Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 14, No. 1, 2022 Article ID IJIM-1451, 10 pages
DOI: http://dx.doi.org/10.30495/ijim.2022.19236
Research Article

Science and Research Branch (IAU)

An Approximate Method for Solving Space-Time Fractional Advection-Dispersion Equation

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Received Date: 2020-07-16 Revised Date: 2021-07-15 Accepted Date: 2021-10-02

Abstract

The present work reports an attempt to show the plausibilility of applying fuzzy transform method (FTM) to work out an approximate solution for space-time differential of non integer order which is named advection-dispersion equation (STFADE). From this perspective, the derivatives of non integer order are viewed from the Caputo sense. In approximate approaches, just a certain limitted number of points are utilized to work out an approximate function for a given distance. Nontheless, the main reason why the F-transform is prefered to other similar approaches is that it takes advantage of whole the existing points in the distance. The achieved numerical results indicate that the suggested algorithm results in appropriate approximate solutions.

Keywords : Fuzzy-transform; Space-time fractional differential equation; Advection-dispersion differential equation; Caputo derivative.

1 Introduction

It is a well-known fact that Fractional arithmetic and fractional differential equations have an important part to play many scientific areas such as medical science [14], economic studies [26], dynamical science [4, 20], mathematical physics [7], the field of chemistry [24], patterns of traffic [19] and fluid flow [17], to name a few. It is suggested those who are interested, including researchers, have a review of the papers and books that are available in order to get a full grasp of fractional arithmetic [1, 11]. Numerical methods have been utilized by scholars and scientists in order to work out a way to solve equations of the STFADE [30, 31, 6]. We have attempted in the present research report to prove the plausibility of applying F-transform method (FTM) to address non-homogeneous STFADE of the following form:

$$D_t^{\sigma} u(\omega,t) + \nu D_{\omega}^{\beta} u(\omega,t) - \kappa D_{\omega}^{\gamma} u(\omega,t) = r(\omega,t), \tag{1.1}$$

$$0 < \omega \leq L, 0 < t \leq T,$$

exposed to the following condition:

$$u(\omega,0) = g(\omega), \quad u(0,t) = h(t),$$

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$$u_\omega(0,t) = k(t), \tag{1.2}$$

where $r(\omega,t)$ is given, $\gamma \leq 1,\ 0 < \sigma,\ 1 < \beta \leq 2$, u is solute concentration, the positive constants ν and κ stand for the median velocity of fluid and the coefficience of dispersion, in the order mentioned, t is time, ω is the spatial domain and parameter σ illustrates the sequence of derivative in time, β and γ are parameters signifying the sequence of the space derivatives of non integer order. Based on time Caputo differential of order σ the definition is [11]

$$D_t^\sigma u(\omega,t) = \frac{1}{\Gamma(m-\sigma)}\int_0^t (t-s)^{m-\sigma-1}\times u^{(m)}(\omega,s)ds, \tag{1.3}$$
$$m-1 < \sigma \leq m,\ m \in \mathbb{Z}^+.$$

The STFADE is an extension of the classical advection-dispersion equation which includes non-integer differential operators in time and/or space. On the importance and motivation for STFADE, it is emphasized that it has a siginificant part to play in a number of physical phenomena, chemical models and several other similar scientific areas. The exemplary applications covers scientific areas of studies such as patterns of transport diffusive problems [8], illustration of dynamical transport in intricate systems controled via atypical dispersion andrelaxation [16], hydrology to simulate solute movement [34], estimation of the outcomes of cold-water injection into an advection dominated geothermal tank in fault-related structures in geothermal domains [28], particles motion in crowded cellular environments [27], charge carrier transport in disordered semiconductors [32], passive tracers transfer conducted via fluid flow in a porous medium for the groundwater hydrology analysis [2]. In a series of studies, the authors in [9, 10, 29] have utilized the FTM to work out a solution with approximate figures not only for the equations of first order fuzzy differential nature but also for problems of two-point boundary values. Following those research works, Chen and his colleagues [3] have devised an algorithm to attain the approximate solutions of initial value problems. It should also be emphasized that within the recent two decades researchers have employed certain schemes to work out solutions for space-time advection-dispersion equation feacharing of differential of integer and non integer orders. A number of similar approaches can be mentioned in this regard which encompasses finite difference method [15], Adomian's decomposition method [5], variational iteration method [18], homotopy perturbation method [33], homotopy analysis method [22], to name a few [12, 25, 35].

2 Discretization of the Caputo derivative

When we consider the discretization of the Caputo derivative [13] of Eq. (1.3), the following is achieved:

$$D^\vartheta u(t_{k+1}) \approx \frac{1}{\tau^\vartheta \Gamma(2-\vartheta)}\times \sum_{j=0}^{k}(u(t_{j+1})-u(t_j))\times \left((k-j+1)^{1-\vartheta}-(k-j)^{1-\vartheta}\right), \tag{2.4}$$

in which $0 < \vartheta \leq 1$, $u(t_0) = u_0$ and

$$D^\vartheta u(t_{k+1}) \approx \frac{1}{\tau^\vartheta \Gamma(3-\vartheta)} \sum_{j=0}^{k}(u(t_{j+1})-2u(t_j)+u(t_{j-1})) \left((k-j+1)^{2-\vartheta}-(k-j)^{2-\vartheta}\right), \tag{2.5}$$

in which $1 < \vartheta \leq 2$, $u(t_{-1}) = u(t_0) - \tau\ u_t(t_0)$.

3 Basics of Fuzzy partition and Fuzzy transform

For the rest of the paper, just the major definitions of F-transform will be used to elaborate the numerical implementations.

Definition 3.1. *[21] If we consider that $a = t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ be some given nodes, it can be concluded that that fuzzy sets*

$X_1, \cdots, X_n$ in $[a, b]$ with their membership functions $X_1(t), \cdots, X_n(t)$, for $n \geq 2$, build up a fuzzy partition of $[a, b]$ on condition that they bear the features below:

(1) X_k of $[a, b]$ to $[0, 1]$ is continuous, $\sum_{k=1}^{n} X_k(t) = 1$ for all $t \in [a, b]$ and $X_k(t_k) = 1$, $k = 1, 2, \cdots, n$.

(2) $X_k(t) = 0$ if $t \notin (t_{k-1}, t_{k+1})$, with $t_0 = a$ and $t_{n+1} = b$,

(3) On $[t_{k-1}, t_{k+1}]$, for $k = 2, \cdots, n-1$, $X_k(t)$, there surely exists a growing function on $[t_{k-1}, t_k]$ and a declining function on $[t_k, t_{k+1}]$.

The membership functions $X_1, X_2, \cdots, X_n$ are dubbed basic functions (BFs). The proceeding formulas are triangular membership functions:

$$X_1(t) = \begin{cases} 1 - \frac{t-t_1}{h_1}, & t_1 \leq t \leq t_2 \\ 0, & otherwise, \end{cases}$$

$$X_k(t) = \begin{cases} \frac{t-t_{k-1}}{h_{k-1}}, & t_{k-1} \leq t \leq t_k \\ 1 - \frac{t-t_k}{h_k}, & t_k \leq t \leq t_{k+1}, \\ 0, & otherwise, \end{cases}$$

$$X_n(t) = \begin{cases} \frac{t-t_{n-1}}{h_{n-1}}, & t_{n-1} \leq t \leq t_n, \\ 0, & otherwise. \end{cases} \tag{3.6}$$

The proceeding formulas for $k = 2, \cdots, n-1$ are sinusoidal membership functions:

$$X_1(t) = \begin{cases} 0.5\left(1 + cos\frac{\pi}{h}(t - t_1)\right), \\ t_1 \leq t \leq t_2 \\ 0, \\ \quad otherwise, \end{cases}$$

$$X_k(t) = \begin{cases} 0.5\left(1 + cos\frac{\pi}{h}(t - t_k)\right), \\ t_{k-1} \leq t \leq t_{k+1}, \\ 0, \\ \quad otherwise, \end{cases}$$

$$X_n(t) = \begin{cases} 0.5\left(1 + cos\frac{\pi}{h}(t - t_n)\right), \\ t_{n-1} \leq t \leq t_n \\ 0, \\ otherwise, \end{cases} \tag{3.7}$$

in which $h_k = t_{k+1} - t_k$ for $k = 1, \cdots, n-1$. It can be argued that fuzzy component of $[a, b]$, is uniform in case $t_{k+1} - t_k = h = \frac{b-a}{n-1}$ and two additional characteristics coincide:

(4) $X_k(t_k - t) = X_k(t_k + t)$, for all $t \in [0, h]$, for $k = 2, \cdots, n-1$,

(5) $X_k(t) = X_{k-1}(t - h)$ and $X_{k+1}(t) = X_k(t - h)$, for $k = 2, \cdots, n-1$, and $t \in [t_k, t_{k+1}]$.

Definition 3.2. *[21] Imagine f be any function related to $C([a, b])$, be the BFs which builds up a fuzzy partition of $[a, b]$. Then, the n-tuple $[F_1, F_2, \cdots, F_n]$ can be defined by real numbers attained by*

$$F_k = \frac{\int_a^b r(t) X_k(t) dt}{\int_a^b X_k(t) dt}, \quad k = 1, 2, \cdots, n, \tag{3.8}$$

as the F-transform of f with regard to $X_1, X_2, \cdots, X_n$.

Definition 3.3. *[21] Presupposing $[F_1, F_2, \cdots, F_n]$ are considered as F-transform of function f relative to BFs, $X_1, X_2, \cdots, X_n$. Then, on $[a, b]$*

$$f_n(t) = \sum_{k=1}^{n} F_k X_k(t),$$

is termed as the inverted F-transform (IFT) of function f.

Theorem 3.1. *[21] Imagine f be a continuous function on $[a, b]$ and $X_1, X_2, \cdots, X_n$ be the BFs which builds up a fuzzy member of $[a, b]$. Following that, the kth memebr of the integral F-transform signified over $[r(a), r(b)]$, gives the lowest figure to the function*

$$\phi(y) = \int_a^b (r(t) - y)^2 X_k(t) dt.$$

Lemma 3.1. *[21] Assuming that f be a continuous function on $[a, b]$. Hence, for $\epsilon > 0$, there exist n and a fuzzy partition such that for all $t \in [a, b]$*

$$|r(t) - f_{n_\epsilon}(t)| \leq \epsilon. \tag{3.9}$$

Definition 3.4. *Assuming that u be an arbitrary continuous function in $\mathfrak{D} = [a, b] \times [c, d]$ and let $Y_0, Y_1, \cdots, Y_n$ be BFs in $[a, b]$ and $X_0, X_1, \cdots, X_m$*

be BFs in $[c,d]$ builds up uniform fuzzy partitions, it can be stated the real matrix $U_{n\times m}$ with entries

$$U_{p,k} = \frac{\int_a^b \int_c^d u(\omega,t)Y_p(\omega)X_k(t)dxdt}{\int_a^b \int_c^d Y_p(\omega)X_k(t)dxdt}, \quad (3.10)$$

named the F-transform of $u(\omega,t)$ regard to the given fuzzy partitions, and the matrix entries are termed as the constituents of the F-transform of $u(\omega,t)$.

Evidently to transform the F-transform back, the proceeding inverted F-transform formula can be used [23]

$$u_{n,m}(\omega,t) = \sum_{p=0}^{n}\sum_{k=0}^{m} U_{p,k}Y_i(\omega)X_j(t). \quad (3.11)$$

It is evident that $u_{n,m}(\omega_p,t_k) = U_{p,k}$ for $p = 0,1,\cdots,n$ and $k = 0,1,\cdots,m$.

4 Presentation of the new approach

Presupposing that u be the continuous solution of (1.1) on $\mathfrak{D} = [0,L]\times[0,T]$ satisfying (1). Also, Matrix Unm of F-transform $u(\omega,t)$ in (3.10) in which $U_{n\times m} = [U_{i,j}]_{n\times m}$, calculated by using BFs, $Y_0, Y_1, \cdots, Y_n$ in $[0,L]$ with $x_{k+1} - x_k = \tau$ and BFs, $X_0, X_1, \cdots, X_m$ in $[0,T]$ with $t_{p+1} - t_p = h$ which are uniform fuzzy partitions. Accordingly for gives the approximation $u_{n,m}(\omega,t)$, through using inverse F-transform on the function u

$$u_{n,m}(\omega,t) = \sum_{p=0}^{n}\sum_{k=0}^{m} U_{p,k}Y_p(\omega)X_k(t). \quad (4.12)$$

Then, we should calculate $U_{p,k}$ for $p = 0,1,2,\cdots,n$ and $k = 0,1,2,\cdots,m$. In the proceeding proposition for Eqs.(2.4)-(2.5), the discretization of the Caputo derivative of $u_{n,m}(\omega,t)$ are presented.

Proposition 4.1. *With replacement of $u_{n,m}(\omega,t)$ in Eqs. (2.4)-(2.5), we will have the next equations, respectively:*

$$\begin{aligned} D_t^{\vartheta} u_{n,m}(\omega,t_{k+1}) \approx{}& \frac{1}{h^{\vartheta}\Gamma(2-\vartheta)}\times \\ & \sum_{j=0}^{k} ((U_{j+1}(\omega) - U_j(\omega))\times \\ & \left((k-j+1)^{1-\vartheta} - (k-j)^{1-\vartheta}\right), \\ & 0<\vartheta\le 1, \end{aligned} \quad (4.13)$$

$$\begin{aligned} D_\omega^{\vartheta} u_{n,m}(\omega_{p+1},t) \approx{}& \frac{1}{\tau^{\vartheta}\Gamma(2-\vartheta)} \\ & \sum_{i=0}^{p} \left(U^{i+1}(t) - U^{i}(t)\right) \\ & \left((p-i+1)^{1-\vartheta} - (p-i)^{1-\vartheta}\right), \\ & 0<\vartheta\le 1, \end{aligned} \quad (4.14)$$

where $h = t_{j+1} - t_j$, $\tau = x_{i+1} - x_i$ and $U_j(\omega) := U(\omega,t_j)$, $U^i(t) := U(\omega_i,t)$ for $k = 0,1,2,\cdots,m-1$ and $p = 0,1,2,\cdots,n-1$. And

$$\begin{aligned} D^{\vartheta} u_{n,m}(\omega_{p+1},t) \approx{}& \frac{1}{\tau^{\vartheta}\Gamma(3-\vartheta)}\times \\ & \sum_{i=0}^{k} \left(U^{i+1}(t) - 2U^{i}(t) + U^{i-1}(t)\right)\times \\ & \left((p-i+1)^{2-\vartheta} - (p-i)^{2-\vartheta}\right), \end{aligned} \quad (4.15)$$

where $1 < \vartheta \le 2$, $\tau = x_{i+1} - x_i$, $U^{-1}(t) = u(\omega_0,t) - \tau\, u_t(\omega_0,t)$ and $U^i(t) := U(\omega_i,t)$ for $p = 0,1,2,\cdots,n-1$.

4.1 Approximate solution of STFADE

For the purpose of attaining the approximate solution of (1.1), one can make utilize of $u_{n,m}(t)$; therefore,

$$\begin{aligned} D_t^{\sigma} u_{n,m}(\omega,t) &+ \nu D_\omega^{\beta} u_{n,m}(\omega,t) \\ &- \kappa D_\omega^{\gamma} u_{n,m}(\omega,t) = r(\omega,t), \\ & x>0,\ t>0, \end{aligned} \quad (4.16)$$

in which $0<\sigma,\ \gamma\le 1$, $1<\beta\le 2$, and by putting $x = x_{p+1}$ and $t = t_{k+1}$, we have

$$\begin{aligned} D_t^{\sigma} u_{n,m}(\omega_{p+1},t_{k+1}) &+ \nu D_\omega^{\beta} u_{n,m}(\omega_{p+1},t_{k+1}) \\ &- \kappa D_\omega^{\gamma} u_{n,m}(\omega_{p+1},t_{k+1}) \\ &= r(\omega_{p+1},t_{k+1}). \end{aligned} \quad (4.17)$$

Taking into account Caputo's derivative, using Eqs.(4.13), (4.14) and (4.15), Eq.(4.17) converts to the next form

$$\frac{1}{h^{\sigma}\Gamma(2-\sigma)}\sum_{j=0}^{k}((k-j+1)^{1-\sigma} - (k-j)^{1-\sigma})(U_{p+1,j+1}-U_{p+1,j}) + \frac{\nu}{\tau^{\beta}\Gamma(3-\beta)}\sum_{i=0}^{p}((p-i+1)^{2-\beta} - (p-i)^{2-\beta})(U_{i-1,k+1}+U_{i+1,k+1} - 2U_{i,k+1}) - \frac{\kappa}{\tau^{\gamma}\Gamma(2-\gamma)}\times \sum_{i=0}^{p}\left((p-i+1)^{1-\gamma}-(p-i)^{1-\gamma}\right)\times (U_{i+1,k+1}-U_{i,k+1}) = r(\omega_{p+1},t_{k+1}), \tag{4.18}$$

in which $0<\sigma,\ \gamma\leq 1,\ 1<\beta\leq 2,\ k = 0,1,2,\cdots,m-1,\ i=0,1,2,\cdots,n-1$ featuring boundary conditions $U_{i,0}=g(\omega_i),\ U_{0,j}=h(t_j),\ U_{-1,j}=U_{0,j}-\tau\, k(t_j)$. An algorithm for approximation of FBDEs by this approach will be postulated in the proceeding Algorithm.

Algorithm *1.* An algorithm for approximation of STFADE

Step 1. Given $r(\omega,t)$, ν, κ, m, n, L and T.

Step 2. Determine $\tau\leftarrow\frac{L}{n}$ and $h\leftarrow\frac{T}{m}$.

Step 3. set $x_p\leftarrow p,\ \tau,\ p = 0,1,2,\cdots,n,\ t_k\leftarrow k,\ h,\ k = 0,1,2,\cdots,m.$

Step 4. Select sinusoidal $Y_p(\omega)$ for $p = 0,1,2,\cdots,n$ and *BFs* X_k, for $k = 0,1,2,\cdots,m.$

Step 5. Set recursive equation

$$\frac{1}{h^{\sigma}\Gamma(2-\sigma)}\sum_{j=0}^{k}((k-j+1)^{1-\sigma} - (k-j)^{1-\sigma}big)(U_{p+1,j+1}-U_{p+1,j}) + \frac{\nu}{\tau^{\beta}\Gamma(3-\beta)}\sum_{i=0}^{p}((p-i+1)^{2-\beta} - (p-i)^{2-\beta})(U_{i-1,k+1}+U_{i+1,k+1} - 2U_{i,k+1}) - \frac{\kappa}{\tau^{\gamma}\Gamma(2-\gamma)}\times \sum_{i=0}^{p}\left((p-i+1)^{1-\gamma}-(p-i)^{1-\gamma}\right)\times (U_{i+1,k+1}-U_{i,k+1}) = r(\omega_{p+1},t_{k+1}).$$

in which $0<\sigma,\ \gamma\leq 1,\ 1<\beta\leq 2,$, $k=0,1,2,\cdots,m-1,\ p=0,1,2,\cdots,n-1$ featuring boundary condition $U_{i,0}=g(\omega_i)$, $U_{0,j}=h(t_j),\ U_{-1,j}=U_{0,j}-\tau\, k(t_j)$.

Step 6. Calculate every $U_{p,k}$, $p=0,1,2,\cdots,n$, $k=0,1,2,\cdots,m$.

Step 7. Via *IFT* the approximate solution is

$$u_{n,m}(\omega,t)=\sum_{p=0}^{n}\sum_{k=0}^{m}U_{p,k}Y_p(\omega)X_k(t).$$

5 Examples

In the proceeding part, some exemplary instances will be outlined in order to indicate FTM for STFADE. The software *Mathematica* has been utilized for all the exemplary instances mentioned below.

Example 5.1. *For the first example, the STFADE below is proposed:*

$$D_t^{\sigma}u(\omega,t)+D_{\omega}^{\beta}u(\omega,t)-D_{\omega}^{\gamma}u(\omega,t) = \frac{\pi\csc(\pi\sigma)}{\Gamma(-\sigma)}-\frac{\pi\csc(\pi\beta)}{\Gamma(-\beta)} - \frac{\beta\Gamma(\beta)\sigma^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}, \tag{5.19}$$

in which $0<\sigma,\ \gamma\leq 1$ *and* $1<\beta\leq 2$, *with the succinct solution* $u(\omega,t)=-\gamma^3-t^{\sigma}-\sigma^{\beta}$ *and the* *conditions given initially:*

$$u(\omega,0) = -\gamma^3 - \sigma^\beta, \quad u(0,t) = -\gamma^3 - t^\sigma,$$
$$u_\omega(0,t) = 0. \tag{5.20}$$

On the basis of the FTM, in reference to what has been postulated and stated in part 4 for Eqs. (5.19)-(5.20), one may calculate $U_{p,k}$ for $p = 1,2,\cdots,n$ and $k = 1,2,\cdots,m$ then gain the approximate solution $u_{n,m}(\omega,t)$ of (5.19). As can

Table 1: Results of example 5.1 bearing $\sigma = \gamma = 0.6$ and $\beta = 1.6$.

		FTM			
ω	t	n=50	n=100	n=150	Exact
0.0	0.0	-0.216	-0.216	-0.216	-0.216
0.2	0.2	-0.656772	-0.664785	-0.667457	-0.672877
0.4	0.4	-1.01745	-1.02045	-1.02154	-1.02391
0.6	0.6	-1.39401	-1.39349	-1.39344	-1.39364
0.8	0.8	-1.80182	-1.79567	-1.7938	-1.79044

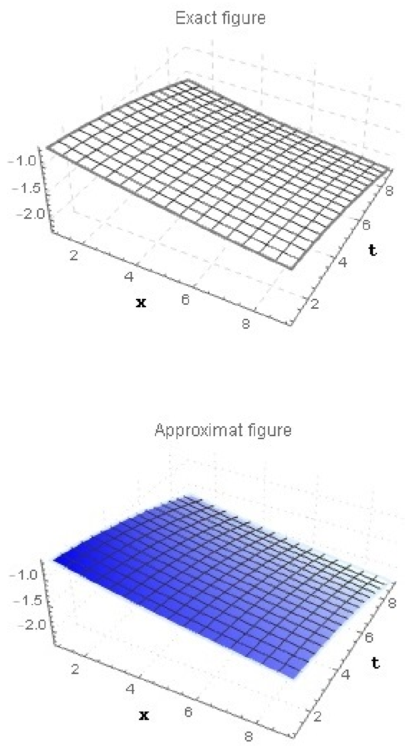

Figure 1: Comparison between the exact and F-transform of example 5.1 for $n = 150$, $\tau = 0.002$ and different values of ϑ and t.

be seen in Table 1, a comparison is given as to the succinct and the F-transform of example 5.1 for $\sigma = \gamma = 0.6$, $\beta = 1.6$ and the different values given for σ, t and n. Also in Figure 1, there can be seen a comparison of the succinct and approximate solution for equations with $\sigma = \gamma = 0.6$, $\beta = 1.6$, $n = 150$, and the different values given for σ and t.

Example 5.2. *Consider the STFADE:*

$$D_t^\sigma u(\omega,t) + D_\omega^\beta u(\omega,t) - D_\omega^\gamma u(\omega,t) =$$
$$1 + \frac{\sigma^\gamma}{\Gamma(\gamma+1)} - \frac{\sigma^\beta}{\Gamma(\beta+1)}, \tag{5.21}$$

in which $0 < \sigma,\ \gamma \le 1$ and $1 < \beta \le 2$, including the conditions given initially:

$$u(\omega,0) = -\frac{\sigma^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)},$$
$$u(0,t) = \frac{t^\sigma}{\Gamma(\sigma+1)}, \quad u_\omega(0,t) = 0. \tag{5.22}$$

On the basis of what was attained in section 4 for Eqs.(5.21)-(5.22), the unknown coefficient $U_{p,k}$ for $p = 1,2,\cdots,n$ and $k = 1,2,\cdots,m$ has been calculated with due attention to the FTM.

In Table 2, a comparison is presented as to the succinct and approximate solution for equations (5.21)-(5.22) with $n = 150$ and the different values given for σ and t, bearing $\sigma = \gamma = 0.6$ and $\beta = 1.6$. Regarding Figure 2, it is noted that a

Table 2: Results of example 5.2 bearing $\sigma = \gamma = 0.6$ and $\beta = 1.6$.

		FTM			
ω	t	n=50	n=100	n=150	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.2	0.391378	0.0174797	0.0174416	0.0173947
0.4	0.4	0.574046	0.080709	0.0804221	0.0799251
0.6	0.6	0.67538	0.197142	0.196391	0.195022
0.8	0.8	0.714371	0.372563	0.367439	0.367239

comparison is presented as to the succinct and the F-transform of example 5.2 for the different values given for σ and t, with $\sigma = \gamma = 0.6$ and $\beta = 1.6$. Toward $\sigma = \gamma = 0.6$ and $\beta = 1.6$, the solution that has been attained conforms to the succinct solution $u(\omega,t) = \frac{t^\sigma}{\Gamma(\sigma+1)} - \frac{\sigma^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}$.

Example 5.3. *Consider the STFADE:*

$$D_t^{\sigma}u(\omega,t)+D_{\omega}^{\beta}u(\omega,t)-D_{\omega}^{\gamma}u(\omega,t)=\frac{\pi\gamma\csc(\pi\beta)t^{\sigma}}{\Gamma(-\beta)}+\frac{\gamma\Gamma(\beta+1)t^{\sigma}\sigma^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}-\frac{\pi\gamma\csc(\pi\sigma)\sigma^{\beta}}{\Gamma(-\sigma)}, \tag{5.23}$$

in which $0<\sigma,\ \gamma\leq 1$ and $1<\beta\leq 2$, bearing

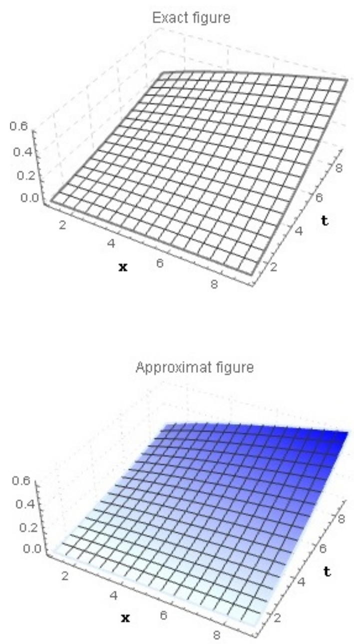

Figure 2: Comparison between the exact and F-transform of example 5.2 for $\sigma=\gamma=0.6$ and $\beta=1.6$, $n=150$ and various values of ω and t.

the conditions given initially:

$$u(\omega,0)=0,\ u(0,t)=0,$$
$$u_{\omega}(0,t)=0. \tag{5.24}$$

The solution that has been resulted here bears the equivalent values as to the succinct solution in a closed configuration $u(\omega,t)=t^{\sigma}\sigma^{\beta}\gamma$, which itself is the exact same approximate solution for Eq. (5.23). In Table 3, one can observe the approximate solutions which have been attained for different values assumed for t. Fig. 3 shows a comparison between the succinct solution and the approximation with F-transform for various values of $\sigma=0.7$, $\gamma=0.8$ and $\beta=1.9$.

Table 3: Results of example 5.3 bearing $\sigma=0.7$, $\gamma=0.8$ and $\beta=1.9$.

		FTM			
ω	t	n=50	n=100	n=150	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.2	0.0127597	0.0124554	0.0123571	0.0121834
0.4	0.4	0.0764292	0.0750537	0.0746256	0.0738662
0.6	0.6	0.21778	0.214674	0.213707	0.211974
0.8	0.8	0.461283	0.454188	0.451955	0.447841

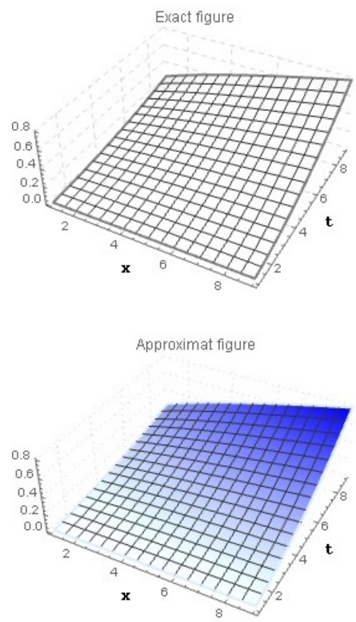

Figure 3: Comparison between the F-transform of example 5.3 for $\sigma=0.7$, $\gamma=0.8$ and $\beta=1.9$, $n=150$ and various values of ω and t.

Example 5.4. *For the fourth example, , the STFADE is proposed:*

$$D_t^{\sigma}u(\omega,t)+D_{\omega}^{\beta}u(\omega,t)-D_{\omega}^{\gamma}u(\omega,t)=\frac{2\sigma\Gamma(2\sigma)t^{\sigma}}{\Gamma(\sigma+1)}+\frac{\Gamma(3\beta+1)\sigma^{3\beta-\gamma}}{\Gamma(3\beta-\gamma+1)}-\frac{\Gamma(3\beta+1)\sigma^{2\beta}}{\Gamma(2\beta+1)}, \tag{5.25}$$

in which $0<\sigma,\ \gamma\leq 1$ and $1<\beta\leq 2$, bearing the conditions given initially:

$$u(\omega,0)=\gamma+\sigma^{3\beta},$$
$$u(0,t)=\gamma+t^{2\sigma},\quad u_{\omega}(0,t)=0. \tag{5.26}$$

The attained solution in this example is similar to the succinct solution in a closed configuration $u(\omega,t)=\gamma+t^{2\sigma}+\sigma^{3\beta}$, which is exactly the same approximate solution for Eq. (5.25).

In Table 4, there can be observed the approximate solutions attained for different values given for ω and t. In Table 4, the approximate solutions which are derived for different values of ω and t, can be seen. The approximation

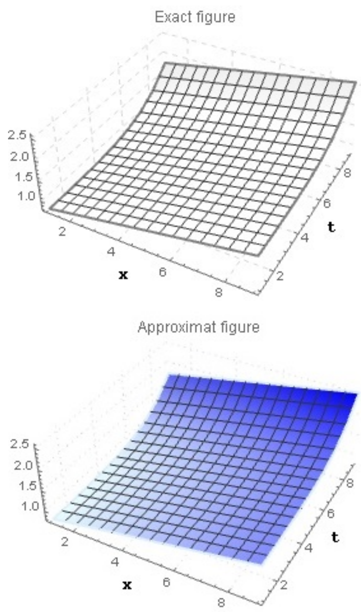

Figure 4: Comparison between the exact and the approximation solution with F-transform of example 5.4 for $\sigma = 0.7$, $\gamma = 0.9$ and $\beta = 1.8$, $n = 150$ and various values of ω and t.

Table 4: The exact and approximate results of example 5.4 bearing $\sigma = 0.7$, $\gamma = 0.9$ and $\beta = 1.8$.

ω	t	*FTM* n=50	n=100	n=150	Exact
0.0	0.0	0.9	0.9	0.9	0.9
0.5	0.1	0.951536	0.957497	0.96449	0.963494
0.2	0.3	1.06965	1.07761	1.08625	1.08551
0.2	0.8	1.60311	1.61759	1.63237	1.63186
0.5	0.9	1.75609	1.77136	1.78672	1.78654

with F-transform for the different values given for $\sigma = 0.7$, $\gamma = 0.9$ and $\beta = 1.8$, is presented in Fig.4.

6 Concluding remarks

The *FTM* has successfully been employed, in the present research report, to attain approximate solution of the STFADE. The results of the investigation demonstrate that a few iterations of *FTM* will outcome in fruitful approximate solutions. Ultimately, it has to be noted that what has been recommended in this approach has the capability to be practically employed as to finding solutions to other similar linear and nonlinear problems in partial differential equations bearing non integer derivative.

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