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Research Article

A Study on Analytical Solutions of the Fuzzy Partial Differential Equations

R. Ghasemi Moghaddam [*], S. Abbasbandy [†‡], M. Rostamy-Malkhalifeh [§]

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Abstract

In the present paper, we obtain the traveling wave fuzzy solution for the fuzzy linear Transport equation and the fuzzy Wave equation by considering the type of generalized Hukuhara differentiability. The d'Alembert's formulas for the fuzzy Wave equation obtained by Considering the type of gH-differentiability of the solution. Also, The existence and the uniqueness of these solutions and the stability of the fuzzy Wave equation are shown. Furthermore, Some examples are solved to illustrate the technique.

Keywords : Linear fuzzy Transport equation; The fuzzy Wave equation; Traveling wave fuzzy solution; d'Alembert's formula; Generalized partial Hukuhara differentiability.

1 Introduction

Any modeling of phenomena is subject to limitations such as correct understanding, ambiguity in the accuracy and uncertainty of the data, and measurement errors that lead to uncertainties in the model. Fuzzy modeling and utilizing fuzzy systems is an effective way that enables researchers to express engineering and other sciences issues by taking into account the uncertainties in the model so that it is closer to its true reality and nature.

Often modeling of many physical phenomena such as dynamical and magnetic systems, engineering and biological and environmental issues, and humanities phenomena result in the use of differential equations, whether ordinary differential equations or partial differential equations. Uncertainties in differential equation models can occur anywhere in the equations, including initial and boundary values, equation coefficients, shape and domain amplitude, and so on.

Starting from the pioneering papers [2, 3, 6, 10, 11, 12], considerable interest has been shown in finding fuzzy solutions to the fuzzy partial differential equation. Maria Bertone [10] obtained the fuzzy solutions of heat, wave, and Poisson equations by using the fuzzification of the deterministic solutions. Allahviranloo [4] converted the fuzzy heat equation with appropriate fuzzy ini-

*Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
†Corresponding author. abbasbandy@ikiu.ac.ir, Tel:+98(912)1305326.
‡Department of Applied Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.
§Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

tial conditions into the fuzzy ordinary differential equation and found the corresponding analytical solution, also proved the existence and uniqueness of the solution to this equation. Furthermore, the authors in [14] presented the fuzzy solutions of the fuzzy Poisson and fuzzy Laplace equation. Recently, in [5] a new approach for the linear partial differential equations with fuzzy coefficients has been presented.

The traveling wave solutions of the partial differential equations can provide physical aspects of the problems, therefore, they play an essential role in applied science fields [1, 13, 15]. In this article, we will obtain the fuzzy traveling wave solution for the fuzzy linear transform equation and fuzzy Wave equation. We will discuss the fuzzy traveling wave solution of these equations by considering the type of $gH-$differentiability.

A brief outline of the contents is now given. Some concepts associated with fuzzy numbers and generalized Hukuhara differentiability and, etc. are expressed, and some new theorems and lemmas to be used in the main part of the paper are proved in Section 2. In Section 3, we develop the ideas of the traveling wave solution for two-variable fuzzy function and depending on the type of gH-differentiability different formulas are obtained. Section 4 then obtains the traveling wave fuzzy solution of the fuzzy linear traveling equation, and the fuzzy Wave equation, and considering the type of $[gH-p]$-differentiability; the corresponding formulas are shown. Additionally, the d'Alembert's formulas for the fuzzy Wave equation are obtained, and the existence and uniqueness of these solutions also the stability of the fuzzy Wave equation are shown.

2 Preliminaries

In this section the basic definitions used in fuzzy operations and the necessary notation which will be used throughout the paper are introduced.

The triangular fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ is defined as an ordered triple $a = (a_1, a_2, a_3)$ with $a_1 \leq a_2 \leq a_3$. The generalized Hukuhara difference of two fuzzy number $a, b \in \mathbb{R}_{\mathcal{F}}$ is the fuzzy number c,(if it exists), such that

$$a \ominus_{gH} b = c \Longleftrightarrow (i).\ a = b \oplus c, \quad or \quad (ii).b = a \oplus (-1)c.$$

Now consider $a, b \in \mathbb{R}_{\mathcal{F}}$, then

$$a \ominus_{gH} b = c \Longleftrightarrow \begin{cases} (i).\ c = (a_1 - b_1, a_2 - b_2, a_3 - b_3); \\ or\ (ii).\ c = (a_3 - b_3, a_2 - b_2, a_1 - b_1). \end{cases}$$

provided that c is a triangular fuzzy number [5, 7]. The results obtained in [7] show that if $a, b \in \mathbb{R}_{\mathcal{F}}$, then $a \ominus_{gH} b$ always exists in $\mathbb{R}_{\mathcal{F}}$.

Definition 2.1 *(See [7]) The fuzzy function $f(t)$ is generalized Hukuhara differentiable ($[gH]-$differentiable) at $t_0 \in \mathbb{J}$ if*

$$f'_{gH}(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h},$$

belongs to $\mathbb{R}_{\mathcal{F}}$. In addition we can say that $f(t)$ is

- *$[(i) - gH]-$differentiable function if and only if for all $t \in \mathbb{J}$*

$$f'_{i.gH}(t) = \Big(f'_1(t), f'_2(t), f'_3(t)\Big),$$

defines a triangular fuzzy number.

- *$[(ii) - gH]-$differentiable function if and only if for all $t \in \mathbb{J}$*

$$f'_{ii.gH}(t) = \Big(f'_3(t), f'_2(t), f'_1(t)\Big),$$

is a triangular fuzzy number.

Definition 2.2 *(See [4]) A triangular fuzzy function $u(x,t) : \mathbb{D} \subseteq \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$, without any switching point on $\mathbb{D}$ is called*

- *$[(i) - p]$-differentiable w.r.t. x at (x_0, t_0) if*

$$\frac{\partial_{i.gH} u(x_0, t_0)}{\partial x} = \Big(\frac{\partial u_1(x_0, t_0)}{\partial x}, \frac{\partial u_2(x_0, t_0)}{\partial x}, \frac{\partial u_3(x_0, t_0)}{\partial x}\Big),$$

- $[(ii)-p]$-*differentiable w.r.t.* x *at* (x_0, t_0) *if*

$$\frac{\partial_{ii.gH} u(x_0,t_0)}{\partial x} = \Big(\frac{\partial u_3(x_0,t_0)}{\partial x}, \frac{\partial u_2(x_0,t_0)}{\partial x}, \frac{\partial u_1(x_0,t_0)}{\partial x}\Big).$$

Moreover, if $\frac{\partial_{gH} u}{\partial x}$ *is* $[gH-p]-$*differentiable at* (x_0, t_0) *with respect to* x *without any switching point on* $\mathbb{D}$ *and*

- *if the type of* $[gH-p]-$*differentiability of both* $u(x,t)$ *and* $\frac{\partial_{gH} u}{\partial x}$ *are the same, then* $\frac{\partial_{gH} u}{\partial x}$ *is* $[(i)-p]$-*differentiable w.r.t* x *and*

$$\frac{\partial^2_{ii.gH} u(x_0,t_0)}{\partial x^2} = \Big(\frac{\partial^2 u_1(x_0,t_0)}{\partial x^2}, \frac{\partial^2 u_2(x_0,t_0)}{\partial x^2}, \frac{\partial^2 u_3(x_0,t_0)}{\partial x^2}\Big),$$

- *if the type of [gH-p]-differentiability* $u(x,t)$ *and* $f_{x_{gH}}(x,t)$ *are different, therefore* $\frac{\partial_{gH} u}{\partial x}$ *is* $[(ii)-p]-$*differentiable w.r.t* x *and*

$$\frac{\partial^2_{ii.gH} u(x_0,t_0)}{\partial x^2} = \Big(\frac{\partial^2 u_1(x_0,t_0)}{\partial x^2}, \frac{\partial^2 u_2(x_0,t_0)}{\partial x^2}, \frac{\partial^2 u_3(x_0,t_0)}{\partial x^2}\Big).$$

Definition 2.3 *(See [7]) Let* $f:(a,b)\to\mathbb{R}_T$ *is a triangular fuzzy-valued function and* $t_0\in(a,b)$ *then*

$$\int_a^b f(t)t = \Big(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \int_a^b f_3(t)dt\Big).$$

Theorem 2.1 *([9]) If f is gH-differentiable with no switching point in the interval* $[a,b]$*, then we have*

$$\int_a^b f'_{gH}(t)dt \;=\; f(b)\ominus_{gH} f(a).$$

Lemma 2.1 *If* $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ *be a triangular fuzzy function with no switching point, then we have*

1. *If* $f(t)$ *is* $[i-gH]-$*differentiable , then*

$$\int_a^b f'_{i.gH}(t)dt = f(b)\ominus f(a).$$

2. *If* $f(t)$ *is* $[ii-gH]-$*differentiable , then*

$$\int_a^b f'_{ii.gH}(t)dt = (-1)f(a)\ominus(-1)f(b).$$

Proof. Suppose that $f(t)$ is $[ii-gH]-$differentiable, in this case by Definition 2.3

$$\int_a^b f'_{ii.gH}(t)dt = \Big(\int_a^b f'_3(t)dt, \int_a^b f'_2(t)dt, \int_a^b f'_1(t)dt\Big) = \Big(f_3(b)-f_3(a), f_2(b)-f_2(a), f_1(b)-f_1(a)\Big) = (-1)f(a)\ominus(-1)f(b).$$

The other case is proved in a similar way. ■

Proposition 2.1 *Let* λ_1 *and* λ_2 *are two real constants such that* $\lambda_1,\lambda_2\geq 0$ *(or* $\lambda_1,\lambda_2\leq 0$ *). If* $f(t)$ *is a triangular fuzzy function, then*

$$\lambda_1 f(t)\ominus_{gH}\lambda_2 f(t) = (\lambda_1-\lambda_2)f(t). \qquad (2.1)$$

Proof. First consider λ_1 and λ_2 are positive constants, then

$$\lambda_1 f(t) \;=\; \Big(\lambda_1 f_1(t), \lambda_1 f_1(t), \lambda_1 f_1(t)\Big),$$

$$\lambda_2 f(t) \;=\; \Big(\lambda_2 f_1(t), \lambda_2 f_1(t), \lambda_2 f_1(t)\Big).$$

Now , we have two cases

i. If $\lambda_1\geq\lambda_2$, we have

$$\lambda_1 f(t)\ominus_{gH}\lambda_2 f(t) = \Big((\lambda_1-\lambda_2)f_1(x), (\lambda_1-\lambda_2)f_2(x), (\lambda_1-\lambda_2)f_3(x)\Big) = (\lambda_1-\lambda_2)f(x),$$

ii. If $\lambda_1\leq\lambda_2$, therefore

$$\lambda_1 f(t)\ominus_{gH}\lambda_2 f(t) = \Big((\lambda_1-\lambda_2)f_3(x), (\lambda_1-\lambda_2)f_2(x), (\lambda_1-\lambda_2)f_1(x)\Big) = (\lambda_1-\lambda_2)f(x).$$

Hence we have Eq. (2.1). The case where λ_1 and λ_2 are negative constants , is similar and we omit the details. ■

Lemma 2.2 *Consider $g : [a,b] \to \mathbb{I} \subseteq \mathbb{R}$ is real and differentiable function at t , and $f : \mathbb{I} \to \mathbb{R}_{\mathcal{F}}$ is gH-differentiable at the point $g(t)$ without any switching points. Then type of $gH-$differentiability for $f(t)$ and $f(g(t))$ is the same if*

$$\Big(f(g(x))\Big)'_{gH} = \begin{cases} g'(x) \odot f'_{gH}(g(x)), \\ \quad \text{If } g(t) \text{ is an increasing function,} \\ \ominus(-1)g'(x) \odot f'_{gH}(g(x)), \\ \quad \text{If } g(t) \text{ is a decreasing function.} \end{cases}$$

Lemma 2.3 *(See [14]) $\int_b^a u(x,t)\ dx = \ominus \int_a^b u(x,t)\ dx$; where $\ominus$ denote Hukuhara deference and $u(x,t)$ be a fuzzy valued function.*

Theorem 2.2 *(See [14]) Let $t \in I \subseteq \mathbb{R}$ and $f : I \Rightarrow \mathbb{R}$ and $g : I \Rightarrow \mathbb{R}$. Suppose that $g(t)$ is real continuous function and fuzzy function $f(t)$ is a fuzzy function gH-differentiable at t. Then*

$$(f \odot g)'_{gH}(t) = f'_{gH}(t) \odot g(t) \oplus f(t) \odot g'(t).$$

Theorem 2.3 *(See [14]) (The Chain rule)Let $Z := F(\xi(t), \eta(t))$ is a fuzzy valued function, where $\xi(t)$ and $\eta(t)$ are differentiable real valued functions of t. Then, F is gH-differentiable function of t and we have:*

$$\frac{\partial Z}{\partial t} = \frac{d_{gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t} \oplus \frac{d_{gH}F}{d\eta} \odot \frac{\partial \eta}{\partial t}.$$

Theorem 2.4 *Let $Z(x,t) = F(\xi)$ is a fuzzy valued function, where $\xi(x,t)$ is differentiable real valued function of x and t. Then, F is gH-differentiable function of ξ and*

$$\frac{\partial Z}{\partial t} = \frac{d_{gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t}$$

and

1. *If $\frac{\partial \xi}{\partial t} > 0$ and*

- **i.** *$F(\xi)$ is $[(i) - gH]-$differentiable then $Z(x,t)$ is $[(i) - p]-$differentiable w.r.t. t.*
- **ii.** *$F(\xi)$ is $[(ii) - gH]-$differentiable then $Z(x,t)$ is $[(ii) - p]-$differentiable w.r.t. t.*

2. *If $\frac{\partial \xi}{\partial t} < 0$ and*

- **i.** *$F(\xi)$ is $[(i) - gH]-$differentiable then $Z(x,t)$ is $[(ii) - p]-$differentiable w.r.t. t.*
- **ii.** *$F(\xi)$ is $[(ii) - gH]-$differentiable then $Z(x,t)$ is $[(i) - p]-$differentiable w.r.t. t.*

Proof. For $Z(x,t) := F(\xi(x,t))$, by the Chain rule 2.3 we have

$$\frac{\partial Z}{\partial t} = \frac{d_{gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t}.$$

Now, let $\frac{\partial \xi}{\partial t} > 0$ and Consider $F(\xi)$ is $[(i) - gH]$-differentiable for all $\alpha \in [0,1]$ for every $t \in J$, it follows that

$$\frac{d_{i.gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t} = \Big(\frac{dF_1(\xi)}{d\xi}, \frac{dF_2(\xi)}{d\xi}, \frac{dF_3(\xi)}{d\xi}\Big) \odot \frac{\partial \xi}{\partial t} = \Big(\frac{dF_1(\xi)}{d\xi}\frac{\partial \xi}{\partial t}, \frac{dF_2(\xi)}{d\xi}\frac{\partial \xi}{\partial t}, \frac{dF_3(\xi)}{d\xi}\frac{\partial \xi}{\partial t}\Big) = \frac{\partial_{i.gH}Z}{\partial t}.$$

Now, if $F(\xi)$ is $[(ii) - gH]$-differentiable for all $\alpha \in [0,1]$ for every $t \in J$ we obtain

$$\frac{d_{ii.gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t} = \Big(\frac{dF_3(\xi)}{d\xi}, \frac{dF_2(\xi)}{d\xi}, \frac{dF_1(\xi)}{d\xi}\Big) \odot \frac{\partial \xi}{\partial t} = \Big(\frac{dF_3(\xi)}{d\xi}\frac{\partial \xi}{\partial t}, \frac{dF_2(\xi)}{d\xi}\frac{\partial \xi}{\partial t}, \frac{dF_1(\xi)}{d\xi}\frac{\partial \xi}{\partial t}\Big) = \frac{\partial_{ii.gH}Z}{\partial t}.$$

This concludes the proof for these cases. The other cases prove in a similar manner. ■

Theorem 2.5 *Let $Z(x,t) := F(\xi(x,t))$ and $Z(x,t)$ is a $[gH - p]-$differentiable function such that the second order generalized partial Hukuhara derivatives w.r.t t and x exist, then*

$$\frac{\partial^2 Z}{\partial t^2} = \frac{d_{gH}F}{d\xi} \odot \frac{\partial^2 \xi}{\partial t^2} \oplus \frac{d^2_{gH}U}{d\xi^2} \odot \Big(\frac{\partial \xi}{\partial t}\Big)^2,$$
$$\frac{\partial^2 Z}{\partial x^2} = \frac{d_{gH}F}{d\xi} \odot \frac{\partial^2 \xi}{\partial x^2} \oplus \frac{d^2_{i.gH}F}{d\xi^2} \odot \Big(\frac{\partial \xi}{\partial x}\Big)^2.$$

Proof. For $Z(x,t) := F(\xi(x,t))$, by the Chain rule 2.3 we have

$$\frac{\partial Z}{\partial t} = \frac{d_{gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t}. \tag{2.2}$$

Then by Theorem 2.2 and (2.2), the desired result is obtained

$$\frac{\partial^2 Z}{\partial t^2} = \frac{d_{gH}F}{d\xi} \odot \frac{\partial^2 \xi}{\partial t^2} \oplus \frac{d^2_{gH}U}{d\xi^2} \odot \Big(\frac{\partial \xi}{\partial t}\Big)^2.$$

Using the same method, the equation $\frac{\partial^2 Z}{\partial x^2}$ can also be proved. ■

3 The Traveling Wave Fuzzy Solution

Consider that we have a linear fuzzy partial differential equation in the following form

$$P\Big(u, \frac{\partial_{gH}u}{\partial t}, \frac{\partial_{gH}u}{\partial x}, \frac{\partial^2_{gH}u}{\partial t^2}, \frac{\partial^2_{gH}u}{\partial x^2}, \ldots\Big) = 0, \tag{3.3}$$

where $u = u(x,t)$ is an unknown fuzzy function, P is a polynomial in $u = u(x,t)$ and its generalized Hukuhara derivatives. Let us now give the main step for solving equation (3.3) using the traveling wave method

Step 1. To find a traveling wave solution for equation (3.3), consider

$$u(x,t) = U(\xi), \quad \xi(x,t) = x - ct, \tag{3.4}$$

where $c \in \mathbb{R}^+$ is arbitrary constant generally termed the wave velocity. In this paper we consider $c > 0$, it means the profile $U(x-ct)$ at a later time t is moving to the positive x direction by a amount ct with speed c.

Step 2. The traveling wave variable $U(\xi)$, permit us reducing equation (3.3) to the following fuzzy ordinary differential equation of ξ

$$P\Big(U, (-1)c \odot \frac{d_{i.gH}U}{d\xi}, \frac{d_{i.gH}U}{d\xi}, \frac{d^2_{i.gH}U}{d\xi^2}, c^2\frac{d^2_{i.gH}U}{d\xi^2}, \ldots\Big), \tag{3.5}$$

and

$$P\Big(U, (-1)c \odot \frac{d_{ii.gH}U}{d\xi}, \frac{d_{ii.gH}U}{d\xi}, \frac{d^2_{ii.gH}U}{d\xi^2}, c^2\frac{d^2_{ii.gH}U}{d\xi^2}, \ldots\Big). \tag{3.6}$$

Since, by using Theorems 2.4, 2.5 and

$$\frac{\partial \xi}{\partial t} = -c, \quad \frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2} = 0, \tag{3.7}$$

and by considering the type of $gH-$differentiability for U, the following cases are obtained

Case i. Let $U(\xi)$ is a $[(i)-gH]-$differentiable fuzzy function, then

- $u(x,t)$ is $[(ii)-p]-$differentiable with respect to t and
$$\frac{\partial_{ii.gH}u}{\partial t} = \frac{d_{i.gH}U}{d\xi} \odot \frac{\partial \xi}{\partial t} = (-1)c \odot \frac{d_{i.gH}U}{d\xi}.$$
- $u(x,t)$ is $[(i)-p]-$differentiable with respect to x and
$$\frac{\partial_{i.gH}u}{\partial x} = \frac{d_{i.gH}U}{d\xi} \odot \frac{\partial \xi}{\partial x} = \frac{d_{i.gH}U}{d\xi}.$$
- $\frac{\partial_{gH}u}{\partial t}$ is $[(i)-p]-$differentiable with

respect to t and

$$\frac{\partial^2_{i.gH}u}{\partial t^2} = \frac{d_{i.gH}U}{d\xi} \odot \frac{\partial^2 \xi}{\partial t^2} \oplus \frac{d^2_{i.gH}U}{d\xi^2} \odot \left(\frac{\partial \xi}{\partial t}\right)^2 = c^2 \odot \frac{d^2_{i.gH}U}{d\xi^2}.$$

- $\frac{\partial_{gH}u}{\partial x}$ is $[(i)-p]-$differentiable with respect to x and

$$\frac{\partial^2_{i.gH}u}{\partial x^2} = \frac{d_{i.gH}U}{d\xi} \odot \frac{\partial^2 \xi}{\partial x^2} \oplus \frac{d^2_{i.gH}U}{d\xi^2} \odot \left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{d^2_{i.gH}U}{d\xi^2}.$$

Case ii. Consider $U(\xi)$ is a $[(ii)-gH]-$differentiable fuzzy function, hence we have

- $u(x,t)$ is $[(i)-p]-$differentiable with respect to t and

$$\frac{\partial_{i.gH}u}{\partial t} = \frac{d_{ii.gH}U}{d\xi} \odot \frac{\partial \xi}{\partial t} = (-1)c \odot \frac{d_{ii.gH}U}{d\xi}.$$

- $u(x,t)$ is $[(ii)-p]-$differentiable with respect to x and

$$\frac{\partial_{ii.gH}u}{\partial x} = \frac{d_{ii.gH}U}{d\xi} \odot \frac{\partial \xi}{\partial x} = \frac{d_{ii.gH}U}{d\xi}.$$

- $\frac{\partial_{gH}u}{\partial t}$ is $[(ii)-p]-$differentiable with respect to t and

$$\frac{\partial^2_{ii.gH}u}{\partial t^2} = \frac{d_{ii.gH}U}{d\xi} \odot \frac{\partial^2 \xi}{\partial t^2} \oplus \frac{d^2_{ii.gH}U}{d\xi^2} \odot \left(\frac{\partial \xi}{\partial t}\right)^2 = c^2 \odot \frac{d^2_{ii.gH}U}{d\xi^2}.$$

- $\frac{\partial_{gH}u}{\partial x}$ is $[(ii)-p]-$differentiable with respect to x and

$$\frac{\partial^2_{ii.gH}u}{\partial x^2} = \frac{d_{ii.gH}U}{d\xi} \odot \frac{\partial^2 \xi}{\partial x^2} \oplus \frac{d^2_{ii.gH}U}{d\xi^2} \odot \left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{d^2_{ii.gH}U}{d\xi^2}.$$

Step 3. To find fuzzy solutions for equations (3.5) and (3.6), we need some initial conditions and some auxiliary conditions. In this paper we consider the following auxiliary boundary conditions

$$\lim_{\xi\to\pm\infty} U(\xi) = 0, \quad \lim_{\xi\to\pm\infty} \frac{dU}{d\xi} = 0, \quad \lim_{\xi\to\pm\infty} \frac{d^2U}{d\xi^2} = 0. \tag{3.8}$$

4 Application

This section examines the traveling wave solutions for two important fuzzy linear partial differential equations, the fuzzy linear transport equation and the fuzzy wave equation, which are very important in the mathematical physics and have been paid attention by many researchers.

4.1 The Fuzzy Linear Transport Equation

Consider u is a quantity to be transported and the positive constant a is the velocity. Consider the following fuzzy transport equation

$$\frac{\partial u}{\partial t} = (-1)a\frac{\partial u}{\partial x},$$
$$(x,t) \in \mathbb{R} \times (0,\infty), \qquad (4.9)$$

with fuzzy initial condition

$$u(x,0) = f(x). \qquad (4.10)$$

To obtain a traveling fuzzy solution for equation (4.9), consider

$$u(x,t) = U(\xi), \quad \xi = x - ct.$$

First, we consider $U(\xi)$ is $[(i)-gH]-$differentiable then

$$(-1)c\frac{d_{i.gH}U}{d\xi} = -a\frac{d_{i.gH}U}{d\xi} \quad \Rightarrow$$
$$c\frac{d_{i.gH}U}{d\xi} \ominus_{gH} a\frac{d_{i.gH}U}{d\xi} = 0.$$

Two constants a and c are positive, therefore Proposition 2.1 implies

$$(c-a)\frac{d_{i.gH}U}{d\xi} = 0.$$

For non-constant U, we have $\frac{d_{i.gH}U}{d\xi} \neq 0$ which implies that $c = a$. So any function $U(x-at)$ with sufficiently smooth U which satisfies in the initial fuzzy value (4.10) and $[(i) - gH]$-differentiable , is a traveling wave solution. In fact, the traveling fuzzy solution of equation (4.9) is $u(x,t) = f(x-at)$, such that $u(x,t)$ is $[(ii) - p]-$differentiable with respect to t and $[(i) - p]-$differentiable with respect to x.

Similarly , consider $U(\xi)$ is $[(ii) - gH]-$differentiable, then in this case, the initial fuzzy value $f(x)$ has to be $[(ii) - gH]-$differentiable at $\xi = x - ct$ and we obtain the traveling wave solution $u(x,t) = f(x-at)$ which is $[(i) - p]-$differentiable with respect to t and $[(ii) - p]-$differentiable with respect to x.

Example 4.1 *Consider the following transport equation*

$$\frac{\partial u}{\partial t} = \frac{-\partial u}{\partial x}, \qquad (4.11)$$

with the fuzzy initial condition

$$u(x,0) = (1.2, 3.5, 6)e^{-x^2}. \qquad (4.12)$$

Then according to the method described above, equation (4.11) has the following traveling wave solution

$$u(x,t) = (1.2, 3.5, 6)e^{-(x-t)^2}.$$

4.2 The Fuzzy Wave Equation

We want to find traveling wave fuzzy solution of the fuzzy Cauchy one-dimensional homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} \ominus_{gH} a^2 \odot \frac{\partial^2 u}{\partial x^2} = 0,$$
$$(x,t) \in \mathbb{R} \times (0,\infty). \qquad (4.13)$$

Assuming that $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ are $[(i) - p]-$differentiable with respect to t and x, respectively. Now, set $u(x,t) = U(\xi)$, where $\xi = x - ct$. We obtain

$$c^2\frac{d^2_{i.gH}U}{d\xi^2} \ominus_{gH} a^2\frac{d^2_{i.gH}U}{d\xi^2} = 0,$$

then by Proposition 2.1

$$(c^2 - a^2)\frac{d^2U}{d\xi^2} = 0. \qquad (4.14)$$

One possibility is for $\frac{d^2_{i.gH}U}{d\xi^2} = 0$ in which case we have

$$U(\xi) = A \oplus B\xi \Rightarrow u(x,t) = A + B(x - ct),$$

where A and B are fuzzy integral constants. But the boundary conditions (3.8) cannot be satisfied unless $B = 0$. Thus the only traveling solution in this case is a fuzzy constant. Another possibility is for $c^2 = a^2$. In this case

$$u(x,t) = U(x - at),$$
$$u(x,t) = U(x + at), \qquad (4.15)$$

are traveling wave solution of the wave equation and U can be any two gH-differentiable function. In general, it follows that any solution to the fuzzy wave equation can be obtained as a superposition of two traveling waves,

$$u(x,t) = F(x + at) \oplus G(x - at). \qquad (4.16)$$

Now we would like to satisfy the initial conditions

$$u(x,0) = f(x), \quad u_{t_{gH}}(x,0) = g(x). \qquad (4.17)$$

Since equation (4.16) is a fuzzy solution for equation (4.13), then it must apply to the initial conditions of the equation (4.17), hence the initial condition $u(x,0) = f(x)$ concludes

$$F(x) \oplus G(x) = f(x). \quad (4.18)$$

Differentiating (4.16) with respect to t yields

$$\frac{\partial u(x,t)}{\partial t} = a \odot F'_{gH}(x+at) \ominus a \odot G'_{gH}(x-at).$$

In the following, all fuzzy solutions of the fuzzy wave equation in different type of $[gH-p]-$differentiability will be examined.

Case 1. If $u(x,t)$ is $[(i)-p]-$differentiable with respect to t, then F and G are $[(i)-gH]$-differentiable with respect to $(x+at)$ and $(x-at)$, respectively. Thus by Lemma 2.2 we have

$$\frac{\partial u(x,t)}{\partial t} = a \odot F'_{i.gH}(x+at) \ominus a \odot G'_{i.gH}(x-at),$$

so that at $t=0$ by initial condition, we obtain

$$aF'_{i.gH}(x) \ominus aG'_{i.gH}(x) = g(x).$$

Dividing this last equation by a and after integration using Lemma 2.1

$$\Big(F(x) \ominus F(0)\Big) \ominus \Big(G(x) \ominus G(0)\Big) = \frac{1}{a}\int_0^x g(s)ds,$$

$$\Rightarrow F(x) \ominus G(x) = \Big(F(0) \ominus G(0)\Big) \oplus \frac{1}{a}\int_0^x g(s)ds. \quad (4.19)$$

By Eqs.(4.18) and (4.19) we obtain the following system of equations

$$\begin{cases} F(x) \oplus G(x) = f(x), \\ F(x) \ominus G(x) = \Big(F(0) \ominus G(0)\Big) \oplus \\ \frac{1}{a}\int_0^x g(s)ds. \end{cases}$$

The solution of this system of equations is given by

$$F(x) = \frac{1}{2}f(x) \oplus \frac{1}{2}\Big(F(0) \ominus G(0)\Big) \oplus \frac{1}{2a}\int_0^x g(s)ds,$$

$$G(x) = \frac{1}{2}f(x) \ominus \frac{1}{2}\Big(F(0) \ominus G(0)\Big) \ominus \frac{1}{2a}\int_0^x g(s)ds.$$

But according to Lemma 2.3 we can write

$$G(x) = \frac{1}{2}f(x) \ominus \frac{1}{2}\Big(F(0) \ominus G(0)\Big) \oplus \frac{1}{2a}\int_x^0 g(s)ds.$$

By substituting these equations for F and G into the general solution (4.16) we observe that

$$u(x,t) = \frac{1}{2}\Big(f(x+at) \oplus f(x-at)\Big) \oplus \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds. \quad (4.20)$$

Case 2. If $u(x,t)$ is $[(ii)-p]-$differentiable with respect to t, then F and G are $[(ii)-gH]$-differentiable with respect to $(x+at)$ and $(x-at)$, respectively. Thus we get

$$u_{t_{ii.gH}}(x,t) = a \odot F'_{ii.gH}(x+at) \ominus a \odot G'_{ii.gH}(x-at),$$

by initial value at $t=0$ we conclude that

$$aF'_{ii.gH}(x) \ominus aG'_{ii.gH}(x) = g(x).$$

Using Lemma 2.1 we have

$$\Big((-1)F(0)\ominus(-1)F(x)\Big)\ominus \Big((-1)G(0)\ominus(-1)G(x)\Big) = \frac{1}{a}\int_0^x g(s)ds,$$

$$\Rightarrow \quad G(x)\ominus F(x) = \Big(G(0)\ominus F(0)\Big)\oplus \frac{(-1)}{a}\int_0^x g(s)ds.$$

Consequently, we find that

$$\begin{cases} F(x)\oplus G(x) = f(x), \\ G(x)\ominus F(x) = \\ \Big(G(0)\ominus F(0)\Big)\oplus \frac{(-1)}{a}\int_0^x g(s)ds. \end{cases}$$

Solving this system get the following solution

$$G(x) = \frac{1}{2}f(x)\oplus\frac{1}{2}\Big(G(0)\ominus F(0)\Big)\oplus \frac{(-1)}{2a}\int_0^x g(s)ds,$$
$$F(x) = \frac{1}{2}f(x)\ominus\frac{1}{2}\Big(G(0)\ominus F(0)\Big)\ominus \frac{(-1)}{2a}\int_0^x g(s)ds.$$

On the other hand, we can write

$$G(x) = \frac{1}{2}f(x)\oplus\frac{1}{2}\Big(G(0)\ominus F(0)\Big)\ominus \frac{(-1)}{2a}\int_x^0 g(s)ds.$$

Then the general solution (4.16), we have that

$$u(x,t) = \frac{1}{2}\Big(f(x+at)\oplus f(x-at)\Big)\ominus \frac{(-1)}{2a}\int_{x-at}^{x+at} g(s)ds. \quad (4.21)$$

Theorem 4.1 *The homogeneous Wave equation (4.17) in the domain $-\infty < x < \infty$, $0\le t\le T$ is well-posed for $f\in\mathcal{C}^2(\mathbb{R},\mathbb{R}_{\mathcal{F}})$, $g\in\mathcal{C}^1(\mathbb{R},\mathbb{R}_{\mathcal{F}})$.*

Proof. The d'Alembert's formula shows the existence and the uniqueness of the solution of (4.17). Actually, it was shown that any solution of the homogeneous fuzzy wave equation, by attention the type of $[gH-p]-$differentiability, will be obtained by using the d'Alembert formula. Now, it sufficient that the problem (4.17) is stable. Let u_1 and u_2 be two fuzzy solution of (4.17) with fuzzy initial condition given by $f_i(x)$ and $g_i(x)$, where $i=1,2$. Moreover for all $x\in\mathbb{R}$

$$D\Big(f_1(x),f_2(x)\Big)<\delta, \quad D\Big(g_1(x),g_2(x)\Big)<\delta. \quad (4.22)$$

First, suppose that $u_1(x,t)$ and $u_2(x,t)$ are $[(i)-p]-$differentiable with respect to t, hence for $(x,t)\in\mathbb{R}\times[0,T]$ and by using properties of Hausdorff distance D

$$\begin{aligned} D\Big(u_1(x,t),u_2(x,t)\Big) \le{}& \frac{1}{2}D\Big(f_1(x+ct),f_2(x+ct)\Big)+ \frac{1}{2}D\Big(f_1(x-ct),f_2(x-ct)\Big)+ \\ & \frac{1}{2c}\int_{x-ct}^{x+ct} D\Big(g_1(s),g_2(s)\Big)ds \le \frac{1}{2}(\delta+\delta)+\frac{1}{2c}(2ct)\delta = \\ & (1+t)\delta \le (1+T)\delta, \end{aligned}$$

then, for a given $\varepsilon>0$, we consider $\delta<\frac{\varepsilon}{(1+T)}$, therefore

$$D\Big(u_1(x,t),u_2(x,t)\Big) \le (1+T)\delta<\varepsilon. \quad (4.23)$$

With the same procedure, we can be prove that if $u_1(x,t)$ and $u_2(x,t)$ are $[(ii)-p]-$differentiable with respect to t, the same result is still valid. ■

Example 4.2 *We want to find a $[(i)-gH]-$differentiable solution for the following fuzzy Wave equation*

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2}\ominus_{gH}\frac{\partial^2 u(x,t)}{\partial x^2}=0, \\ u(x,0)=(2.7,5,9.8)x^2, \\ \frac{\partial u(x,0)}{\partial t}=0. \end{cases}$$

So using equation (4.20) we have

$$u(x,t) = \frac{1}{2}\Big(f(x+at) \oplus f(x-at)\Big) \oplus \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds =$$

$$\frac{1}{2}\Big((2.7,5,9.8)(x+t)^2 \oplus (2.7,5,9.8)(x-t)^2\Big) = (2.7,5,9.8)(x^2+t^2).$$

Example 4.3 *Consider the following wave equation*

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} \ominus_{gH} 4\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \\ u(x,0) = (1.1,3,6)e^x, \\ \frac{\partial u(x,0)}{\partial t} = (-12,-6,-2.2)e^x. \end{cases}$$

We want to find a $[(ii)-gH]$*−differentiable solution for this problem. By equation (4.21) we have*

$$u(x,t) = \frac{1}{2}\Big(f(x+at) \oplus f(x-at)\Big) \ominus \frac{(-1)}{2a}\int_{x-at}^{x+at} g(s)ds$$

$$= \frac{1}{2}\Big((1.1,3,6)e^{x+2t} \oplus (1.1,3,6)e^{x-2t}\Big) \ominus \frac{1}{4}\int_{x-2t}^{x+2t} (2.2,6,12)e^s ds$$

$$= (1.1,3,6)e^{x-2t}.$$

5 Conclusion

In this paper, we obtain the fuzzy traveling wave solution of the partial differential equation by considering the type of gH-differentiability. To demonstrate the efficiency of the method, the fuzzy traveling wave solutions of the fuzzy transport equation and fuzzy Wave equation are obtained. All results show that this method is a very powerful and efficient method for obtaining an analytical solution for the fuzzy linear partial differential equation.

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Razieh Ghasemi Moghaddam received her Ph.D. degree in Applied Mathematics in Numerical Analysis area from IAU- Science and Research Branch, Tehran, Iran, in 2018. Her research interests include numerical and analytical solutions of fuzzy differential and integral equations, inverse problem methods, fuzzy decision making, and risk analysis methods.

Saeid Abbasbandy is Professor of applied mathematics and Faculty of Mathematical sciences, Imam Khomeini international University, Qazvin, Iran. Interested in numerical solution of functional equations and fuzzy numerical analysis.

Mohsen Rostamy- Malkhalifeh is an Associate professor at the Applied Mathematics Operations Research group in Islamic Azad University in Science and Research Branch in Tehran, Iran. His research interests include performance management, efficiency, Fuzzy and especially those based on the broad set of methods known as Data Envelopment Analysis (DEA). Dr. Rostamy- Malkhalifeh research studies are published in such journals as Applied Mathematics and Computation, International Journal of Fuzzy Systems, International Journal of Applied Decision Sciences, European Journal of Operational Research, Mathematics Scientific Journal, The Scientific World Journal and etc.