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Research Article

Construction of Pseudospectral Meshless Radial Point Interpolation for Sobolev Equation with Error Analysis

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Abstract

This work studies (2D) Sobolev equations by pseudospectral meshless radial point interpolation (PSM-RPI). The Sobolev equations which are arisen in the fluid flow penetrating rocks, soils, or different viscous media do not have an exact solution except in some special cases. Approximation of high-order derivatives through operational matrices is possible. It is proved that the method is convergent and unconditionally stable. Also, the important results for the Sobolev equation are validated for the purpose of showing the ability of this technique.

Keywords : Pseudospectral method; Sobolev equation; Radial point interpolation (RPI); Radial basis function.

1 Introduction

THe methods of meshfree or meshless are in competition with mesh-based methods [10, 34]. Three kind of these methods are exist:

- These methods as weak forms are constructed in [1, 12, 14, 16, 24, 40, 46, 50].
- Those Meshless techniques benefit from collocation approach for instance, the Kansa's method via radial basis functions (RBFs) [2, 3, 8, 32, 30, 31, 51].
- Meshless approaches of weak forms and collocation tool [9, 44, 4, 5, 18, 36, 33, 45, 47, 57, 48, 58, 60, 28, 29, 59].

The problem of (2D) Sobolev equations is introduced by

$$\frac{\partial}{\partial t}(\nabla^2 w - w) + a(\nabla^2 w - w) = f(\mathbf{x}), \quad \mathbf{x} \in \Lambda \subseteq \mathbb{R}^2, \tag{1.1}$$

with conditions

$$w(\mathbf{x}, 0) = w_0(\mathbf{x}), \ \mathbf{x} \in \partial\Lambda, \tag{1.2}$$

$$\frac{\partial w(\mathbf{x})}{\partial n} = 0, \ \mathbf{x} \in \partial\Lambda, \tag{1.3}$$

where n is the outward unit normal vector on the boundary and $\mathbf{x} = (x, y)$. The existence and uniqueness of such equations have been studied

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in [17, 21]. For its applications, see [13, 42, 62] and for discussion about the exact analytical solution refer to[6]. Therefore, applying numerical techniques are unavoidable, regarding this issue some studies are well done [20, 7, 22, 65, 39, 27, 41, 11, 19, 26, 66, 15, 61, 25].

In Ref. [37], authors studied a kind of Crank-Nicolson finite volume element extrapolating algorithm (CNFVEEA) holding fully second-order accuracy based on suitable orthogonal decomposition in order to numerically solving two-dimensional (2D) Sobolev equations in the case of Dirichlet boundary conditions. Also, for this kind of problem, another type of extrapolated finite difference method has been discussed in [38]. In this study, we develop PSMRPI method. We conclude that it is accurate through some examples. Moreover, it is proved that the method is convergence and unconditionally stable.

2 Brief review of pseudo-spectral (PS) methods

In this method the unknown function is written in terms of A_j, $j = 1, ..., N$, as [63]

$$\hat{w}(x) = \sum_{j=1}^{N} s_j A_j(x), x \in \mathbb{R}. \tag{2.4}$$

PS is based on so-called *differentiation matrices* or *operational matrices*, and approximation of first derivative is done through the matrix D as

$$\mathbf{w}' = D\mathbf{w}, \tag{2.5}$$

in where $\mathbf{w} = [\hat{w}(x_1), \hat{w}(x_2), ..., \hat{w}(x_N)]^T$ is the vector of unknowns $\hat{w}$ at the collocation nodal points. Take into account the expansion (2.4) and assume $A_j, j = 1, ..., N$, be a set of arbitrary linearly independent functions. Setting $x = x_i, i = 1, ..., N$, in Eq. (2.4) we obtain

$$\hat{w}(x_i) = \sum_{j=1}^{N} s_j A_j(x_i), \quad i = 1, ..., N, \tag{2.6}$$

or equivalently in matrix-form

$$\mathbf{w} = \mathbf{H}\mathbf{c}, \tag{2.7}$$

where $\mathbf{c} = [s_1, ..., s_N]$ is the vector of coefficient, the matrix $\mathbf{H}$ is defined by $\mathbf{H}_{ij} = A_j(x_i)$, and $\mathbf{w}$ is as defined previously. Benefiting linearity of (2.4), computing the derivative of $\hat{w}$ via differentiating the basis functions gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat{w}(x) = \sum_{j=1}^{N} s_j \frac{\mathrm{d}}{\mathrm{d}x} A_j(x), \quad i = 1, ..., N, \tag{2.8}$$

in the points x_i and by matrix format, we have

$$\mathbf{w}' = \mathbf{H}_x \mathbf{c}, \tag{2.9}$$

where $\mathbf{H}_x$ are expressed by $\frac{\mathrm{d}}{\mathrm{d}x} A_j(x_i)$. In order to obtain D it is sufficient to be relaxed about inversion of matrix $\mathbf{H}$, if so, then

$$\mathbf{w}' = \mathbf{H}_x \mathbf{H}^{-1} \mathbf{w}, \tag{2.10}$$

where the operational (differentiation) matrix D corresponding to Eq. (2.5) is represented via

$$D = \mathbf{H}_x \mathbf{H}^{-1}. \tag{2.11}$$

3 PSMRPI method

The previously explained method so-called PSMRPI has proved itself as an efficient and more reliable tool some senses [64, 43, 23, 49, 54, 55, 53, 52, 56, 35].

Theorem 3.1. ***(Micchelli)*** *Taking into account $\psi \in C[0, \infty) \cap C^{\infty}(0, \infty)$, the expression $(-1)^m \psi^{(m)}$ is completely monotone on $(0, \infty)$ if and only if the function $\nu = \psi(\|.\|_2^2)$ is strictly conditionally positive definite of order $m \in \mathbb{N}_0$ on $\mathbb{R}^d$.*

Theorem 3.1 is very helpful to find conditionally positive definite functions like for example thin-plate or surface splines $\psi(r) = (-1)^{k+1} r^{2k} \log(r)$ of order $m = k + 1$ on $\mathbb{R}^d$.

Let $w(\mathbf{x})$ at a node of interest $\mathbf{x}$ be approximated by

$$w(\mathbf{x}) = \sum_{i=1}^{n} R_i(\mathbf{x}) a_i + \sum_{j=1}^{np} P_j(\mathbf{x}) b_j = \mathbf{R}^{tr}(\mathbf{x})\mathbf{a} + \mathbf{P}^{tr}(\mathbf{x})\mathbf{b}. \tag{3.12}$$

Since we ensure about invertibility of the matrix system, to extract coefficients a_i and b_j in Eq. (3.12), via the interpolation idea Eq. (3.12) turns to the following form

$$w(\mathbf{x}) = \boldsymbol{\Upsilon}^{tr}(\mathbf{x})\mathbf{W}_s = \sum_{i=1}^{n} \nu_i(\mathbf{x}) w_i, \qquad (3.13)$$

where $\nu_i(\mathbf{x})$ satisfies

$$\nu_i(\mathbf{x}_j) = \begin{cases} 1, & i = j, \quad j = 1, 2, ..., n, \\ 0, & i \neq j, \quad i, j = 1, 2, ..., n, \end{cases} \qquad (3.14)$$

indeed. Moreover, the basis functions hold the partitions of unity, i.e.

$$\sum_{i=1}^{n} \nu_i(\mathbf{x}) = 1. \qquad (3.15)$$

In the next stage, we provide differentiation matrices to make it more appropriate for high-order PDEs. Suppose that the number of total nodes covering the domain of the problem i.e. $\bar{\Lambda} = \Lambda \bigcup \partial\Lambda$, is N. Here, n depends on the point of interest $\mathbf{x}$ (call it $n_{\mathbf{x}}$) in Eq. (3.13) which is the number of nodes included in the support domain $\Lambda_{\mathbf{x}}$ corresponding to the point of interest $\mathbf{x}$ (for example $\Lambda_{\mathbf{x}}$ can be a disk centered at $\mathbf{x}$ with radius r_s). We have $n_{\mathbf{x}} \leq N$ and then Eq. (3.13) can be modified as

$$w(\mathbf{x}) = \boldsymbol{\Upsilon}^{tr}(\mathbf{x})\mathbf{W}_s = \sum_{j=1}^{N} \nu_j(\mathbf{x}) w_j. \qquad (3.16)$$

The derivatives of $w(\mathbf{x})$ are easily obtained as

$$\begin{aligned} \frac{\partial w(\mathbf{x})}{\partial x} &= \sum_{j=1}^{N} \frac{\partial \nu_j(\mathbf{x})}{\partial x} w_j, \\ \frac{\partial w(\mathbf{x})}{\partial y} &= \sum_{j=1}^{N} \frac{\partial \nu_j(\mathbf{x})}{\partial y} w_j, \\ \frac{\partial w(\mathbf{x})}{\partial z} &= \sum_{j=1}^{N} \frac{\partial \nu_j(\mathbf{x})}{\partial z} w_j, \end{aligned} \qquad (3.17)$$

and for higher order derivatives of $w(\mathbf{x})$

$$\begin{aligned} \frac{\partial^s w(\mathbf{x})}{\partial x^s} &= \sum_{j=1}^{N} \frac{\partial^s \nu_j(\mathbf{x})}{\partial x^s} w_j, \\ \frac{\partial^s w(\mathbf{x})}{\partial y^s} &= \sum_{j=1}^{N} \frac{\partial^s \nu_j(\mathbf{x})}{\partial y^s} w_j, \\ \frac{\partial^s w(\mathbf{x})}{\partial z^s} &= \sum_{j=1}^{N} \frac{\partial^s \nu_j(\mathbf{x})}{\partial z^s} w_j. \end{aligned} \qquad (3.18)$$

Denoting $w_x^{(s)}(.) = \frac{\partial^s(.)}{\partial x^s}$, $w_y^{(s)}(.) = \frac{\partial^s(.)}{\partial y^s}$, $w_z^{(s)}(.) = \frac{\partial^s(.)}{\partial z^s}$ and setting $\mathbf{x} = \mathbf{x}_i$ in Eq. (3.18):

$$\begin{aligned} &W_x^{(s)} = D_x^{(s)} W, W_y^{(s)} = D_y^{(s)} W, \\ &W_z^{(s)} = D_z^{(s)} W, \end{aligned} \qquad (3.19)$$

where

$$\begin{aligned} W_x^{(s)} &= \left(w_{x1}^{(s)}, ..., w_{xN}^{(s)}\right)^{tr}, \\ W_y^{(s)} &= \left(w_{y1}^{(s)}, ..., w_{yN}^{(s)}\right)^{tr}, \\ W_z^{(s)} &= \left(w_{z1}^{(s)}, ..., w_{zN}^{(s)}\right)^{tr}, \end{aligned} \qquad (3.20)$$

and

$$\begin{aligned} D_{x_{ij}}^{(s)} &= \frac{\partial^s \nu_j(\mathbf{x}_i)}{\partial x^s}, \\ D_{y_{ij}}^{(s)} &= \frac{\partial^s \nu_j(\mathbf{x}_i)}{\partial y^s}, \\ D_{z_{ij}}^{(s)} &= \frac{\partial^s \nu_j(\mathbf{x}_i)}{\partial z^s}, \end{aligned} \qquad (3.21)$$

and

$$W = (w_1, w_2, ..., w_N)^{tr}. \qquad (3.22)$$

4 Application to Sobolev equation

In this section, we apply the previously explained method to (1.1)-(1.3). Let consider:

$$\frac{\partial w\left(\mathbf{x}, (k + \frac{1}{2})\Delta t\right)}{\partial t} \cong \frac{1}{\Delta t}\left(w^{k+1}(\mathbf{x}) - w^k(\mathbf{x})\right), \qquad (4.23)$$

$$\nabla^2 w\left(\mathbf{x},(k+\frac{1}{2})\Delta t\right)\cong \frac{1}{2}\left(\nabla^2 w^{k+1}(\mathbf{x})+\nabla^2 w^k(\mathbf{x})\right), \quad (4.24)$$

$$\frac{\partial \nabla^2 w\left(\mathbf{x},(k+\frac{1}{2})\Delta t\right)}{\partial t}\cong \frac{1}{\Delta t}\left(\nabla^2 w^{k+1}(\mathbf{x})-\nabla^2 w^k(\mathbf{x})\right), \quad (4.25)$$

where $w^k(\mathbf{x})=w(\mathbf{x},k\Delta t)$, and $\nabla^2 w=\Delta w$:

$$\nabla^2 w=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}. \quad (4.26)$$

4.1 Time Discretization

Using the formulations (4.23)-(4.25), we discretize Eq. (1.1) in the form of finite difference formula at $t=(k+\frac{1}{2})\Delta t$ as:

$$\frac{1}{\Delta t}\left(\nabla^2 w^{k+1}(\mathbf{x})-\nabla^2 w^k(\mathbf{x})\right)-\frac{w^{k+1}(\mathbf{x})-w^k(\mathbf{x})}{\Delta t}+a\left(\frac{1}{2}\left(\nabla^2 w^{k+1}(\mathbf{x})+\nabla^2 w^k(\mathbf{x})\right)-\frac{w^{k+1}(\mathbf{x})+w^k(\mathbf{x})}{2}\right)=\frac{f^{k+1}(\mathbf{x})+f^k(\mathbf{x})}{2}, \quad (4.27)$$

where $f^k=f(\mathbf{x},k\Delta t)$, denoting $\zeta=\frac{a\Delta t}{2}$ and $F^{k+1}(\mathbf{x})=\frac{\Delta t(f^{k+1}(\mathbf{x})+f^k(\mathbf{x}))}{2}$, we get

$$\nabla^2 w^{k+1}(\mathbf{x})-\nabla^2 w^k(\mathbf{x})-(w^{k+1}(\mathbf{x})-w^k(\mathbf{x}))+\zeta(\nabla^2 w^{k+1}(\mathbf{x})+\nabla^2 w^k(\mathbf{x})-(w^{k+1}(\mathbf{x})+w^k(\mathbf{x})))=F^{k+1}(\mathbf{x}), \quad (4.28)$$

or

$$(1+\zeta)\nabla^2 w^{k+1}(\mathbf{x})-(1+\zeta)w^{k+1}(\mathbf{x})=(1-\zeta)\nabla^2 w^k(\mathbf{x})-(1-\zeta)w^k(\mathbf{x})+F^{k+1}(\mathbf{x}). \quad (4.29)$$

Finally Eq. (4.29) can be rewritten as

$$(1+\zeta)\left(\frac{\partial^2 w^{k+1}(\mathbf{x})}{\partial x^2}+\frac{\partial^2 w^{k+1}(\mathbf{x})}{\partial y^2}\right)-(1+\zeta)w^{k+1}(\mathbf{x})(1-\zeta)\times\left(\frac{\partial^2 w^{k}(\mathbf{x})}{\partial x^2}+\frac{\partial^2 w^{k}(\mathbf{x})}{\partial y^2}\right)-(1-\zeta)w^k(\mathbf{x})+F^{k+1}(\mathbf{x}). \quad (4.30)$$

If we apply the approximations (3.16) and (3.18):

$$(1+\zeta)\left(\sum_{j=1}^{N}\frac{\partial^2 \nu_j(\mathbf{x})}{\partial x^2}w_j^{k+1}+\sum_{j=1}^{N}\frac{\partial^2 \nu_j(\mathbf{x})}{\partial y^2}w_j^{k+1}\right)-(1+\zeta)\times\sum_{j=1}^{N}\nu_j(\mathbf{x})w_j^{k+1}=(1-\zeta)\times\left(\sum_{j=1}^{N}\frac{\partial^2 \nu_j(\mathbf{x})}{\partial x^2}w_j^{k}+\sum_{j=1}^{N}\frac{\partial^2 \nu_j(\mathbf{x})}{\partial y^2}w_j^{k}\right)-(1-\zeta)\sum_{j=1}^{N}\nu_j(\mathbf{x})w_j^{k}+F^{k+1}(\mathbf{x}). \quad (4.31)$$

Let us set $\mathbf{x}=\mathbf{x}_i$, $i=1,2,...,N$, then

$$(1+\zeta)\left(\sum_{j=1}^{N}\frac{\partial^2 \nu_j(\mathbf{x}_i)}{\partial x^2}w_j^{k+1}+\sum_{j=1}^{N}\frac{\partial^2 \nu_j(\mathbf{x}_i)}{\partial y^2}w_j^{k+1}\right)-(1+\zeta)w_i^{k+1}=(1-\zeta)\left(\sum_{j=1}^{N}\frac{\partial^2 \nu_j(\mathbf{x}_i)}{\partial x^2}w_j^{k}+\sum_{j=1}^{N}\frac{\partial^2 \nu_j(\mathbf{x}_i)}{\partial y^2}w_j^{k}\right)-(1-\zeta)w_i^k+F_i^{k+1}, \quad (4.32)$$

where $F_i^{k+1} = F^{k+1}(\mathbf{x}_i)$, $i = 1, 2, ..., N$, afterwards by notations (3.21)

$$(1+\zeta)\left(\sum_{j=1}^{N} D^{(2)}x_{ij}w_j^{k+1} + \sum_{j=1}^{N} D^{(2)}y_{ij}w_j^{k+1}\right) - (1+\zeta)w_i^{k+1} = (1-\zeta)\left(\sum_{j=1}^{N} D^{(2)}x_{ij}w_j^{k} + \sum_{j=1}^{N} D^{(2)}y_{ij}w_j^{k}\right) - (1-\zeta)w_i^k + F_i^{k+1}. \quad (4.33)$$

In the matrix form, we have

$$\mathbf{A}W^{k+1} = \mathbf{B}W^k + \mathbf{F}^{k+1}, \quad (4.34)$$

where

$$W^{k+1} = \left(w_1^{k+1}, w_2^{k+1}, \ldots, w_N^{k+1}\right)^T,$$
$$W^{k} = \left(w_1^{k}, w_2^{k}, \ldots, w_N^{k}\right)^T,$$
$$\mathbf{F}^{k+1} = \left(F_1^{k+1}, F_2^{k+1}, \ldots, F_N^{k+1}\right)^T,$$
$$\mathbf{A}_{ij} = (1+\zeta)\left(D^{(2)}x_{ij} + D^{(2)}y_{ij}\right) -(1+\zeta)\delta_{ij},$$
$$\mathbf{B}_{ij} = (1-\zeta)\left(D^{(2)}x_{ij} + D^{(2)}y_{ij}\right) -(1-\zeta)\delta_{ij}.$$

4.2 Imposing the boundary conditions

Imposing boundary conditions (1.3), we have

$$n_1(\mathbf{x}_i)\frac{\partial w(\mathbf{x}_i)}{\partial x} + n_2(\mathbf{x}_i)\frac{\partial w(\mathbf{x}_i)}{\partial y} = 0, \quad i = 1, 2, ..., N_{\partial\Lambda}, \quad (4.35)$$

where $\mathbf{n} = n_1(\mathbf{x}_i)\mathbf{i} + n_2(\mathbf{x}_i)\mathbf{j}$, is the outward unit normal on the boundary $\partial\Lambda$ at $\mathbf{x}_i \in \partial\Lambda$. Eq. (4.35) can be rewritten as

$$n_1(\mathbf{x}_i)\sum_{j=1}^{N}\frac{\partial \nu_j(\mathbf{x}_i)}{\partial x}w_j + n_2(\mathbf{x}_i)\times \sum_{j=1}^{N}\frac{\partial \nu_j(\mathbf{x}_i)}{\partial y}w_j = 0, \quad i = 1, 2, ..., N_{\partial\Lambda}. \quad (4.36)$$

Eq. (4.36) can be given as

$$n_1(\mathbf{x}_i)\sum_{j=1}^{N} D_{x_{ij}}w_j + n_2(\mathbf{x}_i)\sum_{j=1}^{N} D_{y_{ij}}w_j = 0, \quad i = 1, 2, ..., N_{\partial\Lambda}, \quad (4.37)$$

or

$$\sum_{j=1}^{N}\left(n_1(\mathbf{x}_i)D_{x_{ij}} + n_2(\mathbf{x}_i)D_{y_{ij}}\right)w_j = 0, \quad i = 1, 2, ...N_{\partial\Lambda}. \quad (4.38)$$

Now, we consider the block matrices in Eq. (4.34) based on all collocation points and therefore

$$\mathbf{A}'W^{k+1} = \mathbf{B}'W^k + \mathbf{F}'^{k+1}. \quad (4.39)$$

5 Stability of time discretization scheme

Theorem 5.1. *Suppose $w^k \in H^2(\Lambda), k = 0, 1, \ldots, K$ with $K\Delta t = T$, is the solution of Eq. (4.29) satisfying the boundary condition (1.3), if the function w^k and its derivatives with respect to x and y be bounded on Λ then*

$$\|w^{k+1}\|_2 \leq \left(\frac{1-\zeta}{1+\zeta}\right)^{k+1}\|w^0\|_2 + \frac{1}{2\zeta}\left\{1 - \left(\frac{1-\zeta}{1+\zeta}\right)^{k+1}\right\}\times \left((1-\zeta)M + max_{0\leq l\leq K}\|F^l\|_2\right).$$

Proof. Multiplying Eq. (4.29) by w^{k+1} and integrating on Λ we obtain

$$(1+\zeta)(\nabla^2 w^{k+1}, w^{k+1}) -(1+\zeta)(w^{k+1}, w^{k+1}) = (1-\zeta)(\nabla^2 w^k, w^{k+1}) - (1-\zeta)\times (w^k, w^{k+1}) + (F^{k+1}, w^{k+1}). \quad (5.40)$$

By using the following Green's formula:

$$\int_\Lambda \nabla v\cdot\nabla w d\mathbf{x} = \int_{\partial\Lambda} v\frac{\partial w}{\partial \mathbf{n}}d\mathbf{s} - \int_\Lambda v\Delta w d\mathbf{x},$$

where

$$\frac{\partial w}{\partial \mathbf{n}} = \frac{\partial w}{\partial x}n_1 + \frac{\partial w}{\partial y}n_2$$

is the normal derivative, we have

$$-(1+\zeta)(\nabla w^{k+1},\nabla w^{k+1})-(1+\zeta)\|w^{k+1}\|_2^2=-(1-\zeta)(w^k,w^{k+1})-(1-\zeta)(\nabla w^{k+1},\nabla w^k)+(F^{k+1},w^{k+1}),$$

or equivalently

$$(1+\zeta)(\nabla w^{k+1},\nabla w^{k+1})+(1+\zeta)\|w^{k+1}\|_2^2=(1-\zeta)(w^k,w^{k+1})+(1-\zeta)(\nabla w^{k+1},\nabla w^k)-(F^{k+1},w^{k+1}).$$

Obviously the value ζ goes to zero as the time stepping Δt tends to zero. Therefore, having Δt small enough, both $1+\zeta$ and $1-\zeta$ will be positive. Now, we have

$$(1+\zeta)\|w^{k+1}\|_2^2\leq(1-\zeta)\|w^k\|_2\|w^{k+1}\|_2+(1-\zeta)M\|w^{k+1}\|_2+\|F^{k+1}\|_2\|w^{k+1}\|_2,$$

where M is big enough positive constant. By simplifying the above inequality, it turns to

$$(1+\zeta)\|w^{k+1}\|_2\leq(1-\zeta)\|w^k\|_2+(1-\zeta)M+\|F^{k+1}\|_2, \tag{5.41}$$

or equivalently

$$\|w^{k+1}\|_2\leq\frac{1-\zeta}{1+\zeta}\|w^k\|_2+\frac{1-\zeta}{1+\zeta}M+\frac{1}{1+\zeta}max_{0\leq l\leq K}\|F^l\|_2. \tag{5.42}$$

Applying once again the above equation to the first term of the right hand, we have

$$\|w^{k+1}\|_2\leq\left(\frac{1-\zeta}{1+\zeta}\right)^2\|w^{k-1}\|_2+\left(\frac{1-\zeta}{1+\zeta}\right)^2M+\frac{1-\zeta}{1+\zeta}\frac{1}{1+\zeta}\times max_{0\leq l\leq K}\|F^l\|_2+\frac{1-\zeta}{1+\zeta}M+\frac{1}{1+\zeta}max_{0\leq l\leq K}\|F^l\|_2. \tag{5.43}$$

If we continue this strategy, then we get

$$\|w^{k+1}\|_2\leq\left(\frac{1-\zeta}{1+\zeta}\right)^{k+1}\|w^0\|_2+M\left\{\left(\frac{1-\zeta}{1+\zeta}\right)^{k+1}+\cdots+\left(\frac{1-\zeta}{1+\zeta}\right)\right\}+\frac{1}{1+\zeta}max_{0\leq l\leq K}\|F^l\|_2\times\left\{\left(\frac{1-\zeta}{1+\zeta}\right)^{k}+\cdots+\left(\frac{1-\zeta}{1+\zeta}\right)+1\right\}, \tag{5.44}$$

finally

$$\|w^{k+1}\|_2\leq\left(\frac{1-\zeta}{1+\zeta}\right)^{k+1}\|w^0\|_2+\frac{1}{2\zeta}\left\{1-\left(\frac{1-\zeta}{1+\zeta}\right)^{k+1}\right\}\times\left((1-\zeta)M+max_{0\leq l\leq K}\|F^l\|_2\right),$$

and the proof is complete. □

Now, the following stability result holds.

Theorem 5.2. *If the solution of the problem (1.1)-(1.3) with $a>0$ and its derivatives with respect to x and y be bounded on Λ then the numerical method defined by Eq. (4.29) is stable with L^2-norm in the sense that*

$$\|\sigma^k\|_2\leq\|\sigma^0\|_2+Const.$$

Proof. Denote the error

$$\sigma(\mathbf{x})=w^k(\mathbf{x})-W^k(\mathbf{x}),$$

where w^k and W^k are the exact and approximate solutions to Eq. (4.29). Therefore, it satisfies

$$(1+\zeta)\nabla^2\sigma^{k+1}(\mathbf{x})-(1+\zeta)\sigma^{k+1}(\mathbf{x})=(1-\zeta)\nabla^2\sigma^k(\mathbf{x})-(1-\zeta)\sigma^k(\mathbf{x}), \tag{5.45}$$

with

$$\frac{\partial\sigma^k(\mathbf{x})}{\partial n}|_{\partial\Lambda}=0.$$

Now from Theorem 5.1 and having $a > 0$, we obtain

$$\|\sigma^{k+1}\|_2 \leq \left(\frac{1-\zeta}{1+\zeta}\right)^{k+1} \|\sigma^0\|_2 + \frac{(1-\zeta)M}{2\zeta}\left\{1 - \left(\frac{1-\zeta}{1+\zeta}\right)^{k+1}\right\} \leq \|\sigma^0\|_2 + \frac{(1-\zeta)M}{2\zeta}. \quad (5.46)$$

□

6 Numerical Results

For implementation, the radius of support domain (that is a circle), is considered $r_s = 3.2h$, where h is the nodal distance in both directions. This size is significant enough to have sufficient number of nodes ($n_{\mathbf{x}}$) and gives appropriate basis functions. In Eq. (3.12), we set $np = 21$, and define

$$AbsoluteError(\mathbf{x}) = |w_{exact}(\mathbf{x}) - w_{approx}(\mathbf{x})|, \quad (6.47)$$

and relative error as

$$RelativeError(\mathbf{x}) = \frac{AbsoluteError(\mathbf{x})}{\|w_{exact}(\mathbf{x})\|_\infty}. \quad (6.48)$$

Example 6.1. *Consider the problem (1.1)-(1.3) with the exact solution*

$$w(\boldsymbol{x}) = \sin(t)\sinh(x)\sinh(y),$$

$\Lambda = [0,1]^2$ *and* $w_0(\boldsymbol{x}) = 0$. *Fig. 1 shows the numerical PSMRPI solutions with* $\Delta t = 0.01$ *and Fig. 2 reveals the AbsoluteError. Fig. 3 is for* $\Delta t = 0.1$. *The results show the convergence and stability. Stability of the mentioned approach for this example is inspected separately in Fig. 4 when* $\Delta t = 0.01$.

Example 6.2. *Like pervious example, consider*

$$w(\boldsymbol{x}) = \exp(t)\left(1 - x^2 - y^2\right)$$

with $w_0(\boldsymbol{x}) = 1 - x^2 - y^2$. *The domain of the problem as shown in Fig. 5 is defined by* $\Lambda = \{(x,y) : x^2 + y^2 \leq 1\}$. *Fig. 6 shows the numerical solutions for* $\Delta t = 0.001$ *and Fig. 7 reveals the RelativeError. Fig. 8 is for* $\Delta t = 0.01$. *Figures show the convergence and stability. The Fig. 9 is plotted when* $\Delta t = 0.001$.

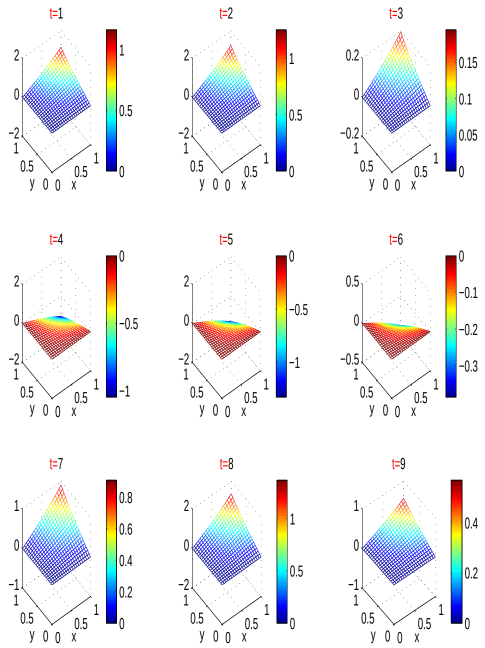

Figure 1: The solution output of PSMRPI for Example 1 with $N = 441$.

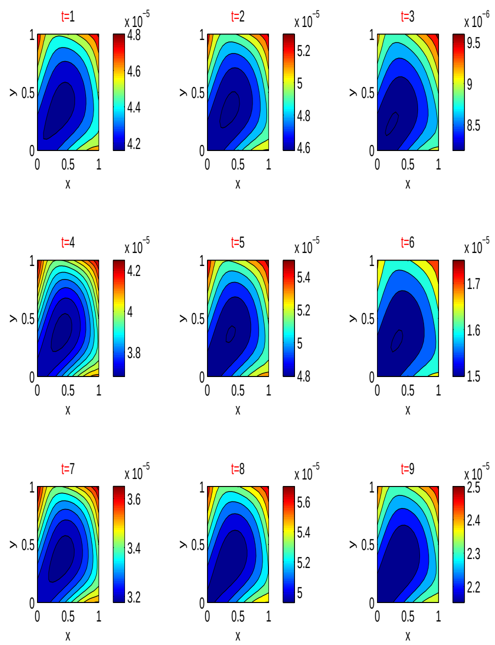

Figure 2: The *AbsoluteError* in Example 1 with $N = 441$ and $\Delta t = 0.01$.

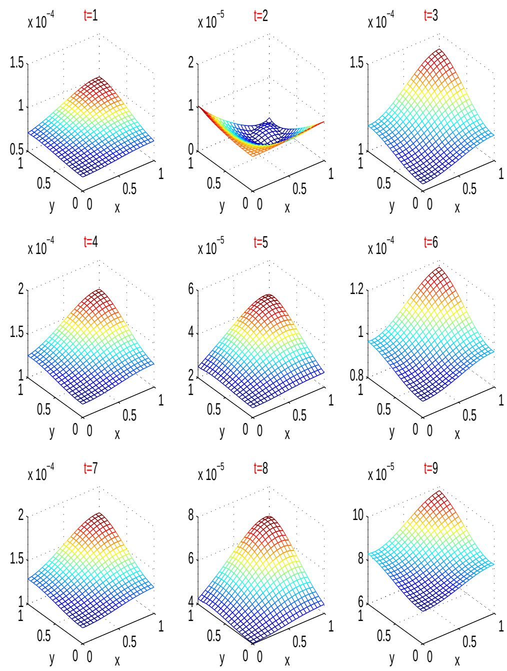

Figure 3: The *AbsoluteError* in Example 1 with $N = 441$ and $\Delta t = 0.1$.

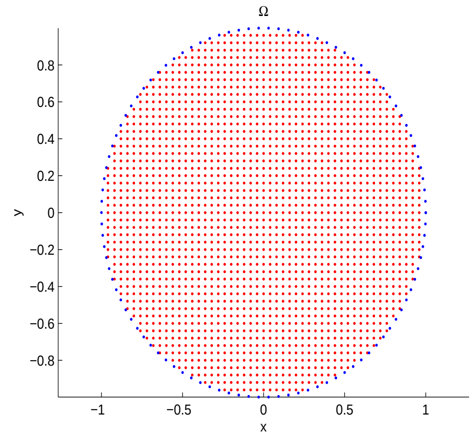

Figure 5: The domain covering by regularly distributed nodal points for Example 2.

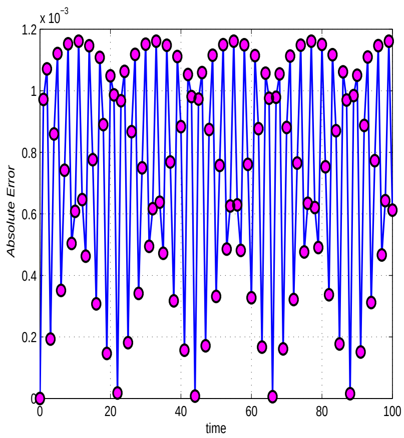

Figure 4: The *AbsoluteError* during the time in the range $[0, 100]$ for Example 1 with $N = 441$.

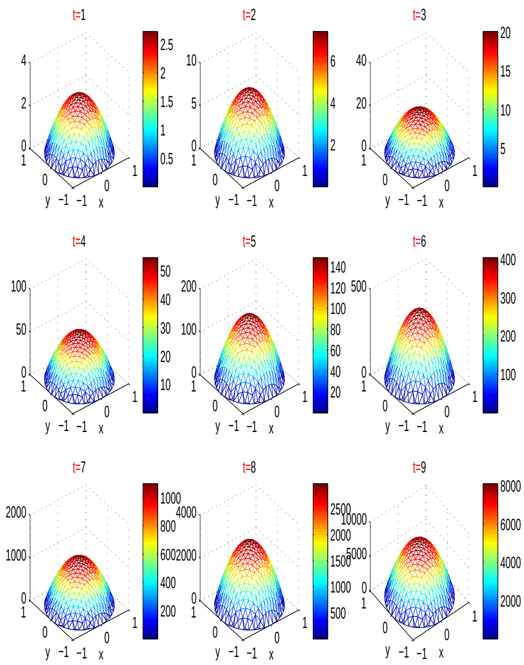

Figure 6: The solution output of PSMRPI for Example 2 with $N = 347$.

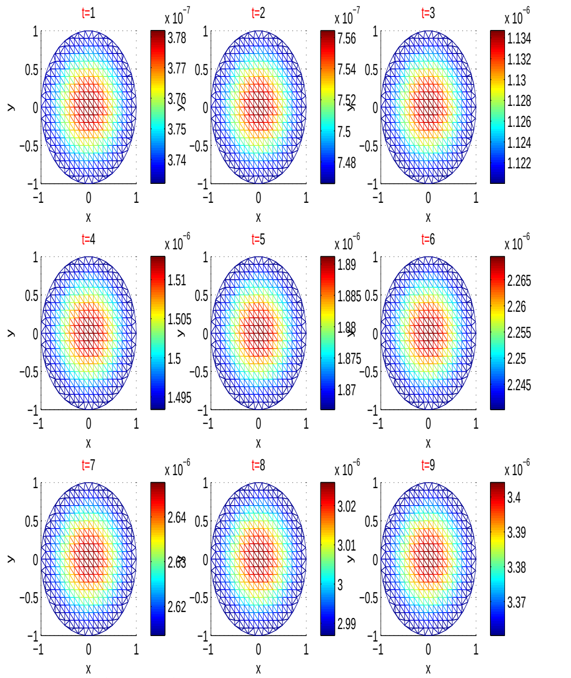

Figure 7: The *RelativeError* in Example 2 with $N = 347$ and $\Delta t = 0.001$.

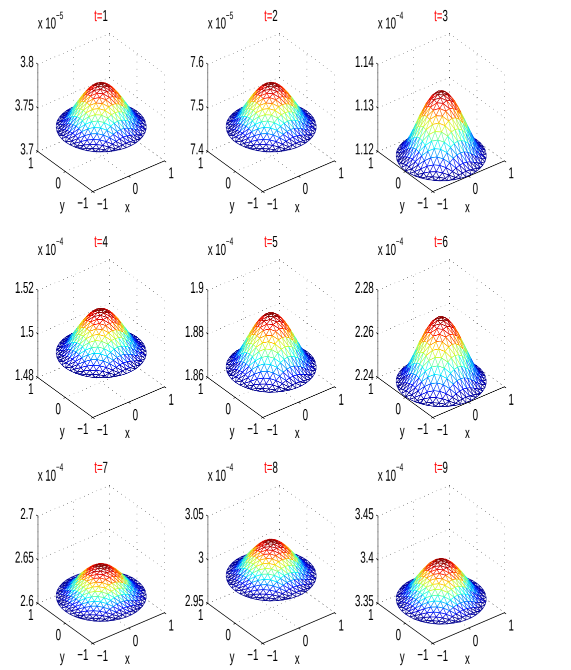

Figure 8: The *RelativeError* in Example 2 with $N = 347$ and $\Delta t = 0.01$.

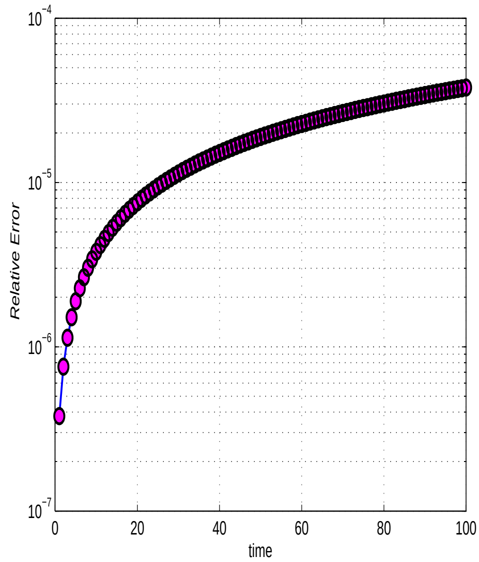

Figure 9: The *RelativeError* during the time in the range $[0, 100]$ for Example 2 with $N = 347$.

7 Conclusions

In this manuscript, quasi-spectral mesh radial point interpolation (PSMRPI) is improved to the Sobolev equations. This is a two-dimensional time-dependent diffusion equation, and dependents on Neumann's type of boundary conditions. Because the method is a truly mesh-free technique, the complexity of the domain is removed by scattered nodal points. It has also been shown that the unconditional method is somehow stable with respect to time.

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