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Research Article

Hadamard Well-posedness for a Family of Mixed Variational Inequalities and Inclusion Problems

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Abstract

In this paper, the concepts of well-posednesses and Hadamard well-posedness for a family of mixed variational inequalities are studied. Some metric characterizations of well-posednesses are presented. Then, one relation between well-posedness and Hadamard well-posedness of the family of mixed variational inequalities is studied. Finally, a relation between well-posedness for the family of mixed variational inequalities and well-posedness for the family of inclusion problems is discussed.

Keywords : Mixed variational inequality; Monotonicity; Approximating sequence; Parametric well-posedness.

1 Introduction

LEt X be a topological space, H be a real Hilbert space and K be a nonempty convex closed subset of H. We suppose in what follows that $F(x,.)$ is a mapping from H to H, for any $x \in X$ and $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functional. Denote by $dom\phi$ the domain of ϕ, i.e.,

$$dom\phi := \{u \in H : \phi(u) < +\infty\}.$$

Now, let $x \in X$ and we consider the following parametric mixed variational inequality associated with (F, ϕ, K) :

$MVI(F, \phi, K)_x$: Find $u \in K$ such that for all $\nu \in K$

$$\langle F(x, u), u - \nu\rangle + \phi(u) - \phi(\nu) \leq 0.$$

When $\phi = \delta_K$, $MVI(F, \phi, K)$ reduces to the classical variational inequality $VI(F, K)$, where δ_K denotes the indicator function of K. Denote by $S_x(F, \phi, K)$ the solutions set of $MVI(F, \phi, K)_x$. When the operator F does not depend on the parameter x, $MVI(F, \phi, K)_x$ reduces to $MVI(F, \phi, K)$ which has been studied intensively (see, e.g. [1, 2, 4, 5, 7, 8, 11, 12, 21, 22, 24, 25, 29]). Recently, Chen et al. [3] proposed a general inertial proximal point algorithm for the mixed variational inequality problems and under certain assumptions, they established the global convergence and nonasymptotic convergence rate result of the proposed general inertial proximal point algorithm.

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There exist two main types of well-posedness for optimization problems: Tykhonov well-posedness [26] and Hadamard well-posedness [18]. In Hadamard types of well-posedness, continuous dependence of the solution on the data of such a problem is important. While, Tykhonov types of well-posedness such as Tykhonov well-posedness and Levitin-Polyak well-posedness deal the behavior of a class of approximating solution sequences. This concept of well-posedness is inspired by the numerical methods producing minimizing sequence and it is important in optimization problems. Lignola and Morgan [17] introduced and studied various notions of well-posedness for a family of variational inequalities and for an optimization problem with constraints defined by variational inequalities. Then, they obtained sufficient conditions for well-posedness of these problems and gave an application. There are various surveys focused with directly on the types of Tykhonov well-posedness and Hadamard well-posedness(see, e.g. [6, 7, 9, 10, 13, 14, 19, 20, 23, 27, 28]). Some papers are devoted to relations between Hadamard well-posedness and Tykhonov well-posedness for different problems. Yang and Yu [28] presented unified approaches to Hadamard and Tykhonov well-posedness and as applications, they deduce Tykhonov well-posedness for optimization problems, Nash equilibrium point problems and fixed point problems. Afterward, Li and Xia [15] introduced the concept of Hadamard well-posedness for a generalized mixed variational inequality problem in Banach spaces and studied relations between Levitin-Polyak well-posedness and Hadamard well-posedness for this problem.
Motivated by the work of Li and Xia [15] and Lignola and Morgan [17], we extend some notions of Tykhonov well-posedness and Hahamard well-posedness for a family of mixed variational inequalities. The paper is organized as follows: Section 2 contains some useful definitions and preliminary results. Also, some metric characterizations of mixed variational inequalities are presented and some classes of well-posed mixed variational inequalities are obtained. In section 3, the concept of Hadamard well-posedness for the family of mixed variational inequalities is defined. This section aims to show that parametric Hadamard well-posedness implies parametric Tykhonov well-posedness for the family of mixed variational inequalities. Finally, under suitable conditions, some classes of Hadamard well-posed mixed variational inequalities are obtained. In Section 4, it is shown that the parametric weak well-posedness for the family of mixed variational inequalities implies the parametric weak well-posedness for the family of inclusion problems.

2 Parametric well-posedness

Let $x \in X$ be an arbitrary element. In this section, we consider the parametric mixed variational inequality $MVI(F, \phi, K)_x$, that is to find $u \in K$ such that

$$\langle F(x,u), u-\nu\rangle + \phi(u) - \phi(\nu) \leq 0, \text{ for all } \nu \in K$$

and we assume that, $MVI(F, \phi, K)_x$ has a unique solution. Denote by $\partial\phi$ the convex subdifferential of ϕ, i.e.,

$$\partial\phi(u) = \{u^* \in H : \phi(\nu) - \phi(u) \geq \langle u^*, \nu - u\rangle, \text{ for all } \nu \in H\},$$

for all $u \in dom\phi$.

Definition 2.1 *Let $F : X \times H \to H$ be a mapping. F is said to be*

1. *monotone with respect to the second argument, if for any $x \in X$ and $u, \nu \in H$ one has*
$$\langle F(x,u) - F(x,\nu), u-\nu\rangle \geq 0;$$
2. *strongly monotone with respect to the second argument, if there exists $\alpha > 0$ such that for any $x \in X$ and $u, \nu \in H$ one has*
$$\langle F(x,u) - F(x,\nu), u-\nu\rangle \geq \alpha\|u-\nu\|^2.$$

Definition 2.2 *Let $F(x,.) : H \to H$ be an operator from H to H. $F(x,.)$ is said to be hemicontinuous if, for any $u, \nu \in H$, the function*

$$\lambda \mapsto \langle F(x, u+\lambda\nu), \nu\rangle$$

from $[0,1]$ into $(-\infty, \infty)$ is continuous at 0_+.

Definition 2.3 *Let $x \in X$ and $\{x_n\}$ be a sequence converging to x. A sequence $\{u_n\} \subset H$ is said to be an approximating sequence with respect to $\{x_n\}$ for the parametric mixed variational inequality $MVI(F, \phi, K)_x$, if $u_n \in K$ for any $n \in \mathbb{N}$ and there exists a positive sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that*

$$\langle F(x_n, u_n), u_n - \nu \rangle + \phi(u_n) - \phi(\nu) \leq \epsilon_n,$$

for all $\nu \in K$.

Now, Let us consider the family

$$(MVI(F, \phi, K)) := \{MVI(F, \phi, K)_x : x \in X\}$$

and we present concept of parametric well-posedness for mixed variational inequality $(MVI(F, \phi, K))$.

Definition 2.4 *The family mixed variational inequalities $(MVI(F, \phi, K))$ is said to be parametrically (weak) well-posed if*

1. *there exists a unique solution $\bar{u}_x \in H$ to $MVI(F, \phi, K)_x$, for all $x \in X$,*
2. *for all $x \in X$ and for all $\{x_n\}$ converging to x, every approximating sequence for the problem $MVI(F, \phi, K)_x$ with respect to $\{x_n\}$ strongly(weakly) converges to $\bar{u}_x$.*

Remark 2.1 *It is obvious that, for the family $(MVI(F, \phi, K))$, the parametric well-posedness implies the parametric weak well-posedness, but the converse is not true in general.*

For any $x \in X$ and $\epsilon > 0$, we define the following set:

$$\Omega(x, \epsilon) := \{u \in K : \langle F(x, u), u - \nu \rangle + \phi(u) - \phi(\nu) \leq \epsilon, \text{ for all } \nu \in K\}.$$

Proposition 2.1 *Let the family $(MVI(F, \phi, K))$ be parametrically well-posed. Then $\Omega(x, \epsilon) \neq \emptyset$, for every $x \in X$ and every $\epsilon > 0$, and $\lim_{n\to\infty} diam\Omega(x_n, \epsilon_n) = 0$, for all $\{x_n\}$ converging to x and all $\{\epsilon_n\}$ converging to 0.*

Proof. The proof is similar to Proposition 2.3 in [17], hence it is omitted.

Lignola and Morgan in [16] investigated the connection between the concept of parametric well-posedness for the family (VI) and the diameter of the set $\Omega(x, \epsilon)$ in which continuity properties have been studied. Then, they proved in [17] that paremetric well-posedness for (VI) implies that the diameter of $\Omega(x, \epsilon)$ converges to 0. In fact, they showed that in general paremetric well-posedness is not equivalent to the convergence of the diameter $\Omega(x, \epsilon)$ to 0. In The following Theorem, we give some conditions under which paremetric well-posedness is equivalent to the convergence of the diameter $\Omega(x, \epsilon)$ to 0. This Theorem extends and improves Proposition 2.3 in [17].

Condition A: Let $F : X \times H \to H$. Then for any $u \in H$, we have

$$\langle F(x, u) - F(y, u), \nu - u \rangle \leq \|x - y\| \|\nu - u\|,$$

for all $x, y \in X$ and $\nu \in H$.

Theorem 2.1 *Let $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functional. Suppose that $F(x, .)$ be hemicontinuous and monotone with respect to the second argument and F satisfies Condition A. Then the family $(MVI(F, \phi, K))$ is parametrically well-posed if and only if $\Omega(x, \epsilon) \neq \emptyset$, for any $x \in X$ and $\epsilon > 0$, and $\lim_{n\to\infty} diam\Omega(x_n, \epsilon_n) = 0$, for all $\{x_n\}$ converging to x and $\{\epsilon_n\}$ converging to 0.*

Proof. Obviously, the necessity follows immediately from Proposition 2.1. For sufficiency, let x be an arbitrary element in X and $\{x_n\} \subset X$ be a sequence converging to x. Assume that $\{u_n\}$ is an approximating sequence for $MVI(F, \phi, K)_x$(w.r.t. $\{x_n\}$). Then there exists a positive sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and

$$\langle F(x_n, u_n), u_n - \nu \rangle + \phi(u_n) - \phi(\nu) \leq \epsilon_n,$$

for all $\nu \in K$, which implies that $u_n \in \Omega(x_n, \epsilon_n)$. Since $\lim_{n\to\infty} diam\Omega(x_n, \epsilon_n) = 0$, $\{u_n\}$ is a Cauchy sequence and therefore u_n converges strongly to some point $u_x \in K$. Furthermore, since ϕ is lower semicontinuous, the mapping $F(x, .)$ is monotone with respect to the second argument and F satisfies Condition A, we obtain

$$\begin{aligned}&\langle F(x,\nu),u_x-\nu\rangle+\phi(u_x)-\phi(\nu)\\&\le\liminf\langle F(x,\nu),u_n-\nu\rangle+\phi(u_n)-\phi(\nu)\\&\le\liminf\langle F(x,u_n),u_n-\nu\rangle+\phi(u_n)-\phi(\nu)\\&\le\liminf\langle F(x_n,u_n),u_n-\nu\rangle+\phi(u_n)-\phi(\nu)\\&+\|x_n-x\|\|u_n-\nu\|\\&\le\liminf\epsilon_n+\|x_n-x\|\|u_n-\nu\|=0,\end{aligned}$$

for all $\nu \in K$. For any $u \in K$ and $\lambda \in [0,1]$, letting $\nu = u_x + \lambda(u - u_x)$ in last inequality, we have

$$\langle F(x,u_x+\lambda(u-u_x)),\lambda(u_x-u)\rangle+\phi(u_x)-\phi(u_x+\lambda(u-u_x))\le 0.$$

It follows from convexity of ϕ that

$$\langle F(x,u_x+\lambda(u-u_x)),u_x-u\rangle+\phi(u_x)-\phi(u)\le 0, \tag{2.1}$$

for all $u \in K$. Taking the limit $\lambda \to 0_+$ in (2.1) and using the hemicontinuity of mapping F, we have

$$\langle F(x,u_x),u_x-u\rangle+\phi(u_x)-\phi(u)\le 0,$$

for all $u \in K$. Hence, u_x solves $MVI(F,\phi,K)_x$.

Example 2.1 *Suppose that $X = H = K = \mathbb{R}$. Consider the single-valued function $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $F(x,u) = u - x$ and $\phi : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ defined by $\phi(u) = 0$, for all $u \in \mathbb{R}$. It is easy to check that the operator F is monotone with respect to the second argument, hemicontinuous and satisfies Condition A. Also, it is obvious that the operator ϕ is proper, convex and lower semicontinuous. We observe that $x \in \Omega(x,\epsilon)$, for all $x \in \mathbb{R}$ and $\epsilon > 0$. Therefore, $\Omega(x,\epsilon) \neq \emptyset$, for all $x \in \mathbb{R}$. It is easy to verify that $diam\Omega(x_n,\epsilon_n) \to 0$, for all $\{x_n\}$ converging to x and all $\{\epsilon_n\}$ converging to 0. Hence, all assumptions of Theorem 2.1 are fulfilled and therefore $(MVI(F,\phi,K))$ is parametrically well-posed.*

Remark 2.2 *Let $F(x,.)$ be monotone and hemicontinuous. Then, the following statements are equivalent:*

1. *u_x is a solution to the mixed variational inequality $MVI(F,\phi,K)_x$.*
2. *u_x is a solution to the following associated mixed variational inequality:*
   $AMVI(F,\phi,K)_x$: find $u_x \in K$ such that
   $$\langle F(x,\nu),u_x-\nu\rangle+\phi(u_x)-\phi(\nu)\le 0,$$
   for all $\nu \in K$.

In fact, we can easily prove this claim by using a similar argument of the proof of Theorem 2.1. Therefore, we omit it here.

Now, we obtain a class of mixed variational inequalities which they are parametrically well-posed.

Theorem 2.2 *Let F be a strongly monotone and hemicontinuous with respect to the second argument on a bounded convex closed subset K, $\phi : H \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous functional and $F(.,u)$ be continuous on X for all $u \in K$. Then $(MVI(F,\phi,K))$ is parametrically well-posed.*

Proof. The proof follows in the similar lines of Proposition 2.9 in [17] and hence being omitted.

3 Hadamard well-posedness

Let U be the collection of all mapping $P : X \times H \to K$, and $\Gamma(X,H)$ be the collection of all mapping $F : X \times H \to H$ such that there exist $P \in U$ and $\lambda \in \mathbb{R}$, for all $x \in X$ and $u \in H$

$$\langle F(x,u),u-\nu\rangle=\langle P(x,u),u-\nu\rangle+\lambda,$$

for all $\nu \in K$. For any $F, G \in \Gamma(X,H)$, it follows that

$$\langle F(x,u),u-\nu\rangle=\langle P_1(x,u),u-\nu\rangle+\lambda_1$$

and

$$\langle G(x,u),u-\nu\rangle=\langle P_2(x,u),u-\nu\rangle+\lambda_2,$$

for all $\nu \in K$. By a similar way as that in [15], we define

$$d_1(F,G)=\begin{cases}|\lambda_1-\lambda_2| & \text{if } P_1=P_2,\\ 1+|\lambda_1-\lambda_2| & \text{if } P_1\neq P_2.\end{cases}$$

It can be easily checked that $(\Gamma(X,H),d_1)$ is a metric space.

Let $C(H)$ be the set of all nonempty closed subsets of H endowed with the usual Hausdorff distance $H(.,.)$. We say that K_n converges to K in the Hausdorff metric iff $H(K_n, K) \to 0$ as $n \to \infty$.

Let $B(H)$ be the family of all real-valued functions on H, we define

$$d'(\phi_1, \phi_2) = \sup_{u \in H} |\phi_1(u) - \phi_2(u)|,$$

where $\phi_1, \phi_2 \in B(H)$. It can be easily checked that $(B(H), d')$ is a metric space. Now, we define

$$M := \{(F, \phi, K) : F \in \Gamma(X, H), \phi \in B(H), K \in C(H)\}$$

and for any $(F_1, \phi_1, K_1), (F_2, \phi_2, K_2) \in M$

$$\rho((F_1, \phi_1, K_1), (F_2, \phi_2, K_2)) := d_1(F_1, F_2) + d'(\phi_1, \phi_2) + H(K_1, K_2).$$

Clearly, (M, ρ) is a metric space.

Definition 3.1 *A general family $(MVI(F, \phi, K))$ determined by $(F, \phi, K) \in M$ is called parametrically Hadamard well-posed if*

1. *there exists a unique solution u_x to $MVI(F, \phi, K)_x$, for all $x \in X$;*

2. *for all $x \in X$, all $\{x_n\}$ converging to x, every sequence $\{(F_n, \phi_n, K_n)\} \subset M$ converging to (F, ϕ, K), and every sequence $\{u_n\}$ such that $u_n \in S_{x_n}(F_n, \phi_n, K_n)$, it follows that $u_n \to u_x$.*

Theorem 3.1 *Let K be a nonempty, closed subset of H and $F : X \times H \to K$ be a mapping. Suppose that $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a function and the family $(MVI(F, \phi, K))$ is parametrically Hadamard well-posed. Then $(MVI(F, \phi, K))$ is parametrically well-posed.*

Proof. Let $\bar{x} \in X$ and $u_{\bar{x}}$ be the unique solution of $MVI(F, \phi, K)_{\bar{x}}$. Suppose that $\{x_n\} \subset X$ is a sequence converging to $\bar{x}$ and $\{u_n\}$ is an approximating sequence (w.r. to $\{x_n\}$) for the $MVI(F, \phi, K)_{\bar{x}}$. Then $u_n \in K$ and one can find a sequence $\{\epsilon_n\}$ converging to zero such that

$$\langle F(x_n, u_n), u_n - \nu \rangle + \phi(u_n) - \phi(\nu) \leq \epsilon_n, \quad (3.2)$$

for all $\nu \in K$. Now, for each $n \in \mathbb{N}$, $x \in X$ and $u \in H$, we construct a sequence $\{(F_n, \phi_n, K_n)\} \subset M$ as follows:

$$\langle F_n(x, u), u - \nu \rangle = \langle F(x, u), u - \nu \rangle - \epsilon_n, \quad (3.3)$$

for all $\nu \in K$. Notice that

$$\phi_n(u) = \phi(u) - \epsilon_n \quad (3.4)$$

and $K_n = K$. Therefore, $d_1(F_n, F) = |\epsilon_n| \to 0$, $H(K_n, K) \to 0$ and $d'(\phi_n, \phi) \to 0$. Thus

$$\rho((F_n, \phi_n, K_n), (F, \phi, K)) \to 0.$$

Now, it sufficient to show that $u_n \in S_{x_n}(F_n, \phi_n, K_n)$. It follows from (3.2), (3.3), (3.4) and $K_n = K$ that

$$\langle F_n(x_n, u_n), u_n - \nu \rangle + \phi_n(u_n) - \phi_n(\nu) = \langle F(x_n, u_n), u_n - \nu \rangle + \phi(u_n) - \phi(\nu) - \epsilon_n \leq 0$$

and $u_n \in S_{x_n}(F_n, \phi_n, K_n)$. From parametric Hadamard well-posedness for the family $(MVI(F, \phi, K))$, we can deduce that $u_n \to u_x$ and therefore $(MVI(F, \phi, K))$ is parametrically well-posed.

Now, we investigate a class of families that are parametrically Hadamard well-posed in the finite dimensional case. In the following theorem, we assume that $MVI(F, \phi, K)_x$ has a unique solution, for all $(F, \phi, K) \in M$ and $x \in X$.

Theorem 3.2 *Suppose that K is a nonempty, closed and convex subset of H and $F : X \times H \to K$ is a continuous and monotone mapping with respect to the second argument which satisfies Condition A. Let $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and uniformly continuous. Then $(MVI(F, \phi, K))$ is parametrically Hadamard well-posed.*

Proof. Arguing by contradiction, let us suppose that $(MVI(F, \phi, K))$ is not parametrically Hadamard well-posed. Then there exist $x \in X$, a unique solution u_x for $MVI(F, \phi, K)_x$, a sequence $\{x_n\} \subset X$ converging to x, a sequence $\{(F_n, \phi_n, K_n)\}$ converging to (F, ϕ, K) and $u_n \in S_{x_n}(F_n, \phi_n, K_n)$ which does not converge to u_x. From $u_n \in S_{x_n}(F_n, \phi_n, K_n)$, we have $u_n \in K_n$ and

$$\langle F_n(x_n, u_n), u_n - \nu \rangle + \phi_n(u_n) - \phi_n(\nu) \leq 0,$$

for all $\nu \in K_n$. Since $d_1(F_n, F) \to 0$, we can deduce that there exists $0 < \epsilon_n \to 0$ such that

$$\langle F_n(x,u), u-\nu\rangle = \langle F(x,u), u-\nu\rangle + \epsilon_n, \quad (3.5)$$

for all $x \in X$, $\nu \in K$, and $u \in H$.
Since F satisfies Condition A and (3.5) holds, we can deduce that F_n satisfies Condition A, for all $n \in \mathbb{N}$. It follows from $H(K_n, K) \to 0$ that $d(u_n, K) \to 0$. Thus, there exists $0 < \epsilon_n \to 0$, for n sufficiently large

$$d(u_n, K) \leq \epsilon_n < \epsilon_n + \frac{1}{n}.$$

Hence, there exists $\{w_n\} \subset K$ such that $\|u_n - w_n\| < \epsilon_n + \frac{1}{n} \to 0$.

We claim that $\{u_n\}$ is bounded on K. In fact, if $\{u_n\}$ is unbounded, then $\{w_n\}$ is an unbounded sequence. we can suppose that $\|w_n\| \to +\infty$. Set

$$t_n := \frac{1}{\|w_n - u_x\|}$$

and

$$z_n := u_x + t_n(w_n - u_x).$$

Without loss of generality, we can suppose that $t_n \in (0,1]$. Since $\{z_n\}$ is a bounded sequence on K, there exists $z \in K$ such that $z_n \rightharpoonup z$ and clearly, $z \neq u_x$. Now, for all $\nu \in K$,

$$\begin{aligned}
&\langle F(x,\nu), z-\nu\rangle = \langle F(x,\nu), z-z_n\rangle \\
&+ \langle F(x,\nu), z_n - u_x\rangle + \langle F(x,\nu), u_x-\nu\rangle \\
&= \langle F(x,\nu), z-z_n\rangle + t_n\langle F(x,\nu), w_n-\nu\rangle \\
&+ (1-t_n)\langle F(x,\nu), u_x-\nu\rangle \\
&= \langle F(x,\nu), z-z_n\rangle + t_n\langle F(x,\nu), u_n-\nu\rangle \\
&+ (1-t_n)\langle F(x,\nu), u_x-\nu\rangle \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle.
\end{aligned}$$

On the other hand, u_x is the unique solution to the mixed variational inequality $MVI(F,\phi,K)_x$, F is monotone with respect to the second argument, F and F_n satisfy Condition A and ϕ is convex function. Therefore, from the last equality we have

$$\begin{aligned}
&\langle F(x,\nu), z-\nu\rangle \leq \langle F(x,\nu), z-z_n\rangle \\
&+ t_n\langle F(x,u_n), u_n-\nu\rangle \\
&+ (1-t_n)\langle F(x,\nu), u_x-\nu\rangle \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle \\
&\leq \langle F(x,\nu), z-z_n\rangle + t_n\langle F(x,u_n), u_n-\nu\rangle \\
&+ (1-t_n)\langle F(x,u_x), u_x-\nu\rangle \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle \\
&\leq \langle F(x,\nu), z-z_n\rangle \\
&+ t_n\langle F(x,u_n), u_n-\nu\rangle \\
&+ (1-t_n)(\phi(\nu)-\phi(u_x)) \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle \\
&\leq \langle F(x,\nu), z-z_n\rangle \\
&+ t_n(\langle F_n(x,u_n), u_n-\nu\rangle - \epsilon_n) \\
&+ (1-t_n)(\phi(\nu)-\phi(u_x)) \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle \\
&\leq \langle F(x,\nu), z-z_n\rangle + t_n(\langle F_n(x_n,u_n), u_n-\nu\rangle \\
&+ \|x_n - x\|\|u_n-\nu\| - \epsilon_n) \\
&+ (1-t_n)(\phi(\nu)-\phi(u_x)) \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle. \\
&\leq \langle F(x,\nu), z-z_n\rangle + t_n(\phi_n(\nu)-\phi_n(u_n) \\
&+ \|x_n - x\|\|u_n-\nu\| - \epsilon_n) \\
&+ (1-t_n)(\phi(\nu)-\phi(u_x)) \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle \\
&\leq \langle F(x,\nu), z-z_n\rangle + t_n(\phi_n(\nu)-\phi_n(u_n) \\
&+ \|x_n - x\|\|u_n-\nu\| - \epsilon_n) + (1-t_n)\phi(\nu) \\
&+ t_n\phi(w_n) - \phi(z_n) \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle \\
&\leq \langle F(x,\nu), z-z_n\rangle + \phi(\nu) - \phi(z_n) \\
&+ t_n(\phi_n(\nu)-\phi(\nu)) \\
&- [t_n(\phi_n(u_n)-\phi(u_n)) + t_n(\phi(u_n)-\phi(w_n))] \\
&- t_n\epsilon_n + t_n\|x_n - x\|\|u_n-\nu\| \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle.
\end{aligned}$$

By using the uniform continuity of ϕ and the last inequality, we obtain

$$\begin{aligned}
&\langle F(x,\nu), z-\nu\rangle \leq \liminf\{\langle F(x,\nu), z-z_n\rangle \\
&+ \phi(\nu) - \phi(z_n) + t_n(\phi_n(\nu)-\phi(\nu)) \\
&- [t_n(\phi_n(u_n)-\phi(u_n)) + t_n(\phi(u_n)-\phi(w_n))] \\
&- t_n\epsilon_n + t_n\|x_n - x\|\|u_n-\nu\| \\
&+ t_n\langle F(x,\nu), w_n-u_n\rangle\} = \phi(\nu) - \phi(z),
\end{aligned}$$

for all $\nu \in K$. Now, Remark 2.2 implies that z solves $MVI(F,\phi,K)_x$ which is a contradiction. Hence, $\{u_n\}$ is bounded and there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to \bar{u}$. Since ϕ

is uniformly continuous, F is monotone with respect to the second argument and satisfies Conditio A, we have

$$\begin{aligned}&\langle F(x,\nu),\bar{u}-\nu\rangle+\phi(\bar{u})-\phi(\nu)\\&\leq \liminf\langle F(x,\nu),u_{n_k}-\nu\rangle+\phi(u_{n_k})-\phi(\nu)\\&\leq \liminf\langle F(x,u_{n_k}),u_{n_k}-\nu\rangle+\phi(u_{n_k})-\phi(\nu)\\&\leq \liminf\langle F(x_{n_k},u_{n_k}),u_{n_k}-\nu\rangle+\phi(u_{n_k})-\phi(\nu)\\&-\|x_{n_k}-x\|\|u_{n_k}-\nu\|\\&\leq \liminf\langle F_{n_k}(x_{n_k},u_{n_k}),u_{n_k}-\nu\rangle-\epsilon_{n_k}+\phi(u_{n_k})\\&-\phi(\nu)-\|x_{n_k}-x\|\|u_{n_k}-\nu\|\\&\leq \liminf\{\phi_{n_k}(\nu)-\phi_{n_k}(u_{n_k})-\epsilon_{n_k}\\&+\phi(u_{n_k})-\phi(\nu)\}=0,\end{aligned}$$

for all $\nu\in K$. Therefore $\bar{u}$ solves $MVI(F,\phi,K)_x$ and $u_n\to\bar{u}$, which is a contradiction. This complete the proof.

4 Mixed variational inequalities and inclusion problems

In this section, we present the concept of parametric well-posedness for inclusion problems and investigate the relationships between the parametric well-posedness of mixed variational inequality and the parametric well-posedness of inclusion problem. Let $x\in X$ and $F_x:=F(x,.):H\rightrightarrows H$ be a set-valued mapping from the real Hilbert space H to H. The inclusion problem associated with mapping F_x is denoted by $IP(F_x)$ and defined by:
IP(F_x): Find $u\in H$ such that $0\in F_x(u)$.

Definition 4.1 *Let $x\in X$ and $\{x_n\}$ be a sequence converging to x. A sequence $\{u_n\}\subset H$ is called an approximating sequence (w.r. to $\{x_n\}$) for inclusion problem $IP(F_x)$ if $d(0,F_{x_n}(u_n))\to 0$, or equivalently, there exists a sequence $\nu_n\in F_{x_n}(u_n)$ such that $\|\nu_n\|\to 0$ as $n\to\infty$.*

Now, we define

$$(IP(F)):=\{IP(F_x):x\in X\}.$$

Definition 4.2 *The family $(IP(F))$ is parametrically strong (weak) well-posed if*

1. *there exists a unique solution $\bar{u}_x\in H$ to $IP(F_x)$, for all $x\in X$,*
2. *for all $x\in X$ and for all $\{x_n\}$ converging to x, every approximating sequence for the problem $IP(F_x)$ with respect to $\{x_n\}$ strongly(weakly) converges to $\bar{u}_x$.*

Theorem 4.1 *Let $x\in X$, $F_x:H\to H$ be a mapping such that $F_x(u)=F(x,u)$ and $\phi:H\to\mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous functional. Then $u_x\in K$ solves $MVI(F,\phi,K)_x$ if and only if u_x solves $IP(F_x+\partial\phi)$.*

Proof. Let $x\in X$ and $u_x\in K$ be a solution to the mixed variational inequality $MVI(F,\phi,K)_x$, which means

$$\langle F(x,u_x),u_x-\nu\rangle+\phi(u_x)-\phi(\nu)\leq 0, \quad (4.6)$$

for all $\nu\in K$. For any $w\in K$ and $\lambda\in[0,1]$, letting $\nu=u_x+\lambda(w-u_x)$ in the (4.6) yields

$$\begin{aligned}\langle F(x,u_x),\ &-\lambda(w-u_x)\rangle+\phi(u_x)\\ -\ &\phi(u_x+\lambda(w-u_x))\leq 0,\end{aligned}$$

for all $w\in K$. From convexity of ϕ and above inequality, we can deduce that

$$\langle F(x,u_x),-\lambda(w-u_x)\rangle+\lambda(\phi(u_x)-\phi(w))\leq 0.$$

Hence

$$\langle -F_x(u_x),(w-u_x)\rangle\leq\phi(w)-\phi(u_x),$$

for all $w\in K$. Now, definition of convex subdifferential implies that $-F_x(u_x)\in\partial\phi(u_x)$.
Conversely, suppose that $u_x\in K$ be a solution to the inclusion problem $IP(F_x+\partial\phi)$, which means that $0\in F(x,u_x)+\partial\phi(u_x)$. Thus, there exists $u^*\in\partial\phi(u_x)$ such that $0=F(x,u_x)+u^*$ and

$$\langle u^*,\nu-u_x)\rangle\leq\phi(\nu)-\phi(u_x),$$

for all $\nu\in K$. Therefore

$$\begin{aligned}0&=\langle F(x,u_x)+u^*,\nu-u_x\rangle\\&\leq\langle F(x,u_x),\nu-u_x\rangle+\phi(\nu)-(u_x),\end{aligned}$$

for all $\nu\in K$. Hence,

$$\langle F(x,u_x),u_x-\nu\rangle+\phi(u_x)-(\nu)\leq 0,$$

for all $\nu\in K$, which implies that u_x is a solution to the mixed variational inequality

$MVI(F,\phi,K)_x$. This completes the proof.

The following theorem establish the relation between parametric weak well-posedness of family $(MVI(F,\phi,K))$ and parametric weak well-posedness of family inclusion problems.

Theorem 4.2 *Let $F : X \times H \to H$ be hemicontinuous, monotone with respect to the second argument. Suppose that F satisfies Condition A and $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. If the family $(MVI(F,\phi,K))$ is parametrically weak well-posed, then the family $(IP(F+\partial\phi))$ is parametrically weak well-posed.*

Proof. Suppose that $(MVI(F,\phi,K))$ is parametrically weak well-posed. Assume that x is an arbitrary element in X and u_x is the unique solution for $MVI(F,\phi,K)x$. From Theorem 4.1, u_x is also the unique solution of $IP(F_x+\partial\phi)$. Now, suppose that $\{x_n\} \subset X$ converges to x and $\{u_n\}$ is an approximating sequence for $IP(F_x+\partial\phi)$(w.r. to $\{x_n\}$). Hence, it suffices to show that $u_n \rightharpoonup u_x$. Since $\{u_n\}$ is approximating sequence, there exists $u_n^* \in F(x_n,u_n)+\partial\phi(u_n)$ such that $\|u_n^*\| \to 0$. Hence,

$$\langle u_n^* - F(x_n,u_n), \nu - u_n\rangle \leq \phi(\nu) - \phi(u_n),$$

for all $\nu \in K$ and therefore,

$$\langle F(x_n,u_n), u_n - \nu\rangle \leq \langle u_n^*, u_n-\nu\rangle + \phi(\nu) - \phi(u_n), \tag{4.7}$$

for all $\nu \in K$. We claim that the approximating sequence $\{u_n\}$ is bounded on K. In fact, if $\{u_n\}$ is unbounded, we can suppose that $\|u_n\| \to +\infty$. Let

$$t_n := \frac{1}{\|u_n - u_x\|}, \quad z_n := u_x + t_n(u_n - u_x).$$

Without loss of generality, we can suppose that $t_n \in (0,1]$. Since $\{z_n\}$ is a bounded sequence on K, there exists $z \in K$ such that $z_n \rightharpoonup z$ and clearly, $z \neq u_x$. Now, for all $\nu \in K$

$$\begin{aligned}
&\langle F(x,\nu), z-\nu\rangle = \langle F(x,\nu), z - z_n\rangle \\
&+ \langle F(x,\nu), z_n - u_x\rangle + \langle F(x,\nu), u_x - \nu\rangle \\
&= \langle F(x,\nu), z - z_n\rangle + t_n\langle F(x,\nu), u_n - \nu\rangle \\
&+ (1-t_n)\langle F(x,\nu), u_x - \nu\rangle.
\end{aligned}$$

On the other hand, u_x is the unique solution to the mixed variational inequality $MVI(F,\phi,K)_x$, F is monotone with respect to the second argument and satisfies Condition A. Therefore, from last equality we have

$$\begin{aligned}
&\langle F(x,\nu), z-\nu\rangle \leq \langle F(x,\nu), z - z_n\rangle \\
&+ t_n\langle F(x,\nu), u_n - \nu\rangle \\
&+ (1-t_n)\langle F(x,u_x), u_x - \nu\rangle \\
&\leq \langle F(x,\nu), z - z_n\rangle + t_n\langle F(x,\nu), u_n - \nu\rangle \\
&+ (1-t_n)(\phi(\nu) - \phi(u_x)) \\
&\leq \langle F(x,\nu), z - z_n\rangle + t_n\langle F(x,u_n), u_n - \nu\rangle \\
&+ (1-t_n)(\phi(\nu) - \phi(u_x)) \\
&\leq \langle F(x,\nu), z - z_n\rangle + t_n\langle F(x_n,u_n), u_n - \nu\rangle \\
&+ t_n\|x_n - x\|\|\nu - u_n\| + (1-t_n)(\phi(\nu) - \phi(u_x)) \\
&\leq \langle F(x,\nu), z - z_n\rangle + t_n\langle u_n^*, u_n - \nu\rangle \\
&+ t_n(\phi(\nu) - \phi(u_n)) + t_n\|x_n - x\|\|\nu - u_n\| \\
&+ (1-t_n)(\phi(\nu) - \phi(u_x)) \\
&= \langle F(x,\nu), z - z_n\rangle + \frac{\langle u_n^*, u_n - \nu\rangle}{\|u_n - u_x\|} \\
&+ \phi(\nu) - [t_n\phi(u_n) + (1-t_n)\phi(u_x)] \\
&+ t_n\|x_n - x\|\|\nu - u_n\|
\end{aligned}$$

From convexity of ϕ and above inequality, we obtain

$$\begin{aligned}
\langle F(x,\nu), z-\nu\rangle &\leq \langle F(x,\nu), z - z_n\rangle \\
&+ \frac{\langle u_n^*, u_n - \nu\rangle}{\|u_n - u_x\|} + \phi(\nu) - \phi(z_n) \\
&+ t_n\|x_n - x\|\|\nu - u_n\|
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle F(x,\nu), z-\nu\rangle &\leq \liminf\{\langle F(x,\nu), z - z_n\rangle \\
&+ \frac{\langle u_n^*, u_n - \nu\rangle}{\|u_n - u_x\|} + \phi(\nu) - \phi(z_n) \\
&+ t_n\|x_n - x\|\|\nu - u_n\|\} = \phi(\nu) - \phi(z).
\end{aligned}$$

Hence,

$$\langle F(x,\nu), z-\nu\rangle + \phi(z) - \phi(\nu) \leq 0$$

which implies that z is a solution to the mixed variational inequality $MVI(F,\phi,K)_x$. From uniqueness of solution for $MVI(F,\phi,K)_x$, we can deduce that $z = u_x$ which is contradiction. Thus, the approximating sequence $\{u_n\}$ is bounded on K. Since H is reflexive and $\{u_n\}$ is bounded, there exists subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup \hat{u}$ as $k \to \infty$. Therefore, from (4.7), we have

$$\langle F(x_{n_k}, u_{n_k}), u_{n_k} - \nu\rangle + \phi(u_{n_k}) - \phi(\nu)$$

$$\leq \langle u^*_{nk}, u_{n_k} - \nu \rangle, \quad (4.8)$$

for all $\nu \in K$. Since ϕ is lower semicontinuous, F is monotone with respect to the second argument, satisfies Condition A and (4.8), we have

$$\begin{aligned}
&\langle F(x,\nu), \bar{u} - \nu \rangle + \phi(\bar{u}) - \phi(\nu) \\
&\leq \liminf \langle F(x,\nu), u_{n_k} - \nu \rangle + \phi(u_{n_k}) - \phi(\nu) \\
&\leq \liminf \langle F(x,u_{n_k}), u_{n_k} - \nu \rangle + \phi(u_{n_k}) - \phi(\nu) \\
&\leq \liminf \langle F(x_{n_k},u_{n_k}), u_{n_k} - \nu \rangle + \phi(u_{n_k}) - \phi(\nu) \\
&+ \|x_{n_k} - x\| \|\nu - u_{n_k}\| \\
&\leq \liminf \langle u^*_{nk}, u_{n_k} - \nu \rangle + \|x_{n_k} - x\| \|\nu - u_{n_k}\| \\
&= 0,
\end{aligned}$$

for all $\nu \in K$. Therefore, $\bar{u}$ is solution to the associated mixed variational inequality $AMVI(F,\phi,K)_x$. Now, by using Remark 2.2, $\bar{u}$ also solves the mixed variational inequality $MVI(F,\phi,K)_x$ and so we have $\bar{u} = u_x$ in terms of the uniqueness of solution to the mixed variational inequality $MVI(F,\phi,K)_x$. This complete the proof.

Remark 4.1 *Theorem 4.2, generalize [7, Theorem 4.1] for a family of mixed variational inequalities.*

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