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Research Article

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Numerical Solution of Second-Order Hybrid Fuzzy Differential Equations by Generalized Differentiability

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Abstract

In this research paper, a numerical method is presented for solving second-order hybrid fuzzy differential equations by using fuzzy Taylor expansion under generalized Hukuhara differentiability and also with convergence theorem. Also, the method is illustrated by solving several numerical examples. The final results showed that the solution of the second-order hybrid fuzzy differential equations.

Keywords : Fuzzy differential equations; Hybrid fuzzy differential equations; Fuzzy Taylor expansion; Generalized Hukuhara differentiability; gH-differentiability.

1 Preliminaries and problem formulation

THe study of fuzzy differential equations (FDEs) forms a suitable setting for the mathematical modeling of real world problems in which uncertainly or vagueness pervades. There are several approaches to studying fuzzy differential equations [19]. Historically, the first approach was the use of the Hukuhara differentiability for fuzzy-number-valued functions. Under this setting, mainly the existence and uniqueness of the solution of a fuzzy differential equation were studied [11, 16]. The strongly generalized differentiability was introduced in [7] and studied in [8, 9, 12, 17].

Particularly, the use of hybrid fuzzy differential equations (HFDE) is a natural way to model control systems with embedded uncertainty that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics. In recent years, many works have performed by several authors in numerical solutions of fuzzy differential equations. Furthermore, there are some numerical techniques to solve hybrid fuzzy differential equations [20, 21]. The paper is structured as follows; In Section 2, we list some basic definitions for fuzzy-valued functions. In Section 3, we develop a numerical solution for 2-order fuzzy hybrid differential equations. We define Taylor series expansion for fuzzy-valued functions such that f is generalized Hukuhara differentiable. According to types of differentiability, the Taylor expansion of the fuzzy function is investigated in different scenarios. Additionally, in Section 3, the uniqueness and exis-

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tence of second-order fuzzy differential equations are studied. Section 4, contains numerical examples to illustrate the method, and conclusions are drawn in Section 5.

2 Preliminaries

In what follows, we briefly recall the basic definitions and properties of the generalized Hukuhara derivative. We denote by $\mathbb{E}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which are defined over the real line.

Definition 2.1. *(See [14]) A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0;*
2. *$\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0;*
3. *$\underline{u}(r) \leq \overline{u}(r)$, for all $0 \leq r \leq 1$.*

A crisp number k is simply represented by $\underline{u}(r) = \overline{u}(r) = k, \ 0 \leq r \leq 1$.
For arbitrary $u, v \in \mathbb{E}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u + v]^r = [u]^r + [v]^r$, $[ku]^r = k[u]^r$ respectively.
In this paper, we follow [1] and represent an arbitrary fuzzy number with compact support by a pair of functions $\big(\underline{u}(r), \overline{u}(r)\big)$, $0 \leq r \leq 1$. Also, we use the Hausdorff distance between fuzzy numbers. This fuzzy number space as shown in [8] can be embedded into the Banach space $B = \overline{c}^2[0, 1] \times \overline{c}^2[0, 1]$ with the usual metric defined as follows: let $\mathbb{E}$ be the set of all upper semi-continuous normal convex fuzzy numbers with bounded $r-$level sets. Since $r-$cut of fuzzy numbers are always closed and bounded, the intervals are written as $u[r] = [\underline{u}(r), \overline{u}(r)]$, for all r. We denoted by ω the sets of all nonempty compact subsets of $\mathbb{R}$, and by ω_c the subsets of ω consisting of nonempty convex compact sets. The Hausdorff metric d_H on ω is defined by

$$d_H(K, S) = \max\{ \sup_{k \in K} \inf_{s \in S} \|K - S\| \ , \ \sup_{s \in S} \inf_{k \in K} \ \|K - S\|\}, \ K, S \in \omega,$$

where $K = (x, x')$, $S = (\lambda(x), \lambda(x'))$.

The metric D is defined on $\mathbb{E}$ as

$$D(u, v) = sup\{d_H(u[r], v[r]) : 0 \leq r \leq 1\}, \quad u, v \in \mathbb{E}.$$

For arbitrary $(u, v) \in \overline{c}^2[0, 1] \times \overline{c}^2[0, 1]$. The following properties are well-known:

1. $(\mathbb{E}, D)$ is a complete metric space;
2. $D(u \oplus w, v \oplus w) = D(u, v)$ for all $u, v, w \in \mathbb{E}$;
3. $D(ku, kv) = |k|D(u, v)$ for all $u, v \in \mathbb{E}$ and $k \in \mathbb{R}$;
4. $D(u \oplus w, v \oplus t) \leq D(u, v) + D(w, t)$, for all $u, v, w, t \in \mathbb{E}$;
5. $D(u \ominus w, v \ominus t) \leq D(u, v) + D(w, t)$, as long as $u \ominus w$ and $v \ominus t$ exist , $u, v, w, t \in \mathbb{E}$. Where, $\ominus$ is the Hukuhara difference (H-difference), it means that $u \ominus w = v$ if and only if $v \oplus w = u$.

Definition 2.2. *(See [10]) The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathbb{E}$ is defined as follows*

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} \quad (i) \ \ u = v + w; \\ or \ \ (ii) \ \ v = u + (-1)w. \end{cases}$$

In terms of r-levels we have $[u \ominus_{gH} v]^r = [min\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}, max\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}]$ and if the H-difference exists, then $u \ominus_H v = u \ominus_{gH} v$; the conditions for the existence of $w = u \ominus_{gH} v \in \mathbb{E}$ are

$$case(i) \begin{cases} \underline{w}(r) = \underline{u}(r) - \underline{v}(r) \ \ and \\ \overline{w}(r) = \overline{u}(r) - \overline{v}(r), \forall r \in [0, 1], \\ with \ \underline{w}(r) \ \ increasing, \\ \overline{w}(r) \ \ decreasing, \ \ \underline{w}(r) \leq \overline{w}(r), \end{cases}$$

$$case(ii) \begin{cases} \underline{w}(r) = \overline{u}(r) - \overline{v}(r) \ \ and \\ \overline{w}(r) = \underline{u}(r) - \underline{v}(r), \forall r \in [0, 1], \\ with \ \underline{w}(r) \ \ increasing, \\ \overline{w}(r) \ \ decreasing, \ \ \underline{w}(r) \leq \overline{w}(r). \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.

Remark 2.1. *Throughout the rest of this paper, we assume that* $u \ominus_{gH} v \in \mathbb{E}$.

Note that a function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{E}$ is called fuzzy-valued function. The $r-$level representation of this function is given by $f(t; r) = [\underline{f}(t; r), \overline{f}(t; r)]$, for all $t \in [a, b]$ and $r \in [0, 1]$.

Definition 2.3. *(See [15]) A fuzzy valued function* $f : [a, b] \to \mathbb{E}$ *is said to be continuous at* $t_0 \in [a, b]$ *if for each* $\epsilon > 0$ *there is* $\delta > 0$ *such that* $D(f(t), f(t_0)) < \epsilon$. *Whenever* $t \in [a, b]$ *and* $|t - t_0| < \delta$. *We say that* f *fuzzy continuous on* $[a, b]$ *if* f *is continuous at each* $t_0 \in [a, b]$.

Definition 2.4. *(See [23]) The generalized Hukuhara derivative of the fuzzy-valued function* $f : (a, b) \to \mathbb{E}$ *at* $t_0 \in (a, b)$ *is defined as*

$$f'_{gH}(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h}, \qquad (2.1)$$

if $f'_{gH}(t_0) \in \mathbb{E}$ *satisfying (2.1) exists, we say that* f *is generalized Hukuhara differentiable (gH-differentiable for short) at* t_0.

Definition 2.5. *(See [14]) Let* $f : [a, b] \to \mathbb{E}$ *and* $t_0 \in (a, b)$, *with* $\underline{f}(t; r)$ *and* $\overline{f}(t; r)$ *both differentiable at* t_0 *for all* $r \in [0, 1]$. *We say that*

- f *is* $[(i) - gH]-$*differentiable at* t_0 *if*

$$f'_{i.gH}(t_0; r) = [(\underline{f})'(t_0; r), (\overline{f})'(t_0; r)], \quad (2.2)$$

- f *is* $[(ii) - gH]$*-differentiable at* t_0 *if*

$$f'_{ii.gH}(t_0; r) = [(\overline{f})'(t_0; r), (\underline{f})'(t_0; r)]. \quad (2.3)$$

Definition 2.6. *(See [14]) We say that a point* $t_0 \in (a, b)$, *is a switching point for the differentiability of* f, *if in any neighborhood* V *of* t_0 *there exist point* $t_1 < t_0 < t_2$ *such that*

type(I) *at* t_1 *(2.2) holds while (2.3) does not hold and at* t_2 *(2.3) holds and (2.2) does not hold, or*

type(II) *at* t_1 *(2.3) holds while (2.2) does not hold and at* t_2 *(2.2) holds and (2.3) does not hold.*

Definition 2.7. *(See [2]) Let* $f : (a, b) \to \mathbb{E}$. *We say that* $f(x)$ *is gH-differentiable of the 2th-order at* $t_0 \in (a, b)$ *whenever the function* $f(x)$ *is gH-differentiable of the order* $i, i = 0, 1$, *at* t_0, $\left((f^{(i)}(t_0))_{gh} \in \mathbb{E}\right)$, *moreover there isn't any switching point on* (a, b). *Then there exists* $f''_{gH}(t_0) \in \mathbb{E}$ *such that* $f''_{gH}(t_0) = lim_{h\to 0} \frac{f'_{gH}(t_0+h) \ominus_{gH} f'_{gH}(t_0)}{h}$ *if* $f'_{gH}(t_0 + h) \ominus_{gH} f'_{gH}(t_0) \in \mathbb{E}$.

Definition 2.8. *(See [17]) Let* $f : [a, b] \to \mathbb{E}$ *and* $f'_{gH}(x)$, *gH-differentiable at* $t_0 \in (a, b)$, *moreover there isn't any switching point on* (a, b) *and* $(\underline{f})'(t; r)$ *and* $(\overline{f})'(t; r)$ *both differentiable at* t_0.

We say that

1. $f'_{gH}(x)$ *is* $[(i) - gH]$*-differentiable whenever the type of gH-differentiability of* $f(x)$ *and* $f'_{gH}(x)$ *is the same:* $f''_{i.gH}(t_0; r) = [(\underline{f})''(t_0; r), (\overline{f})''(t_0; r)], \quad 0 \leq r \leq 1$,

2. $f'_{gH}(x)$ *is* $[(ii) - gH]$*-differentiable if the type of gH-differentiability of* $f(x)$ *and* $f'_{gH}(x)$ *is different:* $f''_{i.gH}(t_0; r) = [(\overline{f})''(t_0; r), (\underline{f})''(t_0; r)], \quad 0 \leq r \leq 1$.

Definition 2.9. *(See [13]) Let* $f : [a, b] \to \mathbb{E}$. *We say that* $f(t)$ *is Fuzzy Riemann integrable ((FR)-integrable for short) in* $I \in \mathbb{E}$ *if for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any division* $P = \{[u, v]; \xi\}$ *with the norms* $\triangle(P) < \delta$, *we have*

$$D\left(\sum_P^* (v - u) \odot f(\xi), I\right) < \varepsilon,$$

where $\sum_P^*$ *denotes the fuzzy summation. We choose to write* $I := \int_a^b f(t)dt$.

Lemma 2.1. *(See [5]) Let* $f : [a, b] \subseteq \mathbb{R} \to \mathbb{E}$ *be continuous. Then* $\int_a^t f(t)dt$. *is a continuous function in* $t \in [a, b]$.

Lemma 2.2. *(See [5]) Let* $f \in \mathcal{C}_{\mathcal{F}}(\mathbb{R}, \mathbb{E}), r \in \mathbb{N}$.

Then the following integrals

$$\int_a^{s_r-1} f(s_r)ds_r,$$
$$\int_a^{s_r-2}\left(\int_a^{s_r-1} f(s_r)ds_r\right)ds_{r-1},$$
$$\ldots,\int_a^{s}\left(\int_a^{s_1}\cdots\left(\int_a^{s_r-2}\left(\int_a^{s_r-1} f(s_r)ds_r\right)ds_{r-1}\right)\ldots\right)ds,$$

are continuous functions in $s_{r-1}, s_{r-2}, \ldots, s$ respectively. Here $s_{r-1}, s_{r-2}, \ldots, s \geq a$ and all are real numbers.

Now, let $T = [c, d] \subset \mathbb{R}$ be a compact interval.

Definition 2.10. *(See [16]) A mapping $F : T \to \mathbb{E}$ is strongly measurable if for all $r \in [0,1]$ the set valued function $F_r : T \to \rho_k(\mathbb{R})$ defined by $F_r(t) = [F(t)]^r$ is Lebesgue measurable.*

A mapping $F : T \to \mathbb{E}$ is called integrable bounded if there exists an integrable function k such that $\|x\| \leq k(t)$ for all $x \in F_0(t)$.

Definition 2.11. *(See [16]) Let $F : T \to \mathbb{E}$, then the integral of F over T, denote by $\int_T F(t)dt$ or $\int_c^d F(t)dt$, is defined by the equation*

$$\left[\int_T F(t)dt\right]^r = \int_T F_r(t)dt; \qquad r \in [0,1],$$

i.e.,

$$\left[\int_T F(t)dt\right]^r = \left\{\int_T f(t)dt \mid f : T \to \mathbb{R} \text{ is a measurable selection for } F_r\right\}.$$

Also, a strongly measurable and integrable bounded mapping $F : T \to \mathbb{E}$ is said to be integrable over T if $\int_T F(t)dt \in \mathbb{E}$.

Proposition 2.1. *(See [6]) If $F : T \to \mathbb{E}$ is strongly measurable and integrable bounded, then F is integrable.*

Theorem 2.1. *(See [24], [25]) Let $f(x)$ be a fuzzy valued-function on $[a, \infty[$ which is represented by $(\underline{f}(x,r), \overline{f}(x,r))$. For any fixed $r \in [0,1]$, assume $\underline{f}(x,r), \overline{f}(x,r)$ are Riemann integrable on $[a,b]$ for every $b \geq a$, and assume there are two positive constants $\underline{M}(r)$ and $\overline{M}(r)$ such that $\int_a^b |\underline{f}(x,r)|dx \leq \underline{M}(r)$ and $\int_a^b |\overline{f}(x,r)|dx \leq \overline{M}(r)$ for every $b \geq a$. Then $f(x)$ is improper fuzzy Riemann integrable on $[a, \infty[$ and the improper fuzzy Riemann integral is a fuzzy number. Furthermore, we have*

$$\int_a^\infty f(x)dx = \left(\int_a^\infty \underline{f}(x,r)dx , \int_a^\infty \overline{f}(x,r)dx\right).$$

Proposition 2.2. *(See [24]) If each of $f(x)$ and $g(x)$ is a fuzzy valued function and fuzzy Riemann integrable on $[a, \infty[$ then $f(x)+g(x)$ is fuzzy Riemann integrable on $[a, \infty[$. Moreover, we have*

$$\int_a^\infty (f(x)+g(x))dx = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx.$$

For $u, v \in \mathbb{E}$, if there exists $w \in \mathbb{E}$ such that $u = v + w$, then w is the Hukuhara difference of u and v denoted by $u \ominus v$.

Theorem 2.2. *(See [3]) Consider $f : [a,b] \to \mathbb{E}$ is gH-differentiable such that type of differentiability f in $[a,b]$ don't change. Then for $a \leq t_0 \leq b$,*

(i) If $f(t)$ is $[(i) - gH]$-differentiable then $f'_{i-gH}(t)$ is (FR)-integrable over $[a,b]$ and $f(t_0) = f(a) \oplus \int_a^{t_0} f'_{i-gH}(t)dt$,

(ii) If $f(t)$ is $[(ii) - gH]$-differentiable then $f'_{ii-gH}(t)$ is (FR)-integrable over $[a,b]$ and $f(a) = f(t_0) \oplus (-1)\int_a^{t_0} f'_{ii-gH}(t)dt$.

Theorem 2.3. *(See [3]) Let $f^{(i)} : [a,b] \to \mathbb{E}$ and $f \in C^n_{gH}([0,T], \mathbb{E})$. For all $t_0 \in [a,b]$*

(i) Consider $f^{(i)}_{gH}$, $i = 1, \ldots, n$ are $[(i) - gH]$-differentiable and type of differentiability don't change in interval $[a,b]$, then

$$f^{(i-1)}_{i.gH}(s) = f^{(i-1)}_{i.gH}(a) \oplus \int_a^s f^{(i)}_{i.gH}(t)dt.$$

(ii) If $f^{(i)}_{gH}$, $i = 1, \ldots, n$ are $[(ii) - gH]$-differentiable and type of differentiability don't change in interval $[a,b]$, then

$$f^{(i-1)}_{ii.gH}(s) = f^{(i-1)}_{ii.gH}(a) \oplus \int_a^s f^{(i)}_{ii.gH}(t)dt.$$

(iii) *Suppose that* $f^{(i)}$, $i = 2k-1, k \in \mathbb{N}$ *are* $[(i)-gH]$*-differentiable and* $f(t), f^{(i)}$, $i = 2k, k \in \mathbb{N}$ *are* $[(ii)-gH]$*- differentiable, so*

$$f^{(i-1)}_{i.gH}(s) = f^{(i-1)}_{i.gH}(a) \ominus(-1)\int_a^s f^{(i)}_{ii.gH}(t)dt.$$

(iv) *Consider for* $f^{(i)}$, $i = 2k-1, k \in \mathbb{N}$ *are* $[(ii)-gH]$*-differentiable and* $f(t), f^{(i)}$ *are* $[(i)-gH]$*-differentiable for* $i = 2k, k \in \mathbb{N}$, *then*

$$f^{(i-1)}_{ii.gH}(s) = f^{(i-1)}_{ii.gH}(a) \ominus(-1)\int_a^s f^{(i)}_{i.gH}(t)dt.$$

3 Second-order fuzzy hybrid differential equations

First we define a second- order fuzzy hybrid differential equation by

$$x''(t) = f(t, x(t), x'(t), \lambda(x), \lambda(x')).$$

Where $x(t) = (\underline{x}(t,r), \overline{x}(t,r))$ and the fuzzy variables $x'(t)$ and $x''(t)$ are the defined derivatives of $x(t,r)$ and $x'(t,r)$, respectively. Given initial-values $x(t_0) = \alpha_1$ and $x'(t_0) = \alpha_2$, we obtain a fuzzy Cauchy problem of the second-order

$$\begin{cases} x''(t) = f(t, x(t), x'(t), \lambda(x), \lambda(x')), \\ x(t_0) = \alpha_1, \\ x'(t_0) = \alpha_2. \end{cases} \tag{3.4}$$

Theorem 3.1. *(See [22]) Suppose that for* $k = 0, 1, 2, \ldots$ *that each* $f_k : [t_k, t_{k+1}] \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ *is such that*

$$[f_k(t,x,r)] = \Big[\underline{f_k}\Big(t, \underline{x}(t,r), \overline{x}(t,r), \Big(\underline{x}(t,r)\Big)', \Big(\overline{x}(t,r)\Big)'\Big), \overline{f_k}\Big(t, \underline{x}(t,r), \overline{x}(t,r), \Big(\underline{x}(t,r)\Big)', \Big(\overline{x}(t,r)\Big)'\Big)\Big].$$

If for each $k = 0, 1, 2, \ldots$.

There exists $L_k > 0$ *such that*

$$\Big|\underline{f_k}\Big(t, \underline{x}(t), \underline{y}(t), \Big(\underline{x}(t)\Big)', \Big(\underline{y}(t)\Big)'\Big) - \underline{f_k}\Big(t, \overline{x}(t), \overline{y}(t), (\overline{x}(t))', (\overline{y}(t))'\Big)\Big| \leq L_k \max\Big\{|\overline{x}-\underline{x}|, |\overline{y}-\underline{y}|, |\overline{x}'-\underline{x}'|, |\overline{y}'-\underline{y}'|\Big\}.$$

And

$$\Big|\overline{f_k}\Big(t, \underline{x}(t), \underline{y}(t), \Big(\underline{x}(t)\Big)', \Big(\underline{y}(t)\Big)'\Big) - \overline{f_k}\Big(t, \overline{x}(t), \overline{y}(t), (\overline{x}(t))', (\overline{y}(t))'\Big)\Big| \leq L_k \max\Big\{|\overline{x}-\underline{x}|, |\overline{y}-\underline{y}|, |\overline{x}'-\underline{x}'|, |\overline{y}'-\underline{y}'|\Big\}.$$

For all $r \in [0,1]$ *then Eq.* (3.4) *and the hybrid system of ODEs*

$$\begin{cases} \Big(\underline{x}_k(t,r)\Big)'' & = \underline{f_k}\Big(t, \underline{x}_k(t,r), \overline{x}_k(t,r), \underline{x}'_k(t,r), \overline{x}'_k(t,r)\Big), \\ \Big(\overline{x}_k(t,r)\Big)'' & = \overline{f_k}\Big(t, \underline{x}_k(t,r), \overline{x}_k(t,r), \underline{x}'_k(t,r), \overline{x}'_k(t,r)\Big), \\ \underline{x}_k(t_k,r) & = \underline{x}_{k-1}(t_k,r), if k > 0, \ \underline{x}_0(t_0,r) = \underline{x}_0(r), \\ \overline{x}_k(t_k,r) & = \overline{x}_{k-1}(t_k,r), if \ k > 0, \ \overline{x}_0(t_0,r) = \overline{x}_0(r), \\ \Big(\underline{x}_k(t_k,r)\Big)' & = \Big(\underline{x}_{k-1}(t_k,r)\Big)', if \ k > 0, \Big(\underline{x}_0(t_0,r)\Big)' = \underline{x}'_0(r), \\ \Big(\overline{x}_k(t_k,r)\Big)' & = \Big(\overline{x}_{k-1}(t_k,r)\Big)', if \ k > 0, \Big(\overline{x}_0(t_0,r)\Big)' = \overline{x}'_0(r), \end{cases}$$

are equivalent when $x(t)$ *is* $[i-gH]-$*differentiable. The Eq.*(3.4) *and the* *hybrid system of ODEs*

$$\begin{cases} \left(\overline{x}_k(t,r)\right)'' & = \underline{f_k}\Big(t, \underline{x}_k(t,r), \\ & \overline{x}_k(t,r), \underline{x}'_k(t,r), \overline{x}'_k(t,r)\Big), \\ \left(\underline{x}_k(t,r)\right)'' & = \overline{f_k}(t, \underline{x}_k(t,r), \\ & \overline{x}_k(t,r), \underline{x}'_k(t,r), \overline{x}'_k(t,r)\Big), \\ \underline{x}_k(t_k,r) & = \underline{x}_{k-1}(t_k,r), if k>0, \\ & \underline{x}_0(t_0,r) = \underline{x}_0(r), \\ \overline{x}_k(t_k,r) & = \overline{x}_{k-1}(t_k,r), if \ \ k>0, \\ & \overline{x}_0(t_0,r) = \overline{x}_0(r), \\ \left(\underline{x}_k(t_k,r)\right)' & = \left(\underline{x}_{k-1}(t_k,r)\right)', if \\ & k>0, \left(\underline{x}_0(t_0,r)\right)' = \underline{x}'_0(r), \\ \left(\overline{x}_k(t_k,r)\right)' & = \left(\overline{x}_{k-1}(t_k,r)\right)', if \\ & k>0, \left(\overline{x}_0(t_0,r)\right)' = \overline{x}'_0(r), \end{cases}$$

are equivalent when $x(t)$ is $[ii - gH]-$differentiable.

Now we are going to study the uniqueness and existence to second-order fuzzy differential equations.

Theorem 3.2. *(See [4]) Let $t_0 \in [a,b]$, and assume that $f : [a,b] \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ is continuous. A mapping $x : [a,b] \to \mathbb{E}$ is a solution to the initial value problem (3.4) if and only if x and x' are continuous and satisfy one of the following conditions:*

(a)

$$x(t) = \alpha_2(t-t_0) + \int_{t_0}^{t}\Big(\int_{t_0}^{t} f(s, x(s), x'(s), \lambda_k(x), \lambda_k(x'))ds\Big)ds + \alpha_1,$$

where x' and x'' are $[(i)-gH]-$differentials, or

(b)

$$x(t) = \ominus(-1) + \Big(\alpha_2(t-t_0) \ominus (-1) \int_{t_0}^{t}\Big(\int_{t_0}^{t} f\Big(s, x(s), x'(s), \lambda_k(x), \lambda_k(x')\Big)ds\Big)ds\Big) + \alpha_1,$$

where x' and x'' are $[(ii)-gH]-$differentials, or

(c)

$$x(t) = \ominus(-1) + \Big(\alpha_2(t-t_0) + \int_{t_0}^{t}\Big(\int_{t_0}^{t} f\Big(s, x(s), x'(s), \lambda_k(x), \lambda_k(x')\Big)ds\Big)ds\Big) + \alpha_1,$$

where x' is the $[(i)-gH]-$differential and x'' is the $[(ii)-gH]-$differential, or

(d)

$$x(t) = \alpha_2(t-t_0) \ominus (-1) \int_{t_0}^{t}\Big(\int_{t_0}^{t} f(s, x(s), x'(s), \lambda_k(x), \lambda_k(x'))ds\Big)ds + \alpha_1,$$

where x' is the $[(ii)-gH]-$differential and x'' is the $[(i)-gH]-$differential.

Lemma 3.1. *(See [4]) For arbitrary $(u,v) \in \overline{c}^2[0,1] \times \overline{c}^2[0,1]$, we have*

$$D(u \ominus w, u \ominus v) = D(w,v), \quad \forall u,v,w \in \mathbb{E}.$$

Theorem 3.3. *(See [4]) Let: $f : [t_0,T] \times (\mathbb{E})^4 \to \mathbb{E}$ be continuous, and suppose that exist $M_1, M_2 > 0$ such that*

$$d(f\Big(t, x_1, x_2, \lambda(x_1), \lambda(x_2)\Big), f\Big(t, y_1, y_2, \lambda(y_1), \lambda(y_2)\Big)) \le M_1 d(x_1,y_1) + M_2 d(x_2,y_2),$$

for all $t \in [t_0,T]$, x_1, x_2, y_1, y_2, $\lambda(x_1)$, $\lambda(x_2)$, $\lambda(y_1)$, $\lambda(y_2) \in \mathbb{E}$. Then the initial-value problem (3.4) has a unique solution on $[t_0,T]$ for each case.

Our aim now is to solve the following second-order fuzzy hybrid differential equations, using the Taylor expansion under strongly generalized differentiability.
Consider the second-order hybrid fuzzy differential system equation

$$\begin{cases} x''(t) = f(t, x(t), x'(t), \lambda_k(x_k), \lambda_k(x'_k)), \\ x'(t_k) = x'_k, \\ x(t_k) = x_k. \end{cases} \tag{3.5}$$

Where, $0 \leq t_0 < t_1 < \ldots < t_k < \ldots,\ t_k \to \infty$, $t \in [t_k, t_{k+1}]$, $f \in C[R^+ \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E}, \mathbb{E}]$, $\lambda_k \in [\mathbb{E} \times \mathbb{E}, \mathbb{E}]$.

Here, we assume that the existence and uniqueness of solution of the hybrid system hold on each $[t_k, t_{k+1}]$ to be specific the system would look like:

$$x''(t) = \begin{cases} x_0''(t) = f(t, x_0(t), x_0'(t), \lambda_0(x_0), \lambda_0(x_0')), \\ x'(t_0) = x_0',\ x(t_0) = x_0\ , t \in [t_0, t_1], \\ x_1''(t) = f(t, x_1(t), x_1'(t), \lambda_1(x_1), \lambda_1(x_1')), \\ x'(t_1) = x_1', \quad x_1(t_1) = x_1\ , t \in [t_1, t_2], \\ \vdots \\ x_k''(t) = f(t, x_k(t), x_k'(t), \lambda_k(x_k), \lambda_k(x_k')), \\ x'(t_k) = x_k',\ x_k(t_k) = x_k\ , t \in [t_k, t_{k+1}]. \\ \vdots \end{cases}$$

By the solution (3.5) we mean the following function:

$$x'(t) = \begin{cases} x_0'(t), & t \in [t_0, t_1], \\ x_1'(t), & t \in [t_1, t_2], \\ \vdots \\ x_k'(t), & t \in [t_k, t_{k+1}]. \end{cases}$$

We note that the solutions of (3.5) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in \mathbb{E}$ and $k = 0, 1, 2, \ldots$.
We define for each t:

$$\begin{cases} x''(t;r) = f[t, x(t;r), x'(t;r), \lambda_k\Big(x(t;r)\Big), \\ \qquad \lambda_k\Big(x'(t;r)\Big)], \\ x'(t;r) = x_k'(r), \\ x(t;r) = x_k(r), \end{cases} \tag{3.6}$$

for $r \in [0,1]$.

Theorem 3.4. *Let $T = [t_0, t_N] \subset \mathbb{R}$, and $t \in T$.*

Case 1. *Let us suppose that the unique solution of problem (3.5), $y(t)$ and $y'(t)$ are $[(i) - gH]$-differentiable and belongs to $\in C^3_{gH}([0,T], \mathbb{E})$.*

Such that the type of differentiability don't change on $[0,T]$. Consider the Taylor series expansion of the unknown fuzzy function $y(t)$ about t_k, for each $k = 0, 1, \ldots, N$,

$$y_{k,n+1}(t;r) = y_{k,n}(t;r) \oplus (t - t_0) \odot f\Big[t, y_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big)\Big] \oplus \frac{(t-t_0)^2}{2} \odot f\Big[t, y_{k,n}(t;r), y'_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big), \lambda_k\Big(y'_{k,n}(t,r)\Big)\Big]. \tag{3.7}$$

Case 2. *Let us suppose that $y'(t)$ is $[(i) - gH]$-differentiable and $y(t)$ is $[(ii) - gH]$-differentiable and belongs to $C^3_{gH}\Big([0,T], \mathbb{E}\Big)$ such that the type of differentiability don't change on $[0,T]$. We have:*

$$y_{k,n+1}(t;r) = y_{k,n}(t;r) \ominus (-1)(t - t_0) \odot f\Big[t, y_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big)\Big] \ominus (-1)\frac{(t-t_0)^2}{2} \odot f\Big[t, y_{k,n}(t;r), y'_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big), \lambda_k\Big(y'_{k,n}(t,r)\Big)\Big]. \tag{3.8}$$

Case 3. *Consider $y'(t)$ is $[(ii) - gH]$-differentiable and $y(t)$ is $[(i) - gH]$-differentiable and belongs to $C^3_{gH}([0,T], \mathbb{E})$ such that the type of differentiability don't change on $[0,T]$. So the Taylors series expansion is constructed by*

$$y_{k,n+1}(t;r) = y_{k,n}(t;r) \oplus (t - t_0) \odot f\Big[t, y_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big)\Big] \ominus (-1) \frac{(t-t_0)^2}{2} \odot f\Big[t, y_{k,n}(t;r), y'_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big), \lambda_k\Big(y'_{k,n}(t,r)\Big)\Big]. \tag{3.9}$$

Case 4. *Finally, consider $y(t)$ and $y'(t)$ are $[(ii) - gH]$-differentiable and belongs to $\in C^3_{gH}([0,T], \mathbb{E})$. Such that the type of differentiability don't change on $[0,T]$. Consider the Taylor series expansion of the unknown fuzzy function $y(t)$ about t_k, for each*

$k = 0, 1, \ldots, N$.

$$y_{k,n+1}(t;r) = y_{k,n}(t;r) \ominus (-1)(t-t_0)\odot f\Big[t, y_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big)\Big] \ominus (-1) \frac{(t-t_0)^2}{2} \odot f\Big[t, y_{k,n}(t;r), y'_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big), \lambda_k\Big(y'_{k,n}(t,r)\Big)\Big]. \quad (3.10)$$

Proof. **Case 1.** Let $y(t)$ and $y'(t)$ are $[(i)-gH]$-differentiable. Since y, y' are continuous and

$$\begin{cases} y''(t;r) = f\Big[t, y(t;r), y'(t;r), \\ \qquad \lambda_k\Big(y(t;r)\Big), \lambda_k\Big(y'(t;r)\Big)\Big], \\ y'(t;r) = y'_k(r), \\ y(t;r) = y_k(r). \end{cases}$$

By Theorem 2.2, we have

$$y_{k,n+1}(t) = y_{k,n}(t_0) \oplus \int_{t_0}^{t} y'_{k,n}(t_1)dt_1.$$

According to Theorem 2.3

$$y'_{k,n}(t_1) = y'_{k,n}(t_0) \oplus \int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2.$$

Therefore

$$\int_{t_0}^{t} y'_{k,n}(t_1)dt_1 = \int_{t_0}^{t} y'_{k,n}(t_0)dt_1 \oplus \int_{t_0}^{t}\left(\int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2\right)dt_1 = y'_{k,n}(t_0) \odot (t-t_0) \oplus \int_{t_0}^{t}\left(\int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2\right)dt_1.$$

Now by Lemma 2.2, the last double (FR)-integral belongs to $\mathbb{E}$. So

$$y_{k,n+1}(t) = y_{k,n}(t_0) \oplus y'_{k,n}(t_0)\odot (t-t_0) \int_{t_0}^{t}\left(\int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2\right)dt_1.$$

Similarly

$$y''_{k,n}(t_2) = y''_{k,n}(t_0) \oplus \int_{t_0}^{t_2} y'''_{k,n}(t_3)dt_3.$$

Furthermore

$$\int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2 = y''_{k,n}(t_0) \odot (t_1 - t_0).$$

$$\int_{t_0}^{t}\left(\int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2\right)dt_1 = y''_{k,n}(t_0) \odot \int_{t_0}^{t}(t_1-t_0)dt_1.$$

By Lemma 2.2, the last triple integral belongs to $\mathbb{E}$. Hence

$$y_{k,n+1}(t;r) = y_{k,n}(t;r) \oplus f\Big[t, y_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big)\Big] \odot(t-t_0) \oplus f\Big[t, y_{k,n}(t;r), y'_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big), \lambda_k\Big(y'_{k,n}(t,r)\Big)\Big] \odot \frac{(t-t_0)^2}{2!}.$$

Case 2. Let $y(t)$ be $[(ii)-gH]$-differentiable and $y'(t)$ be $[(i)-gH]$-differentiable. By Theorem 2.2, we have

$$y_{k,n}(t_0) = y_{k,n+1}(t) \oplus (-1)\int_{t_0}^{t} y'_{k,n}(t_1)dt_1.$$

Hence

$$y_{k,n+1}(t) = y_{k,n}(t_0) \ominus (-1)\int_{t_0}^{t} y'_{k,n}(t_1)dt_1.$$

According to the hypothesis type of differentiability don't change, so by Theorem 2.3 and by attention to (FR)-integrability of y, y' on T, we obtain

$$y'_{k,n}(t_1) = y'_{k,n}(t_0) \oplus \int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2,$$

therefore

$$\int_{t_0}^{t} y'_{k,n}(t_1)dt_1 = \int_{t_0}^{t} y'_{k,n}(t_0)dt_1 \oplus \int_{t_0}^{t}\left(\int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2\right)dt_1 = y'_{k,n}(t_0) \odot (t-t_0) \oplus \int_{t_0}^{t}\left(\int_{t_0}^{t_1} y''_{k,n}(t_2)dt_2\right)dt_1.$$

Lemma 2.2, implies that the last double integral belongs to $\mathbb{E}$. So

$$y_{k,n+1}(t) = y_{k,n}(t_0) \ominus (-1) y'_{k,n}(t_0) \odot (t-t_0) \ominus (-1) \int_{t_0}^{t} \Big(\int_{t_0}^{t_1} y''_{k,n}(t_2) dt_2 \Big) dt_1.$$

Similarly by Theorem 2.3, we have

$$y''_{k,n}(t_2) = y''_{k,n}(t_0) \oplus \int_{t_0}^{t_2} y'''_{k,n}(t_3) dt_3.$$

Hence

$$\int_{t_0}^{t} \left(\int_{t_0}^{t_1} y''_{k,n}(t_2) dt_2 \right) dt_1 = y''_{k,n}(t_0) \odot \int_{t_0}^{t} (t_1 - t_0) dt_1.$$

According to Lemma 2.2, we obtain

$$y_{k,n+1}(t;r) = y_{k,n}(t;r) \ominus (-1) f\Big[t, y_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big)\Big] \odot (t-t_0) \ominus (-1) f\Big[t, y_{k,n}(t;r), y'_{k,n}(t;r), \lambda_k\Big(y_{k,n}(t,r)\Big), \lambda_k\Big(y'_{k,n}(t,r)\Big)\Big] \odot \frac{(t-t_0)^2}{2!}.$$

Case 3, Case 4. Similarly case 1 and case 2. □

However, for a prefixed k and $r \in [0,1]$, proof of convergence of the approximations in (3.7)- (3.10), that is: $\lim_{h_0,\ldots,h_k \to 0} y_{k,N_k}(t;r) = x(t_{k+1};r)$, is an application of theorem 1 in [18] and lemma 3.2 below.

Lemma 3.2. *Suppose that $i \in \mathbb{Z}^+$, $\epsilon_i > 0$, $r \in [0,1]$ and $h_i < 1$, are fixed and $h = t_{k+1} - t_k$. Let $\{z_{i,n}(t;r)\}_{n=0}^{N_i}$ be the Taylor approximation with $N = N_i$ to the fuzzy IVP:*

$$\begin{cases} x''(t;r) = f\Big[t, x(t;r), x'(t;r), \lambda_k\Big(x(t;r)\Big), \lambda_i\Big(x'(t;r)\Big)\Big], \\ x'(t;r) = x'_i(r), \\ x(t;r) = x_i(r), \qquad t \in [t_i, t_{i+1}]. \end{cases} \tag{3.11}$$

If $\{z_{i,n}(t;r)\}_{n=0}^{N_i}$ denotes the result of Eqs. (3.7)-(3.10) from some $y_{i,0}(t;r)$, then there exists a $\delta_i > 0$ such that $D(z_{i,0}(t;r) \ominus y_{i,0}(t;r), 0) < \delta_i$, imply $D(z_{i,N_i}(t;r) \ominus y_{i,N_i}(t;r), 0) < \epsilon_i$.

Proof. Since the proof procedure is similar to each other for all four cases, we consider only case 1, without loss of generality. Fix $i \in \mathbb{Z}^+$, $\epsilon_i > 0$, $r \in [0,1]$ and $h_i < 1$. Let $\{z_{i,n}(t;r)\}_{n=0}^{N_i}$ be the Taylor approximation with $N = N_i$ to the fuzzy IVP (3.11).

First consider $y(t)$ and $y'(t)$ are $[(i) - gH]$-differentiable.

Suppose that $\{y_{i,n}(t;r)\}_{n=0}^{N_i}$ denotes the result of Eq. (3.7) from some $y_{i,0}(t;r)$. By Eq. (3.7), for each $l = 0, \ldots, N_i - 1$,

$$\begin{aligned} &D(z_{i,l+1}(t;r) \ominus y_{i,l+1}(t;r), 0) = D\Big(z_{i,l}(t;r) \oplus h_i \odot f\Big[t_{i,l}, z_{i,l}(t;r), \Big(z_{i,l}(t;r)\Big)', \lambda_i\Big(z_{i,l}(t;r)\Big), \lambda_i\Big(z_{i,l}(t;r)\Big)'\Big] \ominus y_{i,l}(t;r) \ominus f\Big[t_{i,l}, y_{i,l}(t;r), \Big(y_{i,l}(t;r)\Big)', \lambda_i\Big(y_{i,l}(t;r)\Big), \lambda_i\Big(y_{i,l}(t;r)\Big)'\Big], 0\Big) \\ &\leq D(z_{i,l}(t;r) \ominus y_{i,l}(t;r), 0) \oplus h_i \odot D\Big(f[t_{i,l}, z_{i,l}(t;r), \Big(z_{i,l}(t;r)\Big)', \lambda_i\Big(z_{i,l}(t;r)\Big), \lambda_i\Big(z_{i,l}(t;r)\Big)'] \ominus f\Big[t_{i,l}, y_{i,l}(t;r), \lambda_i\Big(y_{i,l}(t;r)\Big), \Big(y_{i,l}(t;r)\Big)', \lambda_i\Big(y_{i,l}(t;r)\Big)'\Big], 0\Big). \end{aligned} \tag{3.12}$$

Let $\alpha_{N_i} \equiv \epsilon_i$. Since there exists a $\eta_{N_i} > 0$.

Such that $D(z_{i,N_i-1}(t;r) \ominus y_{i,N_i-1}(t;r), 0) < \eta_{N_i}$.

Imply

$$D\Big(f\Big[t_{i,N_i-1}, z_{i,N_i-1}(t;r), \Big(z_{i,N_i-1}(t;r)\Big)', \lambda_i\Big(z_{i,N_i-1}(t;r)\Big), \lambda_i\Big(z_{i,N_i-1}(t;r)\Big)'\Big] \ominus f\Big[t_{i,N_i-1}, y_{i,N_i-1}(t;r), \Big(y_{i,N_i-1}(t;r)\Big)', \lambda_i\Big(y_{i,N_i-1}(t;r)\Big), \lambda_i\Big(y_{i,N_i-1}(t;r)\Big)'\Big], 0\Big) < \frac{\epsilon_i}{2} = \frac{\alpha_{N_i}}{2}. \tag{3.13}$$

Let $\alpha_{N_i-1} \equiv \min\left\{\frac{\epsilon_i}{2}, \frac{\eta_{N_i}}{2}\right\}$.

If $D(z_{i,N_i-1}(t;r)\ominus y_{i,N_i-1}(t;r),0) < \alpha_{N_i-1}$ then by Eq. (3.9) with $l = N_i - 1$ and Eq. (3.13) we have

$$D(z_{i,N_i}(t;r) \ominus y_{i,N_i}(t;r),0) \leq D(z_{i,N_i-1}(t;r) \ominus y_{i,N_i-1}(t;r),0) \oplus h_i \odot D\Big(f\Big[t_{i,N_i-1}, z_{i,N_i-1}(t;r), \Big(z_{i,N_i-1}(t;r)\Big)', \lambda_i\Big(z_{i,N_i-1}(t;r)\Big), \lambda_i\Big(z_{i,N_i-1}(t;r)\Big)'\Big] \ominus f\Big[t_{i,N_i-1}, y_{i,N_i-1}(t;r), \Big(y_{i,N_i-1}(t;r)\Big)', \lambda_i\Big(y_{i,N_i-1}(t;r)\Big), \lambda_i\Big(y_{i,N_i-1}(t;r)\Big)'\Big],0\Big) < \alpha_{N_i-1} \oplus h_i \odot \frac{\epsilon_i}{2} < \epsilon_i. \quad (3.14)$$

Continue inductively for each $j = 2,\ldots,N_i$ as follows. Since f is continue, there exists a $\eta_{N_i-(j-1)} > 0$ such that $D(z_{i,N_i-j}(t;r) \ominus y_{i,N_i-j}(t;r),0) < \eta_{N_i-(j-1)}$.
Imply

$$D\Big(f\Big[t_{i,N_i-j}, z_{i,N_i-j}(t;r), \Big(z_{i,N_i-j}(t;r)\Big)', \lambda_i\Big(z_{i,N_i-j}(t;r)\Big), \lambda_i\Big(z_{i,N_i-j}(t;r)\Big)'\Big] \ominus f\Big[t_{i,N_i-j}, y_{i,N_i-j}(t;r), \Big(y_{i,N_i-j}(t;r)\Big)', \lambda_i\Big(y_{i,N_i-j}(t;r)\Big), \lambda_i\Big(y_{i,N_i-j}(t;r)\Big)'\Big],0\Big) < \frac{\alpha_{N_i-(j-1)}}{2}. \quad (3.15)$$

Let $\alpha_{N_i-1} = \min\Big\{\frac{\alpha_{N_i-(j-1)}}{2}, \frac{\eta_{N_i-(j-1)}}{2}\Big\}$, if $D(z_{i,N_i-j}(t;r) \ominus y_{i,N_i-j}(t;r),0) < \alpha_{N_i-j}$ then by Eq. (3.12) with $l = N_i - j$ and Eq. (3.15) we have

$$D(z_{i,N_i-(j-1)}(t;r) \ominus y_{i,N_i-(j-1)}(t;r),0) \leq D(z_{i,N_i-(j-1)}(t;r) \ominus y_{i,N_i-(j-1)}(t;r),0) \oplus h_i \odot D\Big(f\Big[t_{i,N_i-j}, z_{i,N_i-j}(t;r), \Big(z_{i,N_i-j}(t;r)\Big)', \lambda_i\Big(z_{i,N_i-j}(t;r)\Big), \lambda_i\Big(z_{i,N_i-j}(t;r)\Big)'\Big] \ominus f\Big[t_{i,N_i-j}, y_{i,N_i-j}(t;r), \Big(y_{i,N_i-j}(t;r)\Big)', \lambda_i\Big(y_{i,N_i-j}(t;r)\Big), \lambda_i\Big(y_{i,N_i-j}(t;r)\Big)'\Big],0\Big) \ominus f\Big[t_{i,N_i-j}, y_{i,N_i-j}(t;r), \Big(y_{i,N_i-j},(t;r)\Big)' \lambda_i\Big(y_{i,N_i-j}(t;r)\Big), \lambda_i\Big(y_{i,N_i-j}(t;r)\Big)'\Big],0\Big) < \frac{\alpha_{N_i(j-1)}}{2} \oplus h_i \odot \frac{\alpha_{N_i(j-1)}}{2} < \alpha_{N_i-(j-1)}. \quad (3.16)$$

Then for $j = N_i$ we see

$$D(z_{i,0}(t;r) \ominus y_{i,0}(t;r),0) < \alpha_0$$

imply

$$D(z_{i,1}(t;r) \ominus y_{i,1}(t;r),0) < \alpha_1.$$

For $j = N_i - 1$ we see

$$D(z_{i,1}(t;r) \ominus y_{i,1}(t;r),0) < \alpha_1$$

imply

$$D(z_{i,2}(t;r) \ominus y_{i,2}(t;r),0) < \alpha_2.$$

Continue decreasing to $j = 2$. To see

$$D(z_{i,N_i-2}(t;r) \ominus y_{i,N_i-2}(t;r),0) < \alpha_{N_i-2}$$

imply

$$D(z_{i,N_i-1}(t;r) \ominus y_{i,N_i-1}(t;r),0) < \alpha_{N_i-1}.$$

But it was already shown in Eq. (3.14) that

$$D(z_{i,N_i-1}(t;r) \ominus y_{i,N_i-1}(t;r),0) < \alpha_{N_i-1}.$$

Imply

$$D(z_{i,N_i}(t;r) \ominus y_{i,N_i}(t;r),0) < \epsilon_i$$

This proves the lemma with $\delta_i = \alpha_0$. □

Theorem 3.5. *(See [21]) Consider the systems (3.5) and Eqs. (3.7)-(3.10). For a fixed $k \in \mathbb{Z}^+$ and $r \in [0,1]$,*

$$\lim_{h_0,\ldots,h_k \to 0} y_{k,N_k}(t;r) = x(t_{k+1};r).$$

4 Numerical Example

In this section, we are going to use the Taylor expansion to solve the following examples.

Example 4.1. *Consider the following fuzzy hybrid differential equation*

$$\begin{cases} y''(t) = y'(t) \oplus m(t) \odot y(t) \\ \qquad \odot \lambda_k(y(t_k), y'(t_k)), \\ y'(0;r) = [0.75 + 0.25r, 1.125 - 0.125r], \\ y(0;r) = [0.75 + 0.25r, 1.5 - 0.5r]. \end{cases} \tag{4.17}$$

Where $t_k \in [t_k, t_{k+1}], t_k = k, m(t) = |sin(\pi t)|,\ k = 0, 1, 2, \ldots.$

$$\lambda_k(\mu, \nu) = \begin{cases} \widehat{0}, & if\ \ k = 0, \\ \mu\nu, & if\ \ k \in \{1, 2, \ldots\}. \end{cases} \tag{4.18}$$

Case (1): *By applying the Taylor method which is discussed in detail in Theorem 3.4, we have fig. 1,*

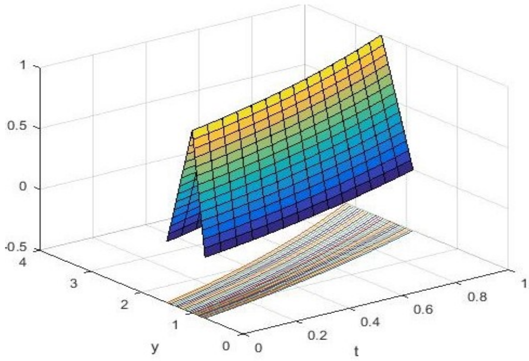

Figure 1: The approximate solution for Example 4.1 at case (1).

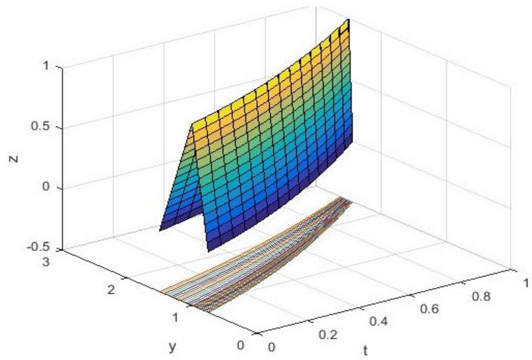

Figure 2: The approximate solution for Example 4.1 at case (2).

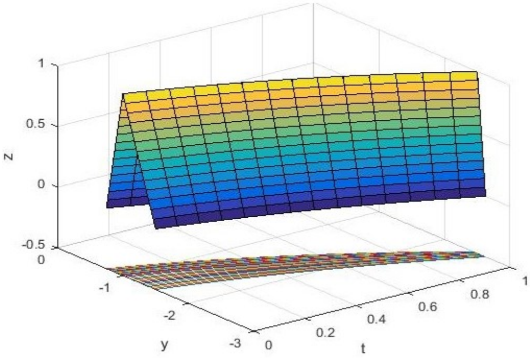

Figure 3: The approximate solution for Example 4.1 at case (3).

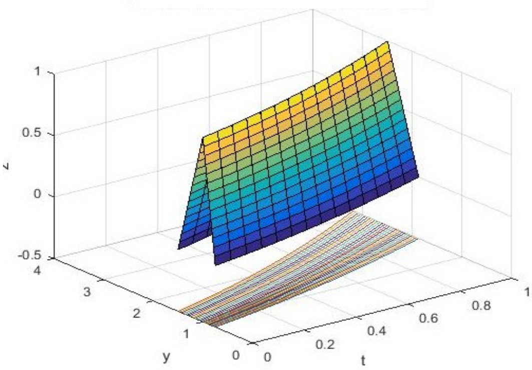

Figure 4: The approximate solution for Example 4.1 at case (4).

Case (2): *Consider* $y(t)$ *is* $[(ii) - gH]$*-differentiable and* $y'(t)$ *is* $[(i) - gH]$*-differentiable.*

Case (3): *Now, consider* $y(t)$ *is* $[(i) - gH]$*-differentiable and* $y''(t)$ *is* $[(ii) - gH]$*-differentiable.*

Case (4): *Finally, if* $y(t)$ *and* $y'(t)$ *are* $[(ii) - gH]$*-differentiable.*

Example 4.2. *Next consider the following HFDE*

$$\begin{cases} y''(t) = (y'(t))^2 \oplus m(t) \odot y(t) \\ \qquad \odot \lambda_k(y(t_k).y'(t_k)), \\ y'(0;r) = [0.75 + 0.25r, 1.125 - 0.125r], \\ y(0;r) = [0.75 + 0.25r, 1.5 - 0.5r]. \end{cases} \tag{4.19}$$

Where $t_k \in [t_k, t_{k+1}], t_k = k, \quad k = 0, 1, 2, \ldots.$

$$m(t) = \begin{cases} 2(t(mod\ 1)), \\ if\ t(mod\ 1) \leq 0.5. \\ \\ 2(1 - t(mod\ 1)), \\ \ if\ t(mod\ 1) > 0.5, \end{cases} \tag{4.20}$$

$$\lambda_k(\mu, \nu) = \begin{cases} \widehat{0}, & if \ \ k = 0, \\ \mu\nu, & if \ \ k \in \{1, 2, \ldots\}. \end{cases} \tag{4.21}$$

Case (1): *Consider $y(t)$ and $y'(t)$ are $[(i) - gH]$-differentiable. Hence we have fig. 5,*

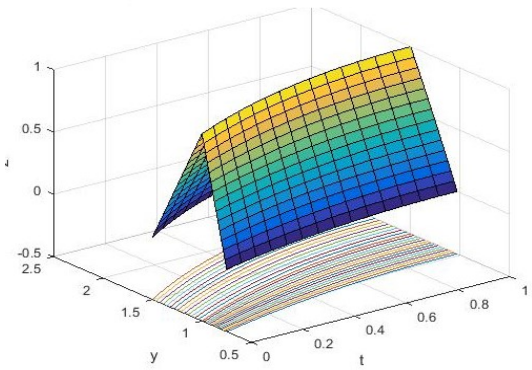

Figure 5: The approximate solution for Example 4.2 at case (1).

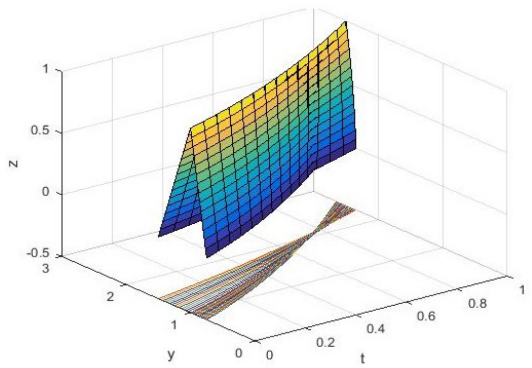

Figure 6: The approximate solution for Example 4.2 at case (2).

Case (2): *Consider $y'(t)$ is $[(i) - gH]$-differentiable and $y(t)$ is $[(ii) - gH]$-differentiable.*

Case (3): *consider $y'(t)$ is $[(ii) - gH]$-differentiable and $y(t)$ is $[(i) - gH]$-differentiable.*

Case (4): *Consider $y(t)$ and $y'(t)$ are $[(ii) - gH]$-differentiable.*

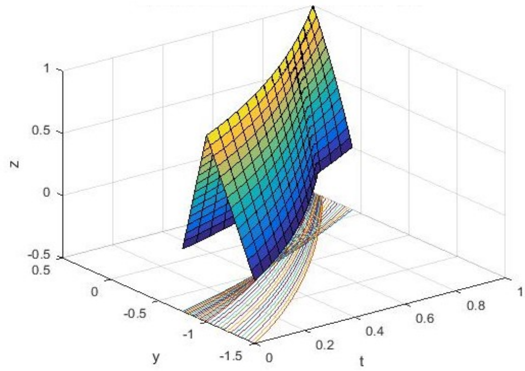

Figure 7: The approximate solution for Example 4.2 at case (3).

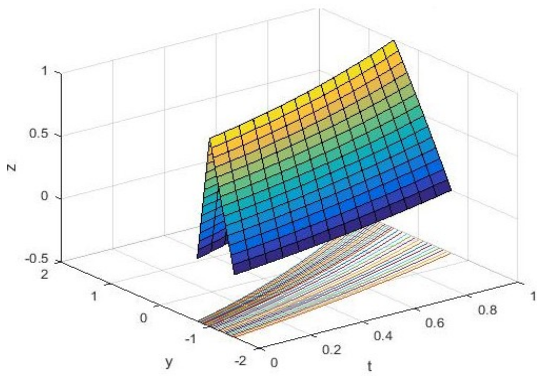

Figure 8: The approximate solution for Example 4.2 at case (4).

5 Conclusion

In this paper, a new approach was introduced in the hybrid fuzzy second order differential equations by presenting the fuzzy Taylor expansion based on $gH-$differentiability. According to the type of gH-differentiability, the fuzzy Taylor expansion was obtained in four cases, and the convergence of the proposed method is proved. The final results showed that the solution of the second order hybrid fuzzy differential equations.

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