid of block-pulse functions and Bernstein polynomials, *J. Comput. Methods, DEs.* 1 (2013) 78-95.

[4] A. Bellen, M. Zennaro, Adaptive integration of delay differential equations, In proceding of the Workshop CNRS-NSF, *Advances in Time-Delay Systems., Paris,* (2003).

[5] A. Bellen, M. Zennaro, Numerical methods for delay differential equations, *Num. Math. Sci. Comput Series, Oxford university press, New York,* (2003).

[6] R. Bellman, K. L. Cooke, Differential-Difference Equations, *Math. Sci. Eng 6. Academic Press, New York,* (1963).

[7] M. Cakir, D. Arslan, The Adomian decomposition method and the differential transform method for numerical solution of multi-pantograph delay differential equations, *J. Appl. Math.* 6 (2015) 1332-1343.

[8] O. Danek, Selbsterregte schwingungen an werkzeugmaschine, *VEB Verlag Technik, Berlin, Germany,* (1962).

[9] M. R. Doostdar, A. R. Vahidi, T. Damercheli , E. Babolian, A hybrid functions method for solving linear and non-linear systems of ordinary differential equations, *Math. Commun.* 26 (2021) 197-213.

[10] L. Dugard, E. Verriest, Stability and control of time-delay systems, *Lecture Notes in control and Information Sciences* 228 (1998) 259-282.

[11] A. El-Safty, M. S. Salim, M. A. El-Khatib, Convergence of the spline function for delay dynamic system, *J. Int. Comput. Math.* 80 (2003) 509-518.

[12] L. Fox, D. F. Mayers, J. R. Ockendon, A. B. Tayler, On a functional differential equation, *J. Inst. Math. Appl.* 8 (1971) 271-307.

[13] M. S. Hafshejani, S. Karimi Vanani, J. S. Hafshejani, Numerical solution of delay differential equations using Legendre wavelet method, *J. World. Appl. Sci.* 13 (2011) 27-33.

[14] J. M. Heffernan, R. M. Corless, Solving some delay differential equations with computer algebra, *J. Math. Sci.* 31 (2006) 21-34.

[15] S. Karimi Vanani, A. Aminataei, On the numerical solution of neutral delay differential equations using multiquadric approximation scheme, *J. Bull. Korean Math. Soc.* 45 (2008) 663-670.

[16] S. Karimi Vanani, A. Aminataei, On the numerical solution of nonlinear delay differential equations, *J. Concrete. Appl. Math.* 8 (2010) 568-576.

[17] H. Kocak, A. Yildirim, Series solution for a delay differential equation arising in electrodynamics, *Commun. Numer. Methods Eng.* 25 (2009) 1084-1096.

[18] M. Malek-Zavarei, M. Jamshidi, Time-Delay systems: Analysis, Optimization and Applications, North-Holland System and Control Series, *J. Else. Sci.* 9 (1987).

[19] H. R. Marzban, Parameter identification of linear multi-delay systems via a hybrid of block-pulse functions and Taylor's polynomials, *J. Int. Cont.* 90 (2017) 504-518.

[20] N. Minorsky, Self-excited oscillations in dynamical systems possessing retarded action, *J. Appl. Mech.* 9 (1942) 65-71.

[21] G. P. Rao, T. Srinivasan, An optimal method of solving differential equations characterizing the dynamics of a current collection system for an electric locomotive, *J. IMA. Appl. Math.* 25 (1980) 329-342.

[22] M. Razzaghi, H. R. Marzban, A hybrid domain analysis for systems with delays in state and control, *J. Math. Eng.* 7 (2001) 337-353.

[23] A. Saadatmandi, M. Dehghan, Variational iteration method for solving a generalized pantograph equation, *J. Comput. Math. Appl.* 58 (2009) 2190-1296.

[24] B. V. Schlippe, R. Dietrich, Shimmying of a pneumatic wheel, Lilienthal-Gesellschaft für Luftfahrtforschung, *Bericht.*, 140 (1941) 125-160. (transled for the AAF in 1947 by Meyer & Company)

[25] M. Shadia, Numerical solution of delay differential and neutral differential equations using Spline methods, *Ph.D Thesis, Assuit university, Egypt* (1992).

[26] L. F. Shampine, S. Thompson, Solving DDEs in Matlab, *J. Appl. Numer. Math.* 37 (2001) 441-458.

[27] H. L. Smith, An introduction to delay differential equations with applications to the life sciences, Texts In Applied Mathematics, *Springer, New York* (2011).

[28] A. R. Vahidi, Z. Azimzadeh, M. Didgar, An efficient method for solving Riccati equation using homotopy perturbation method, *Indian Journal of Physics* 87 (2013) 447-454.

[29] A. R. Vahidi, E. Babolian, Z. Azimzadeh, An improvement to the homotopy perturbation method for solving nonlinear Duffings equations, *Bulletin of the Malaysian Mathematical Sciences Society* 2 (2018) 1105-1117.

[30] A. R. Vahidi, E. Babolian, G. A. Cordshooli, F. Samiee, Restarted Adomian's decomposition method for Duffing's equation, *J. Math. Anal.* 3 (2009) 711-717.

[31] E. Zeynal, E. Babolian, T. Damercheli, Direct methods for solving time-varying delay systems, *Mathematical Sciences* 14 (2020) 159-166.

[32] E. Zeynal, E. Babolian, T. Damercheli, A Direct Method For Solving Linear Delay Differential Equations, *International Journal of Industrial Mathematics* 12 (2020) 273-382.

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