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Research Article

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Application of Variational Calculus to Integrability of Differential Equations with Physical Applications

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Abstract

Variational calculus is used to determine the integrability of differential equations. A remarkable unified approach is presented by a single theorem employing variational calculus to determine the integrability of any ordinary differential equation whether linear or nonlinear. The theorem is also used to determine the integrating factor for a given equation if it is not directly integrable. Well established results for determining integrating factors obtained by various methods can be combined in a single equation of variational calculus. Many sample problems are extensively treated to show the power and applicability of the theorem. The method is applied to a variety of problems stemming from physical phenomena.

Keywords : Variational Calculus; Integrability; Differential Equations; Integrating Factor; Mathematical Physics Applications.

1 Introduction

One of the most important issues in solving ordinary differential equations is their integrability. A first integral of the equation reduces the order of the equation by one and if successive integrals can be taken one by one, the final solution can be achieved. If the integral of the equation exists, then we can speak of the integrability of the differential equation. Direct integrability is rarely encountered in differential equations. If the first integral is possible, then the next integration may not be performed for most of the time because the outcoming differential equation is not integrable. Non-integrable equations usually accept an integration factor to make it integrable. The integrating factor is mainly exploited under the topics of first order linear non-homogenous differential equations and the so-called exact nonlinear differential equations of first order. See Edwards et al. [9], Bissell [5], Roman-Miller & Smith [16] and Cortez and Oliveria [7] for information on the topic.

In this work, a unified approach employing the calculus of variations is presented in determining the integrability of any ordinary differential equation of arbitrary order, whether linear or nonlinear. Calculus of variations is usually employed in determining the extrema of a functional, mostly in integral form and found application areas such

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as the optimization problems, entropy, mechanics of motion, elastic behaviour of structural systems etc. (O'Neil, [12]). In fact, calculus of variations is a highly effective tool in determining the integrability of an ordinary differential equation which is not emphasised in relevant textbooks. With the aid of a single theorem, for the first order, second order and higher order linear differential equations, the integrability conditions are given. Then the main theorem which is valid for linear and nonlinear equations are given and nonlinear examples are treated. For equations which are not integrable, the systematic way of determining an integral factor is depicted using the main theorem. The ideas presented here are basic, easily applicable, valid for all equations and do not require extensive calculations in the case of advanced methods such as Lie Group Theory (Bluman and Kumei, [6]). Finally, mathematical models varying from heat transfer to dynamics and astrophysics are treated using the formalism developed here.

2 Linear Ordinary Differential Equations

In this section, first the linear equations will be treated starting from the first order and advancing towards the arbitrary orders.

2.1 Linear First Order Equations

For a first order linear equation, the following theorem is proposed:

Theorem 2.1. *If for the first order linear homogenous differential equation*

$$F\left(x, y, y\ \right) = A\left(x\right) y' + B\left(x\right) y = 0 \tag{2.1}$$

the variation of its integral is identically equivalent to zero, i.e.,

$$\delta \int F\left(x, y, y'\right) dx \equiv 0 \tag{2.2}$$

then the equation is exactly integrable with no need for an integrating factor and the coefficients satisfy the relationship

$$B - A' = 0 \quad . \tag{2.3}$$

Proof. The variational equation (2.2) leads to the Euler equation (O'Neil, [12])

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{2.4}$$

or

$$B - A' = 0 \tag{2.5}$$

which can be verified by substituting for B into the original equation

$A\left(x\right) y' + A'y = \left(A\left(x\right) y\right)' = 0$, the integral of which is $Ay = c$ with c an arbitrary constant

2.2 Linear Second Order Equations

Theorem 2.2. *If for the second order linear homogenous differential equation*

$$F\left(x, y, y', y''\right) = A\left(x\right) y'' + B\left(x\right) y' + C(x)y = 0 \tag{2.6}$$

the variation of its integral is identically equivalent to zero, i.e.,

$$\delta \int F\left(x, y, y', y''\right) dx \equiv 0 \tag{2.7}$$

then the equation is exactly integrable once with no need for an integrating factor and the coefficients satisfy the relationship

$$C\left(x\right) - B'\left(x\right) + A''(x) = 0 \quad . \tag{2.8}$$

Proof. $F\left(x, y, y', y''\right)$ satisfies the Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \tag{2.9}$$

or substituting (2.6) yields

$$C\left(x\right) - B'\left(x\right) + A''(x) = 0 \tag{2.10}$$

which is the integrability condition. Under this condition, the equation can be cast into the directly integrable form

$$\left(A\left(x\right) y'\ \right)' + \left(D\left(x\right) y\right)' = 0 \tag{2.11}$$

where

$$D\left(x\right) = \int C\left(x\right) dx \tag{2.12}$$

If one differentiates (2.11) and compares with the original equation, then

$$B = A' + D, \quad C = D' \tag{2.13}$$

But (2.13) is obtained by integrating (2.10) once with D given in (2.12).

Note that condition (2.10) for integrability as well as the suitable form (2.11) for integration was already given by O'Neil ([12]) obtained by some other methods.

Example 2.1. Consider the equation

$$2x^2y'' + 5xy' + y = 0 \tag{2.14}$$

which is integrable since $A''(x) = 4$, $B'(x) = 5$ and C= 1, and the condition (2.8) is already satisfied. Since $D\,(x) = \int C\,(x)\,dx = x$, the integrable form is from (2.11)

$$(2x^2y'\ )' + (xy)' = 0 \tag{2.15}$$

for which the first integral is

$$2x^2y' + xy = \bar{c}_1 \tag{2.16}$$

From Theorem 2.1, for $A = 2x^2$, $B = x$, since $A' \neq B$, this first order equation is not directly integrable. Nevertheless, multiplying with the integrating factor of $\sqrt{x}$, the equation becomes integrable with the solution

$$y = \frac{c_1}{x} + \frac{c_2}{\sqrt{x}} \tag{2.17}$$

Integrating factors will be discussed later in detail.

The following theorem states the necessary conditions for a second order linear equation to be completely integrable:

Theorem 2.3. *The second order linear homogenous differential equation*

$$A\,(x)\,y'' + B\,(x)\,y' + C(x)y = 0 \tag{2.18}$$

is completely integrable (two times integrable) with no need for an integrating factor at each step if the coefficients satisfy the relationships

$$A''(x) = C(x), \qquad B'(x) = 2C(x) \tag{2.19}$$

Proof. From Theorem 2.2, for the first integration, the condition is

$$C\,(x) - B'\,(x) + A''(x) = 0 \tag{2.20}$$

and the first integral of the original equation is

$$Ay' + Dy = c_1 \tag{2.21}$$

with $D = \int Cdx$. For (2.21) to be integrable, from Theorem 2.1, $A' = D = \int Cdx$. Differentiate once

$$A''(x) = C(x) \tag{2.22}$$

and substitute into (2.20) yielding

$$B'(x) = 2C(x) \tag{2.23}$$

with (2.22) and (2.23) satisfying both conditions of the Theorems 2.1 and 2.2 and hence constitutes the necessary conditions for the twice integrability.

Example 2.2. Consider the equation

$$x^3y'' + 6x^2y' + 6xy = 0 \tag{2.24}$$

which is twice integrable since the conditions in (2.19) are satisfied for $A = x^3$, $B = 6x^2$, $C = 6x$. The suitable form for the first integration is

$$(x^3y')' + (3x^2y)' = 0 \tag{2.25}$$

for which the first integral is

$$x^3y' + 3x^2y = c_1 \tag{2.26}$$

This can be expressed as

$$\left(x^3y\right)' = c_1 \tag{2.27}$$

with a final solution

$$y = \frac{c_1}{x^2} + \frac{c_2}{x^3} \quad . \tag{2.28}$$

2.3 Higher Order Linear Equations

Theorems for the third order and for the arbitrary orders are given in this section with applications.

Theorem 2.4. *If for the third order linear homogenous differential equation*

$$F\,(x, y, y', y'', y''') =$$

$$A\,(x)\,y''' + B\,(x)\,y'' + C\,(x)\,y' + D(x)y = 0 \tag{2.29}$$

the variation of its integral is identically equivalent to zero, i.e.,

$$\delta \int F\left(x, y, y^{'}, y^{'}{'}, y^{'}{''}\right) dx \equiv 0 \tag{2.30}$$

then the equation is exactly integrable once with no need for an integrating factor and the coefficients satisfy the relationship

$$D(x) - C'(x) + B''(x) - A'''(x) = 0 \quad . \tag{2.31}$$

Proof. $F(x, y, y', y'', y''')$ satisfies the Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) = 0 \tag{2.32}$$

or substituting (2.29) yields

$$D(x) - C'(x) + B''(x) - A'''(x) = 0 \tag{2.33}$$

which is the integrability condition. Under this condition, the equation can be cast into the directly integrable form

$$(A(x)\, y'')' + (E(x)\, y')' + (G(x)\, y)' = 0 \tag{2.34}$$

where

$$E(x) = \int C(x)\, dx - \int\int D(x)\, dx, \tag{2.35}$$

$G(x) = \int D(x)\, dx$

If one differentiates (2.34), compares the coefficients with the original equation, and uses (2.35), eventually (2.33) is retrieved.

Theorem 2.5. *If for the k'th order linear homogenous differential equation*

$$F\left(x, y, y', y'', \ldots, y^{(k)}\right) = A_k(x)\, y^{(k)} +$$

$$A_{k-1}(x)\, y^{(k-1)} + \cdots + A_1(x)\, y' + A_0(x) y = 0 \tag{2.36}$$

the variation of its integral is identically equivalent to zero, i.e.,

$$\delta \int F\left(x, y, y', y'', \ldots, y^{(k)}\right) dx \equiv 0 \tag{2.37}$$

then the equation is exactly integrable once with no need for an integrating factor and the coefficients satisfy the relationship

$$A_0(x) - A_1'(x) +$$

$$A_2''(x) - \ldots (-1)^{k-1} A_{k-1}^{(k-1)}(x) + (-1)^k A_k^{(k)}(x) = 0 \quad . \tag{2.38}$$

Proof. The theorem is an induction of the previous theorems and can be proven in a similar way as stated before.

3 Nonlinear Ordinary Differential Equations

The main theorem for nonlinear equations of arbitrary order is given in this section. The theorem was already applied to specific linear equations in the previous section. Some nonlinear problems are treated as examples.

Theorem 3.1. *Main Theorem*

If for the k'th order nonlinear differential equation

$$F\left(x, y, y', y'', \ldots, y^{(k)}\right) = 0 \tag{3.39}$$

the variation of its integral is identically equal to zero, i.e.,

$$\delta \int F\left(x, y, y', y'', \ldots, y^{(k)}\right) dx \equiv 0 \tag{3.40}$$

then the equation is exactly integrable once with no need for an integrating factor.

Proof. If F=0 is integrable, then

$$F = \frac{dG}{dx} \tag{3.41}$$

for some function $G\left(x, y, y', y'', \ldots, y^{(k-1)}\right)$ and

$$G\left(x, y, y', y'', \ldots, y^{(k-1)}\right) = c \tag{3.42}$$

is the first integral. Multiplying both sides of (3.41) by dx and integrating yields

$$\int F dx = \int dG = G = c \tag{3.43}$$

Taking the variation of both sides

$$\delta \int F dx = \delta c = 0 \tag{3.44}$$

since the variation of a constant is zero.

Theorem 3.1 is the main theorem which applies to all ordinary differential equations. The theorem is applied to sample problems

Example 3.1. $F = y'' - 2yy' = 0$ possesses a first integral $y' - y^2 = c$. This can be verified by substituting F into the Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \qquad (3.45)$$

or $-2y' + 2y' + 0 = 0$

Example 3.2. $F = y'' + y^2 = 0$ has no first integral since the Euler equation produces $2y \neq 0$. If the equation is of the form $F = y'' + a(x)y^2 = 0$, the condition to have a first integral can be determined from the Euler equation which is $a(x) = 0$.

Example 3.3. $F = sinyy'' + cosyy'^{\ 2} = 0$ possesses a first integral $y'siny = c$. This can be verified by substituting F into the Euler equation

$$\cos yy'' - \sin yy'^2 - \frac{d}{dx}\left(2y'\cos y\right) + \frac{d^2}{dx^2}\left(\sin y\right) = 0$$

or
$\cos yy'' - \sin yy'^2 -$

$$2y''\cos y + 2\sin yy'^2 - \sin yy'^2 + \cos yy'' = 0 \quad .$$

4 Integrating Factors

If the differential equation is not directly integrable, then it can be cast into an integrable form with multiplying by an integrating factor. A unified and simple approach combining the results of various methods for determining the integration factors is presented in this section based on the main theorem.

Theorem 4.1. *Integrating Factor*
For the k'th order nonlinear differential equation

$$F\left(x, y, y', y'', \ldots, y^{(k)}\right) = 0 \qquad (4.46)$$

if there exists an integrating factor $\mu\left(x, y, y', y'', \ldots, y^{(k-1)}\right)$*, then it should satisfy*

$$\delta \int \mu F dx \equiv 0 \qquad (4.47)$$

Proof. From Theorem 3.1 if F=0 is integrable, then $\delta \int F dx \equiv 0$. If F cannot be integrated directly, then multiplying by an integrating factor μ makes the equation μF=0 integrable, hence $\delta \int \mu F dx \equiv 0$ would give the integrating factor for the equation

Note that the theorem does not guarantee the existence of the integrating factor, yet it states that if there exists an integrating factor for the equation, it must obey (4.47). Generally speaking, the form of the integrating factor $\mu\left(x, y, y', y'', \ldots, y^{(k-1)}\right)$ is the most general form which may lead to complex equations hard to solve. Simplifications by dropping some of the dependencies from the integrating function which do not lead to inconsistencies may be possible for the specific form of the equations. To summarize, for the equation $\delta \int \mu F dx \equiv 0$, three distinct cases exist

i) μ=0, hence no integrating factor exists for the equation F=0.

ii) μ=μ_0, a constant or μ=1 without loss of generality and the equation is directly integrable.

iii) μ is a function of its variables, at least one of them, and the equation is integrable after multiplying by μ.

4.1 First Order Equations

The well-established results given for first order equations in standard textbooks on differential equations can be traced through the unified approach presented in this work.

Example 4.1. Consider the first order linear non-homogenous differential equation

$$F = y' + a\left(x\right)y - h\left(x\right) = 0 \qquad (4.48)$$

Multiplying by the integrating factor, the Euler equation should be satisfied by the application of Theorem 4.1

$$\frac{\partial}{\partial y}(\mu F) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}(\mu F)\right) = 0 \qquad (4.49)$$

Rather than taking $\mu = \mu(x, y)$, the most general form, for simplicity, try $\mu = \mu(x)$. Then

$$\mu a - \frac{d}{dx}(\mu) = 0 \tag{4.50}$$

with a solution $\mu = e^{\int a dx}$ which is the well-known result given in textbooks.

Example 4.2. Consider the nonlinear first order differential equation of the form

$$M(x, y)\, dx + N(x, y) dy = 0 \tag{4.51}$$

which is examined under the title of exact differentials. F can be defined by dividing the equation with dx, $F = Ny' + M = 0$ and $\mu F = \mu N y' + \mu M = 0$. If the equation is integrable, then it should satisfy the Euler equation which leads to, after simplification

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \tag{4.52}$$

which is the well-known result derived from the exactness condition in textbooks.

Example 4.3. Bernoulli Equation

For the Bernoulli equation

$$F = P(x)y' + Q(x)\, y - R(x)\, y^{\alpha} = 0 \tag{4.53}$$

assume the integrating factor of $\mu = \mu(x, y)$ in its most general form. Substituting into the Euler equation for μF and simplifying yields

$$\frac{\partial \mu}{\partial y}(Qy - Ry^{\alpha}) - \frac{\partial \mu}{\partial x} P + \mu\left(Q - R\alpha y^{\alpha-1} - P'\right) = 0 \tag{4.54}$$

which is exactly the equation given by O'Neil [12] obtained from exact differential method. The solution of the above equation is (O'Neil [12])

$$\mu = \frac{y^{-\alpha}}{P(x)} e^{(1-\alpha)\int Q(x)/P(x) dx} \quad . \tag{4.55}$$

4.2 Second Order Equations

For the second order differential equations, some examples to determine the integrating factors are depicted.

Example 4.4. Consider the second order differential equation

$$yy'' - y'^2 = 0 \tag{4.56}$$

for which a simplified integrating factor $\mu = \mu(y)$ is suggested. The Euler equation yields

$$\left(2\mu' y + 4\mu\right) y'' + (3\mu' + \mu'' y) y'^2 = 0 \tag{4.57}$$

The equation is a polynomial in terms of y'' and y' for which the coefficients should vanish separating into two equations

$$2\mu' y + 4\mu = 0, \quad 3\mu' + \mu'' y = 0 \tag{4.58}$$

$\mu = 1/y^2$ satisfies both equations which is the integrating factor. The equation

$$\frac{yy'' - y'^2}{y^2} = 0 \tag{4.59}$$

possesses the first integral

$$\frac{y'}{y} = c \quad . \tag{4.60}$$

If one encounters with inconsistencies in the equations with the simpler form of the integrating factors, then resort to selecting a more complex form is required.

Example 4.5. Consider the second order differential equation

$$yy'' + 3y'^2 = 0 \tag{4.61}$$

for which a simplified integrating factor $\mu = \mu(y)$ is suggested. The Euler equation yields

$$\left(2\mu' y - 4\mu\right) y'' + (\mu'' y - \mu'\ ) y'^2 = 0 \tag{4.62}$$

The terms in the parenthesis should vanish leading to a solution $\mu = y^2$. Multiplying by the integration factor

$$y^3 y'' + 3y^2 y'^{\ 2} = 0 \tag{4.63}$$

the first integral comes out to be

$$y^3 y' = c \quad . \tag{4.64}$$

Example 4.6. Consider the second order differential equation

$$y'' + f(y) = 0 \tag{4.65}$$

for which a simplified integrating factor $\mu = \mu(y')$ is suggested. The Euler equation yields

$$\mu f' - \mu'' f y'' - \mu' f' y' = 0 \tag{4.66}$$

Coefficient of y'' should vanish leading to a solution $\mu = y'$ which also satisfies the remaining part of the equations. Hence

$$y' y'' + y' f(y) = 0 \tag{4.67}$$

can be integrable leading to

$$\frac{y'^2}{2} = -\int f dy + c \quad . \tag{4.68}$$

4.3 Complete Integrability

For the tools given in the previous sections, a complete integrability can be defined in the theorem

Theorem 4.2. *Complete Integrability*

If the k'th order nonlinear differential equation

$$F_k\left(x, y, y', y'', \ldots, y^{(k)}\right) = 0 \tag{4.69}$$

is completely integrable with the aid of possible integrating factors, then each integrating factor $\mu_i\left(x, y, y', y'', \ldots, y^{(i-1)}\right)$ *satisfies*

$$\delta \int \mu_i F_i dx \equiv 0 \qquad i = k, k-1, k-2, \ldots, 2, 1 \tag{4.70}$$

with

$$F_{i-1} = \int \mu_i F_i dx \equiv 0 \qquad i = k, k-1, k-2, \ldots, 2, 1 \tag{4.71}$$

and the final result is

$$F_0\left(x, y\right) = 0 \quad . \tag{4.72}$$

Proof is the successive application of Theorem 4.1 which is omitted here.

Example 4.7. Consider the second order differential equation

$$y'' + \frac{y'^2}{y} = 0 \tag{4.73}$$

The equation is completely integrable. Assume $\mu_2 = \mu_2(y)$. The Euler equation yields

$$\mu_2 = y \tag{4.74}$$

for the integrating factor. The first integral is $yy' = \bar{c}_1$. This equation is directly integrable without an integration factor or $\mu_1 = 1$ leading to the final solution

$$F_0 = y^2 - c_1 x - c_2 = 0 \quad . \tag{4.75}$$

5 Mathematical Physics Problems

In this section, real-life problems arising in mathematical physics are treated with respect to their integrability.

5.1 Emden-Fowler Type Equations

Emden-Fowler type equations (Khalique et al., [11]; Wazwaz, [20], Dolapci and Pakdemirli, [8]) may stem from the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents

$$y'' + \frac{\alpha}{x} y' + f\left(x\right) g\left(y\right) = h(x) \quad . \tag{5.76}$$

Applying Theorem 3.1 to the equation leads to a condition

$$\frac{dg}{dy} = -\frac{\alpha}{x^2 f(x)} \quad . \tag{5.77}$$

The left-hand side of the equation is a pure function of y whereas the right-hand side is a pure function of x. For the equation to hold, both parts should be equivalent to a constant. This may happen if $g = ay$ and $f = b/x^2$ for some constants a and b with a relationship between the constants $a = -\alpha/b$. The equation reduces to

$$y'' + \frac{\alpha}{x} y' - \frac{\alpha}{x^2} y = h(x) \tag{5.78}$$

which is the Cauchy-Euler equation (O'Neil, P. V., [12]) well-known for its integrability properties. The first integral of the equation is

$$y' + \frac{\alpha}{x} y = \int h(x)dx \tag{5.79}$$

which now needs an integrating factor $\mu = x^{\alpha}$ for further integration.

5.2 Bratu Type Equations

The Bratu model is encountered in physical applications such as the fuel ignition of the thermal combustion theory, the expansion of the universe (Wazwaz, [20]). Bratu's initial value problem is (Wazwaz, [20], Aksoy and Pakdemirli, [3])

$$u'' - 2e^u = 0. \tag{5.80}$$

Applying Theorem 3.1 to the equation leads to

$$-2e^u \neq 0 \tag{5.81}$$

which is not identically zero. Hence, the equation is not integrable unless an integration factor is employed. Assuming an integration factor of $\mu = \mu(u')$ and applying Eq. (4.47)

$$-2\mu e^u - \frac{d}{dx}\left(\frac{d\mu}{du}\left(u'' - 2e^u\right)\right) + \frac{d^2\mu}{dx^2} = 0 \tag{5.82}$$

the mid-term vanishes due to the original equation and selecting $\mu = u'$yields

$$-2u'e^u + u''' = 0 \tag{5.83}$$

which is the derivative of the original equation and identically equal to zero. Hence the integrating factor is $\mu = u'$ for the problem with a first integral

$$\frac{u'^2}{2} - 2e^u = c \qquad . \tag{5.84}$$

5.3 Heat Transfer Equations

A number of nonlinear equations arising from heat transfer is tested with respect to their integrability.

The first problem is the combined convection and radiation cooling of a lumped system

$$\frac{du}{dy} + u + \varepsilon u^4 = 0 \tag{5.85}$$

for which the problem is solved by Variational Iteration Method (Ganji and Sadighi, [10]) and the Perturbation Iteration Method (Aksoy et al., [4]). Applying Theorem 3.1. to the equation

$$1 + 4\varepsilon u^3 \neq 0 \tag{5.86}$$

which states that the equation is not integrable in its present form.

For the heat transfer problem

$$\frac{d^2u}{dy^2} - \varepsilon u^4 = 0 \tag{5.87}$$

representing the temperature in a thick rectangular fin with radiation, solutions were already presented approximately (Homotopy Analysis method due to Abbasbandy [1], Perturbation Iteration Method due to Aksoy et al. [4], Variational Iteration Method due to Tari [17]). The integrability theorem yields

$$-4\varepsilon u^3 \neq 0 \tag{5.88}$$

which is not integrable in its present form.

Finally, consider the problem

$$(1 + \varepsilon u)\frac{d^2u}{dy^2} + \varepsilon\left(\frac{du}{dy}\right)^2 = 0 \tag{5.89}$$

which arises in heat transfer of conduction in a slab with variable thermal conductivity (Rajabi et al., [15], Aksoy et al., [4]). The integrability test gives

$$-\varepsilon\frac{d^2u}{dy^2} + \varepsilon\frac{d^2u}{dy^2} = 0 \tag{5.90}$$

which is identically equal to zero with a first integral

$$(1 + \varepsilon u)\frac{du}{dy} = c \qquad . \tag{5.91}$$

5.4 Minimum Drag Work

The mathematical model determining the path of the minimum drag work of a flying object has been proposed and solved by a number of analytical techniques (Pakdemirli, [13]; Abbasbandy et al., [2]; Pakdemirli & Aksoy, [14])

$$y'' - \frac{f'(y)}{f(y)}(1 + y'^2) = 0 \tag{5.92}$$

where $f(y) = \rho(y) C_d(y) A(y) U^2(y)$with all physical parameters depending on the altitude y. The integrability test requires

$$\left(1+3y'^2\right) f'' f + (1-y'^2) f'^2 \neq 0 \qquad (5.93)$$

which cannot be identically equal to zero unless the function f is a constant. Hence, the equation is not integrable in its present form.

6 Concluding Remarks

The integrability of a differential equation is associated with the optimality of the functional form of the equation as depicted with the aid of variational calculus. One can say that if the differential equation is integrable, it is in its optimal form. Otherwise, one needs an integrating factor to transfer the equation into an optimal form. The theorems are presented for determining the direct integrability of the equations. Many problems arising in mathematical physics, such as the Emden-Fowler equation, Bratu equation, equations arising from heat transfer problems and drag induced dynamical problems are tested with respect to their integrability. Before attempting to solve a mathematical physics problem, the readers are recommended to test the integrability of the problem via the theorems presented in this work.

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