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A Direct Method For Solving Linear Delay Differential Equations

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Abstract

In this paper, we propose direct methods to solve linear delay differential equations (LDDEs) based on vector forms of Block-Pulse Functions (BPFs) and Triangular Functions (TFs). Operational matrix of integration of BPFs and TFs is applied to transform LDDE to a linear system of algebraic equations. Furthermore, some numerical examples are presented to indicate the reliability and the accuracy of these methods. Convergence analysis of the present method has been discussed, too.

Keywords : Linear delay differential equations; Block-Pulse Functions; Triangular Functions; Direct method.

1 Introduction

In engineering or practical system, it is totally natural to consider the system that its future evolution depends on the present state and also on a period of its history. Numerical solving of mathematical issues which have been discussed by delay equation is difficult. Many phenomena are modelled by DDEs which have attracted numerical analysts. Delay differential equations with a single constant delay have various applications in different sciences like engineering, physics, mathematics, economics and etc. This kind of equations occur in many dynamical systems such as optics [12, 13, 14], biology [16, 17], economics [21, 22] and ecology [9, 10]. There are particular methods for numerical solutions of these problems. Several LDDEs have been solved based on Runge-Kutta or linear multi-step methods [6, 8, 11, 20]. The basic idea in this paper is using BPFs and TFs for solving LDDEs with a single constant delay as

$$\begin{cases} y'(x)=\alpha y(x) + \beta y(x-\tau) + \gamma y'(x-\tau) \\ +f(x), \qquad x \geq x_0, \\ y(x) = \psi_(x), \qquad x \leq x_0, \end{cases} \tag{1.1}$$

where $y' = \frac{dy}{dx}$ and τ is a positive constant called lag. The parameter τ and functions ψ and f are known but y is not. ψ represents the initial function defined for all $x \leq x_0$.
The structure of this paper is as follows: a review of block-pulse functions and its use for solving linear delay differential equations 4 is presented in section 2. Triangular functions are utilized to

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solve linear delay differential equations 4 in section 3. Then, in section 4, we give the convergence analysis. In section 5, some numerical examples are illustrated to show the efficiency and the accuracy of the methods and finally a brief conclusion is in section 6.

2 Block-pulse functions and solving LDDEs by BPFs

The choice of basic functions is one of the most important steps in any numerical solution. Block-pulse functions have been applied for different problems. For example, see [1, 2, 3].

2.1 Definition and properties

An m-set of BPFs is defined on [0,1) as

$$\varphi_i(x) = \begin{cases} 1, & x \in [ih, (i+1)h), \\ 0, & otherwise, \end{cases}$$

where $i = 0, 1, ..., m-1$, $h = \frac{1}{m}$ and m is a positive integer number. The most important properties of BPFs are disjointness, orthogonality, completeness and partition of unity. The disjointness property is

$$\varphi_i(x)\varphi_j(x) = \begin{cases} \varphi_i(x), & i = j, \\ 0, & i \neq j, \end{cases} \tag{2.2}$$

where $i, j = 0, 1, ..., m-1$. For orthogonality, we have

$$\int_0^1 \varphi_i(x)\varphi_j(x)\,dx = h\delta_{ij},$$

where δ_{ij} is the Kronecker delta. The third property is completeness. For every function f in $\mathcal{L}^2([0,1))$, when m approaches to the infinity, Parseval's identity holds

$$\int_0^1 |f(x)|^2\,dx = \sum_{i=0}^{\infty} f_i^2 \|\ \varphi_i(x)\|\ ^2,$$

where

$$f_i = \frac{1}{h}\int_0^1 f(x)\varphi_i(x)\,dx. \tag{2.3}$$

Also, from the definition of BPFs, these functions form a partition of unity i.e.

$$\sum_{i=0}^{m-1} \varphi_i(x) = 1. \tag{2.4}$$

2.2 Vector forms

An m-vector of BPFs in the semi-open interval [0,1) is given by

$$\Phi(x) = [\phi_0(x), \phi_1(x), ..., \phi_{m-1}(x)]^T,$$

where T stands for transpose.
Using relations 2.2 and 2.4 for all $x \in [0,1)$ we have

$$\Phi(x)\Phi^T(x) = \begin{pmatrix} \phi_0(x) & 0 & \cdots & 0 \\ 0 & \phi_1(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{m-1}(x) \end{pmatrix},$$

$$\Phi^T(x)\Phi(x) = 1,$$

$$\Phi(x)\Phi^T(x)V = \tilde{V}\Phi(x),$$

where V is an m-vector and $\tilde{V} = diag(V)$. Moreover, it can be concluded that for every $m \times m$ matrix B

$$\Phi^T(x)B\Phi(x) = \hat{B}^T\Phi(x),$$

where $\hat{B}$ is a column vector with elements equal to the diagonal entries of the matrix B.

2.3 Expansion of functions by BPFs

An arbitrary function $f \in \mathcal{L}^2([0,1))$ can be expanded by BPFs as:

$$f(x) \simeq \sum_{i=0}^{m-1} f_i\varphi_i(x) = F^T\Phi(x) = \Phi^T(x)F, \tag{2.5}$$

where $F = [f_0, f_1, ..., f_{m-1}]$ and for $i = 0, 1, ..., m-1$ the coefficient f_i is defined by 2.3.

2.4 Operational matrix of integration

We compute $\int_0^x \varphi_i(s)\,ds$ as follows

$$\int_0^x \varphi_i(s)\,ds = \begin{cases} 0, & x < ih, \\ x - ih, & ih \leq x < (i+1)h, \\ h, & (i+1)h \leq x < 1. \end{cases}$$

It is shown that $\frac{h}{2}$ is the least square approximation of $\int_0^x \varphi_i(s)\,ds$ on $[ih, (i+1)h)$, see [5]. So we have

$$\int_0^x \varphi_i(s)\,ds \simeq [0, ..., 0, \frac{h}{2}, h, ..., h]\Phi(x),$$

in which $\frac{h}{2}$ is the ith element. Hence

$$\int_0^x \Phi(s)\, ds \simeq P\Phi(x),$$

where P is the $m \times m$ operational matrix of integration given by

$$P = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

So the integral of every function f can be approximated as follows:

$$\int_0^x f(s)\, ds \simeq \int_0^x F^T\Phi(s)\, ds \simeq F^T P\Phi(x).$$

2.5 Solving LDDEs by BPFs

Now using the obtained results, a direct method to solve LDDEs with a single constant delay is presented.
Approximating $y(x), y^{'}(x), y(x-\tau), y^{'}(x-\tau)$ and $f(x)$ with respect to BPFs according to 2.5 gives

$$\begin{aligned} y(x) &\simeq Y^T\Phi(x) = \Phi^T(x)Y \\ y^{'}(x) &\simeq Y^{'T}\Phi(x) = \Phi^T(x)Y^{'} \\ y(x-\tau) &\simeq \Psi^T\Phi(x) = \Phi^T(x)\Psi \\ y^{'}(x-\tau) &\simeq \Psi^{'T}\Phi(x) = \Phi^T(x)\Psi^{'} \\ f(x) &\simeq F^T\Phi(x) = \Phi^T(x)F, \end{aligned} \tag{2.6}$$

where m-vectors Y,$Y^{'}$,Ψ,$\Psi^{'}$ and F are BPFs coefficients of $y(x)$,$y^{'}(x)$,$y(x-\tau)$,$y^{'}(x-\tau)$ and $f(x)$, respectively. For solving Eq. 4, we substitute relations 2.6 into 4 to obtain

$$\begin{aligned} Y^{'T}\Phi(x) &\simeq \alpha Y^T\Phi(x) + \beta\Psi^T\Phi(x) \\ &+\gamma\Psi^{'T}\Phi(x) + F^T\Phi(x), \end{aligned} \tag{2.7}$$

or, using orthogonality of BPFs,

$$Y^{'} \simeq \alpha Y + \beta\Psi + \gamma\Psi^{'} + F. \tag{2.8}$$

Now, we show that $Y^{'}$ can be computed in terms of Y. Note that

$y(x) - y(0) = \int_0^x y^{'}(s)\, ds \simeq \int_0^x Y^{'T}\Phi(s)\, ds \simeq$ Y^'^TPΦ(x),

therefore

$$y(x) \simeq Y^{'T}P\Phi(x) + Y_0^T\Phi(x), \tag{2.9}$$

where Y_0 is an m-vector of the form $Y_0 = [y_0, y_0, ..., y_0]^T$ with $y_0 = y(0)$. Consequently, using 2.9,

$$Y \simeq P^T Y^{'} + Y_0, \tag{2.10}$$

now combining 2.8 and 2.10 and replacing $\simeq$ with $=$ gives

$$(P^T)^{-1}(Y - Y_0) = \alpha Y + \beta\Psi + \gamma\Psi^{'} + F. \tag{2.11}$$

Relation 2.11 is a system of m linear algebraic equations for the m unknowns $y_0, y_1, ..., y_{m-1}$, components of Y. Hence, an approximate solution $y(x) \simeq Y^T\Phi(x)$ can be computed for Eq. 4.

3 Vector forms of triangular functions and solving LDDEs by TFs

Triangular functions were introduced by Deb et al. [7] and have been applied in different problems [4, 18].

3.1 Definition and properties

Two m-set of TFs are defined over interval [0,1) as

$$T1_i(x) = \begin{cases} 1 - \frac{x-ih}{h}, & x \in [ih, (i+1)h), \\ 0, & otherwise, \end{cases}$$

$$T2_i(x) = \begin{cases} \frac{x-ih}{h}, & x \in [ih, (i+1)h), \\ 0, & otherwise, \end{cases}$$

where $i = 0, 1, ..., m-1$, $h = \frac{1}{m}$ and m is a positive integer number. Now, $2m$-vector of TFs is defined as

$$T(x) = \begin{pmatrix} T1(x) \\ T2(x) \end{pmatrix},$$

where

$$T1(x) = [T1_0(x), T1_1(x), ..., T1_{m-1}(x)]^T,$$

$$T2(x) = [T2_0(x), T2_1(x), ..., T2_{m-1}(x)]^T,$$

in which $T1(x)$ and $T2(x)$ are called the left-handed triangular function (LHTF) vector and the right-handed triangular function (RHTF) vector, respectively and if $X \in R^m$ then X^T is

transpose of X.
Orthogonality of TFs is given in [7]. Also

$$\int_0^1 T1_i(x)T1_j(x)\,dx = \int_0^1 T2_i(x)T2_j(x)\,dx = \frac{h}{3}\delta_{ij},$$

$$\int_0^1 T1_i(x)T2_j(x)\,dx = \int_0^1 T2_i(x)T1_j(x)\,dx = \frac{h}{6}\delta_{ij},$$

where δ_{ij} is the Kronecker delta. Hence

$$\int_0^1 T1(x)T1^T(x)\,dx = \int_0^1 T2(x)T2^T(x)\,dx = \frac{h}{3}I,$$

$$\int_0^1 T1(x)T2^T(x)\,dx = \int_0^1 T2(x)T1^T(x)\,dx = \frac{h}{6}I,$$

where I is the $m \times m$ identity matrix. Also,

$$\int_0^1 T(x)T^T(x)\,dx = D,$$

where D is the following $2m \times 2m$ matrix:

$$D = \begin{pmatrix} \frac{h}{3}I & \frac{h}{6}I \\ \frac{h}{6}I & \frac{h}{3}I \end{pmatrix}.$$

Let V be a $2m$-vector, we have [18]

$$T(x)T^T(x)V \simeq \tilde{V}T(x),$$

where $\tilde{V} = diag(V)$. Moreover, it can be concluded that for every $2m \times 2m$ matrix B

$$T^T(x)BT(x) \simeq \hat{B}^T T(x),$$

where $\hat{B}$ is a column vector with elements equal to the diagonal entries of the matrix B.

3.2 Expansion of functions by TFs

An arbitrary function $f \in \mathcal{L}^2([0,1))$ can be expanded by TFs as

$$f(x) \simeq \sum_{i=0}^{m-1} F1_i T1_i(x) + \sum_{i=0}^{m-1} F2_i T2_i(x) = F1^T T1(x) + F2^T T2(x) = F^T T(x), \tag{3.12}$$

where $F1$ and $F2$ are the coefficients of TFs [19], $F1_i = f(ih)$ and $F2_i = f((i+1)h)$ for $i = 0, 1, ..., m-1$. Also the 2m-vector F is defined as

$$F = \begin{pmatrix} F1 \\ F2 \end{pmatrix},$$

3.3 Operational matrix of integration

Expressing $\int_0^x (s)\,ds$ in terms of $T(x)$ gives

$$\int_0^x T(s)\,ds \simeq PT(x),$$

where P is the $2m \times 2m$ operational matrix of integration of T given by

$$P = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix},$$

where $P1$ and $P2$ are the following $m \times m$ operational matrices of integration of TFs as [7]

$$P1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$P2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

So the integral of every function f can be approximated as follows

$$\int_0^x f(s)\,ds \simeq \int_0^x F^T T(s)\,ds \simeq F^T PT(x).$$

3.4 Solving LDDEs by TFs

Now using the obtained results, a direct method to solve LDDEs with a single constant delay is presented.
Approximating $y(x)$,$y^{'}(x)$,$y(x-\tau)$,$y^{'}(x-\tau)$ and $f(x)$ with respect to TFs according to 3.12 gives

$$\begin{aligned} y(x) &\simeq Y^T T(x) = T^T(x)Y \\ y^{'}(x) &\simeq Y^{'^T} T(x) = T^T(x)Y^{'} \\ y(x-\tau) &\simeq \Psi^T T(x) = T^T(x)\Psi \\ y^{'}(x-\tau) &\simeq \Psi^{'^T} T(x) = T^T(x)\Psi^{'} \\ f(x) &\simeq F^T T(x) = T^T(x)F, \end{aligned} \tag{3.13}$$

where $2m$-vectors $y(x)$, $y^{'}(x)$, $y(x-\tau)$, $y^{'}(x-\tau)$ and $f(x)$ are TFs coefficients of $y(x), y^{'}(x), y(x-\tau), y^{'}(x-\tau)$ and $f(x)$, respectively. For solving

Eq. 4, we substitute relations 3.13 into 4 to obtain

$$Y'^T T(x) \simeq \alpha Y^T T(x) + \beta \Psi^T T(x) + \gamma \Psi'^T T(x) + F^T T(x),$$

or, using orthogonality of TFs,

$$Y' \simeq \alpha Y + \beta \Psi + \gamma \Psi' + F. \quad (3.14)$$

Now, we show that Y' can be computed in terms of Y. Note that

$$y(x) - y(0) = \int_0^x y'(s)\, ds \simeq \int_0^x Y'^T T(s)\, ds \simeq Y'^T PT(x),$$

therefore

$$y(x) \simeq Y'^T PT(x) + Y_0^T T(x), \quad (3.15)$$

where Y_0 is an $2m$-vector of the form $Y_0 = [y_0, y_0, ..., y_0]^T$ with $y_0 = y(0)$. Consequently, using 3.15,

$$Y \simeq P^T Y' + Y_0, \quad (3.16)$$

now combining 3.14 and 3.16 and replacing $\simeq$ with $=$ gives

$$(P^T)^{-1}(Y - Y_0) = \alpha Y + \beta \Psi + \gamma \Psi' + F. \quad (3.17)$$

Relation 3.17 is a system of $2m$ linear algebraic equations for the $2m$ unknowns $y_0, y_1, ..., y_{2m-1}$, components of Y. Hence, an approximate solution $y(x) \simeq Y^T T(x)$ can be computed for Eq. 4.

4 Convergence analysis

In this section, we study the convergence of the approximation solution using BPFs.

Theorem 4.1 *Let $I = [0, 1)$ and $f_m(t) = F^T \Phi(t)$ be the BPF approximation of $f \in C^1(I)$, where $F = [f_0, f_1, ..., f_{m-1}]$ and f_i is defined by 2.3 and suppose that there exists a positive number M such that*

$$|f'(t)| \leq M, \qquad t \in [0, 1). \quad (4.18)$$

Then

$$\| f - f_m \|_\infty \leq Mh. \quad (4.19)$$

Proof. We have

$$|f(t) - f_m(t)| = |f(t) - f_i|, \quad t \in [ih, (i+1)h). \quad (4.20)$$

From Eq. 2.3 and integral mean value theorem

$$f_i = f(\eta_i), \quad \eta_i \in (ih, (i+1)h). \quad (4.21)$$

Substituting 4.21 into 4.20 and using mean value theorem for function f gives

$$|f(t) - f(\eta_i)| = |f'(\gamma_i)(t - \eta_i)|, \quad (4.22)$$

where $\gamma_i \in (ih, (i+1)h)$, now from 4.18 and 4.22 the result can be concluded.
We have $f(x) - f_m(x) = O(h)$ as a result $\int_0^s f(x)\, dx - \int_0^s f_m(x)\, dx = O(h)$. By integrating both sides of relations and and using relation 2.10, we conclude $y(x) - y_m(x) = O(h)$.
Similarly it can be shown that the error of the approximation with triangular functions is of $O(h^2)$.

5 Numerical examples

The methods presented in this paper is applied to some examples. All computations were carried out using a program written in Matlab.

Example 5.1 *[19] Consider the following delay differential equation*

$$\begin{cases} y'(t) = -y(t-1), & t \geq 0, \\ y(t) = 1, & t \leq 0, \end{cases}$$

with the analytical solution $y(t) = \sum_{n=0}^{[t]+1} (-1)^n \frac{(t-n+1)^n}{n!}$ for $t \geq 0$. We obtain the numerical solution in the interval $0 \leq t < 20$ by using presented methods. Table 1 shows the maximum of the absolute errors computed by

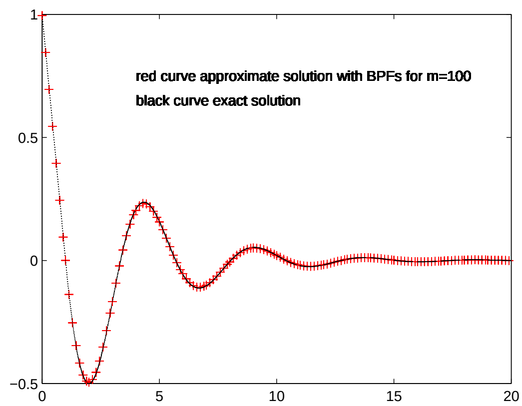

Figure 1: The graph of the exact and approximate solutions using BPFs of example 5.1.

Table 1: Maximum of the absolute errors using BPFs for example 5.1.

$[a, b)$	$m = 10$	$m = 100$
$[0, 1)$	5.0×10^{-2}	5.0×10^{-3}
$[1, 2)$	4.7×10^{-2}	4.5×10^{-3}
$[2, 3)$	5.2×10^{-2}	5.0×10^{-3}
$[3, 4)$	3.8×10^{-2}	3.9×10^{-3}
$[4, 5)$	4.9×10^{-2}	5.0×10^{-3}
$[5, 6)$	2.0×10^{-2}	2.2×10^{-3}
$[6, 7)$	3.3×10^{-2}	3.5×10^{-3}
$[7, 8)$	2.4×10^{-2}	2.4×10^{-3}
$[8, 9)$	2.0×10^{-2}	2.3×10^{-3}
$[9, 10)$	2.0×10^{-2}	2.2×10^{-3}
$[10, 11)$	1.1×10^{-2}	1.3×10^{-3}
$[11, 12)$	1.2×10^{-2}	1.4×10^{-3}
$[12, 13)$	5.0×10^{-3}	6.0×10^{-4}
$[13, 14)$	6.6×10^{-3}	8.0×10^{-4}
$[14, 15)$	5.0×10^{-3}	5.2×10^{-4}
$[15, 16)$	3.5×10^{-3}	4.4×10^{-4}
$[16, 17)$	3.5×10^{-3}	4.1×10^{-4}
$[17, 18)$	1.7×10^{-3}	2.3×10^{-4}
$[18, 19)$	1.8×10^{-3}	2.4×10^{-4}
$[19, 20)$	5.5×10^{-4}	9.8×10^{-5}

Table 2: Maximum of the absolute errors using TFs for example 5.1.

$[a, b)$	$m = 10$	$m = 100$
$[0, 1)$	0	0
$[1, 2)$	2.2×10^{-16}	6.6×10^{-16}
$[2, 3)$	7.5×10^{-4}	7.5×10^{-6}
$[3, 4)$	1.0×10^{-3}	1.0×10^{-5}
$[4, 5)$	8.3×10^{-4}	8.3×10^{-6}
$[5, 6)$	9.9×10^{-4}	9.9×10^{-6}
$[6, 7)$	9.7×10^{-4}	9.7×10^{-6}
$[7, 8)$	6.5×10^{-4}	6.5×10^{-6}
$[8, 9)$	7.0×10^{-4}	7.0×10^{-6}
$[9, 10)$	3.5×10^{-4}	3.5×10^{-6}
$[10, 11)$	4.4×10^{-4}	4.4×10^{-6}
$[11, 12)$	3.6×10^{-4}	3.6×10^{-6}
$[12, 13)$	2.6×10^{-4}	2.6×10^{-6}
$[13, 14)$	2.6×10^{-4}	2.6×10^{-6}
$[14, 15)$	1.3×10^{-4}	1.3×10^{-6}
$[15, 16)$	1.5×10^{-4}	1.5×10^{-6}
$[16, 17)$	8.3×10^{-5}	8.3×10^{-7}
$[17, 18)$	8.2×10^{-5}	8.3×10^{-7}
$[18, 19)$	7.0×10^{-5}	7.0×10^{-7}
$[19, 20)$	4.4×10^{-5}	4.5×10^{-7}

using BPFs with $m = 10, 100$. Table 2 shows the maximum of the absolute errors computed by using TFs with $m = 10$ and $m = 100$. Fig. 1 illustrates the graph of the exact and approximate solutions using BPFs with $m = 100$. Numerical results show that as the upper bound

Table 3: Maximum of the absolute errors using BPFs for example 5.2.

$[a, b)$	$m = 10$	$m = 100$
$[0, 1)$	1.3×10^{-1}	1.4×10^{-2}
$[1, 2)$	6.6×10^{-3}	6.1×10^{-4}
$[2, 3)$	3.2×10^{-4}	2.7×10^{-5}
$[3, 4)$	1.5×10^{-5}	1.3×10^{-6}
$[4, 5)$	7.7×10^{-7}	6.7×10^{-8}
$[5, 6)$	3.8×10^{-8}	3.3×10^{-9}
$[6, 7)$	1.8×10^{-9}	1.6×10^{-10}
$[7, 8)$	9.1×10^{-11}	8.3×10^{-12}
$[8, 9)$	4.4×10^{-12}	4.2×10^{-13}
$[9, 10)$	2.2×10^{-13}	2.1×10^{-14}
$[10, 11)$	1.1×10^{-14}	1.3×10^{-15}
$[11, 12)$	6.6×10^{-16}	2.2×10^{-16}
$[12, 13)$	2.2×10^{-16}	2.2×10^{-16}

Table 4: Maximum of the absolute errors using TFs for example 5.2.

$[a, b)$	$m = 10$	$m = 100$
$[0, 1)$	3.8×10^{-5}	1.6×10^{-7}
$[1, 2)$	2.1×10^{-2}	1.8×10^{-2}
$[2, 3)$	1.1×10^{-3}	9.0×10^{-4}
$[3, 4)$	1.1×10^{-4}	4.5×10^{-5}
$[4, 5)$	6.0×10^{-5}	2.2×10^{-6}
$[5, 6)$	4.0×10^{-5}	1.1×10^{-7}
$[6, 7)$	2.6×10^{-5}	5.6×10^{-9}
$[7, 8)$	1.6×10^{-5}	2.7×10^{-10}
$[8, 9)$	1.0×10^{-5}	1.3×10^{-11}
$[9, 10)$	7.0×10^{-6}	6.7×10^{-12}
$[10, 11)$	4.6×10^{-6}	4.9×10^{-12}
$[11, 12)$	2.9×10^{-6}	2.7×10^{-12}
$[12, 13)$	1.9×10^{-6}	2.1×10^{-12}
$[13, 14)$	1.2×10^{-6}	1.2×10^{-12}
$[14, 15)$	8.1×10^{-7}	5.6×10^{-12}
$[15, 16)$	5.2×10^{-7}	3.2×10^{-12}
$[16, 17)$	3.4×10^{-7}	7.7×10^{-12}
$[17, 18)$	2.2×10^{-7}	8.2×10^{-12}
$[18, 19)$	1.4×10^{-7}	1.5×10^{-12}
$[19, 20)$	9.3×10^{-8}	4.0×10^{-12}

Table 5: Maximum of the absolute errors using BPFs for example 5.3.

$[a, b)$	$m = 100$	$m = 150$
$[0, 1)$	1.2×10^{-2}	8.2×10^{-3}
$[1, 2)$	1.1×10^{-2}	7.6×10^{-3}
$[2, 3)$	1.3×10^{-1}	9.3×10^{-2}

Table 6: Maximum of the absolute errors using TFs for example 5.3.

$[a, b)$	$m = 100$	$m = 150$
$[0, 1)$	1.8×10^{-5}	4.0×10^{-6}
$[1, 2)$	9.8×10^{-4}	4.3×10^{-5}
$[2, 3)$	1.1×10^{-3}	2.9×10^{-4}

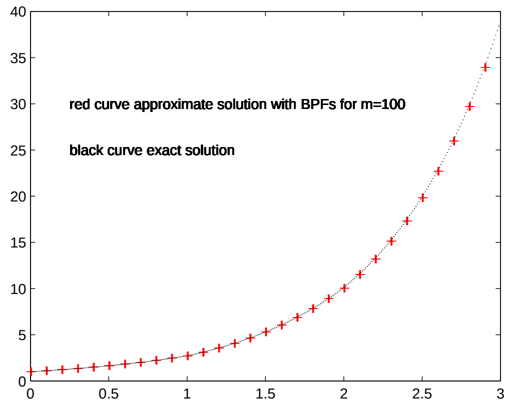

Figure 2: The graph of the exact and approximate solutions using BPFs of example 5.3.

of the interval increases, the results get better. This is because the graph of the exact solution tendes to be a stright line.

Example 5.2 *[23] Consider the following delay differential equation*

$$\begin{cases} y^{'}(t) = -1000y(t) + 997e^{-3}y(t-1) + \\ 1000 - 997e^{-3}, \qquad t \geq 0, \\ y(t) = 1 + e^{-3t}, \qquad t \leq 0, \end{cases}$$

with the exact solution as the initial function. We report the numerical results in the interval $0 \leq t < 20$. Table 3 demonstrates the maximum of the absolute errors through using BPFs with $m = 10, 100$ and Table 4 demonstrates the maximum of the absolute errors using TFs with $m = 10, 100$.

The absolute errors using BPFs for example 5.2 in interval $[13, 20)$ is zero.

Example 5.3 *[15] Consider the following neutral delay differential equation*

$$\begin{cases} y^{'}(t) = y(t) + y^{'}(t-1), & t \geq 0, \\ y(t) = 1, & t \leq 0, \end{cases}$$

with the analytical solution

$$\begin{cases} y(t) = e^t, & 0 \leq t \leq 1, \\ y(t) = (t-1)e^{t-1} + e^t, & 1 \leq t \leq 2, \\ y(t) = 1/2(t^2 - 2t)e^{t-2} + (t-1)e^{t-1} + e^t \\ , 2 \leq t \leq 3. \end{cases}$$

We obtain the numerical solution in the interval $0 \leq t < 3$ using the presented method. Table 5 shows the maximum of the absolute errors computed using BPFs with $m = 100, 150$. Table 6 demonstrates the maximum of the absolute errors computed by using TFs with $m = 100$ and $m = 150$. Fig. 2. illustrates the graph of the exact and approximate solutions using BPFs for $m = 100$.

6 Conclusions

This article presents numerical methods to solve LDDEs using block-pulse and triangular functions. Using vector forms of these functions and their operational matrix of integration, we transform a linear delay differential equation to a system of algebraic equations directly. The approximate solution of Numerical examples are compared with exact solutions only at m specific points. The advantages of these methods are the low cost of setting up the equations without applying any projection methods and finding a global approximation rather than a discrete one on large intervals.

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