Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 12, No. 3, 2020 Article ID IJIM-1364, 9 pages
DOR: http://dorl.net/dor/20.1001.1.20085621.2020.12.3.2.2
Research Article

Science and Research Branch (IAU)

Design of A No-chatter Fractional Sliding Mode Control Approach for Stabilization of Non-Integer Chaotic Systems

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Received Date: 2019-08-17 Revised Date: 2020-04-01 Accepted Date: 2020-05-19

Abstract

A nonlinear chattering-free sliding mode control method is designed to stabilize fractional chaotic systems with model uncertainties and external disturbances. The main feature of this controller is rapid convergence to equilibrium point, minimize chattering and resistance against uncertainties. The frequency distributed model is used to prove the stability of the controlled system based on direct method of Lyapunov theory. Numerical simulations are presented to illustrate the effectiveness of the method.

Keywords : Fractional-order systems; Sliding mode control; Frequency distributed model; Stabilization.

1 Introduction

Historically, the emergence of fractional calculations has been consistent with the invention of the theory of integer calculus; But its applications and development are relate to recent decades [1]. In recent years, differential equations of fractional order have been used to represent and model more accurate real-world nonlinear systems. The control of these type of systems is one of the most important issues in the design of control systems. In addition to the high accuracy of modeling, the fractional equations, also have an interesting feature that linear fractional-order systems, in contrast to integer-order systems, can still remain stable despite the presence of poles on the right side of the complex plane [2]. In the case of nonlinear systems, the stability regions of the fractional systems are different from those with stable positions [3]. So, despite the extensive research and literature on controlling and stabilizing integer-order systems, the study of the stability and control of fractional systems requires separate studies and the design of new methods that have attracted many researchers in recent years [4].

The main technique for controlling and synchronization the fractional-order systems is the use of the Lyapunov direct method for the fractional mode which includes various control method such as sliding mode control [5], optimal control [6], adaptive control [7], nonlinear feedback control method [8] and fuzzy control methods [9].

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Sliding mode control (SMC) is a nonlinear control strategy expressing considerable properties such as robustness, accuracy, simple implementation and immutability to uncertainties. As is known, SMC includes two steps as follows.

- The first one is to design appropriate sliding surface.
- The second one is designing control input for the closed-loop system to change to the desired system specified by the sliding surface.

Varied techniques are available to control the time-finite fractional chaotic systems. The authors of [10] have studied the synchronization of two fractional chaotic systems with unknown parameters using a robust nonlinear adaptive controller and Khanzadeh used the time-variable sliding mode controller [11]. Wang has used the frequency distributed model to control nonlinear fractional-order [12].

But, most of the existing researches use the sign function in controller law which due to discontinuity, it causes chattering and severe fluctuations in the system, which causes the system to be depreciated and permanently degraded. This problem is fixed by replacing a continuous function instead of the sign function [13].

In this paper, we first consider a nonlinear fractional-order system with model uncertainties and external disturbances, then we introduce a suitable sliding surface and prove its stability using the frequency distributed model, using a suitable controller, we prove the convergence of all the system state trajectories to the sliding surface and due to the sliding surface stability, the stability of the initial system results. Finally illustrated examples and numerical simulations are presented to demonstrate the validity of the proposed method in comparison to existing methods. To sum up, the main advantages of this study are as follows:

i designing a novel switching sliding mode surface, which is appropriate for non-integer nonlinear systems;

ii the proposition of a robust switching control law, to force the system states to reach the proposed switching sliding surface;

iii the suggested control approach in this study is robust over the system uncertainties and external disturbances;

iv the suggested non-integer controller can be implemented for non-autonomous fractional-order nonlinear systems;

v the stability of both proposed sliding surface and the global stability of the closed-loop system are presented by using the fractional version of the Lyapunov stability theorem.

The rest of this paper is organized as follows. In Section 2, some preliminaries of fractional calculus are briefly reviewed. Also, the system description and problem formulation are given there. In Section 3, the design procedure of the proposed fractional-order sliding mode approach is presented. Section 4 gives two illustrative examples. Finally, concluding remarks are included in Section 5.

2 Preliminaries

2.1 Basic definitions

Definition 2.1 *The Riemann-Liouville fractional integration of order α is defined as follows [14]:*

$$
\begin{aligned}
{}_{t_0}I_t^{\alpha} f(t) &= {}_{t_0}D_t^{-\alpha} f(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau. \qquad (2.1)
\end{aligned}
$$

In which $\Gamma(.)$ is the Euler's gamma function.

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt. \qquad (2.2)$$

which has the following feature [12]:

$$\Gamma(z).\Gamma(1-z) = \frac{\pi}{\sin \pi z}; \; (0 < Re(z) < 1) \qquad (2.3)$$

Definition 2.2 *The αth order Caputo fractional derivative of a continuous function $f(t) : R^+ \to R$ is defined as [14]:*

$$
\begin{aligned}
{}_{t_0}{}^{C}D_t^{\alpha} f(t) &= {}_{t_0}D_t^{-(m-\alpha)} \frac{d^m}{dt^m} f(t) \\
&= \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{1-m+\alpha}} d\tau. \qquad (2.4)
\end{aligned}
$$

In this paper Caputo fractional derivative is used and D^{α} denote capuot fractional derivative.

2.2 System Descriptions

Because in practical applications, system dynamics is often influenced by model uncertainties and external disturbances [15], in this paper, we consider the following nonlinear fractional order system with model uncertainties and external disturbances:

$$\begin{cases} D^{\alpha}x_1(t) = f_1(t,x(t)) + \Delta f_1(t,x(t)) \\ \qquad +d_1(t) + u_1(t) \\ D^{\alpha}x_2(t) = f_2(t,x(t)) + \Delta f_2(t,x(t)) \\ \qquad +d_2(t) + u_2(t) \\ \qquad \vdots \\ D^{\alpha}x_n(t) = f_n(t,x(t)) + \Delta f_n(t,x(t)) \\ \qquad +d_n(t) + u_n(t) \end{cases} \tag{2.5}$$

Where $\alpha \in (0,1)$ is the order of system, $x(t) = [x_1, x_2, .., x_n]^T \in R^n$ denote the state vector, $f(t,x(t)) = [f_1(t,x(t)), f_2(t,x(t)), ..., f_n(t,x(t))]^T \in R^n$ is the given nonlinear function, $\Delta f(t,x(t)) = [\Delta f_1(t,x(t)), \Delta f_2(t,x(t)), ..., \Delta f_n(t,x(t))] \in R^n$ is the uncertainty term of system, $d(t) = [d_1(t), d_2(t), ..., d_n(t)] \in R^n$ is the external disturbance term of the system, and $u(t) = [u_1(t), u_2(t), ..., u_n(t)] \in R^n$ is the control input.

Assumption 1: In this paper, the uncertainty term and external disturbance are considered bounded as follows [17]:

$$|\Delta f(t,x(t))| + |d(t)| \leq \rho. \tag{2.6}$$

Where $\rho = [\rho_1, \rho_2, ..., \rho_n]$ is vector of positive constants.

Suppose that:

$$F_i(t,x(t)) = f_i(t,x(t)) + \Delta f_i(t,x(t)) + d_i(t) + u_i(t). \tag{2.7}$$

The fractional-order system (2.5) is equally written as [12]:

$$D^{\alpha}X(t) = F(t,x(t)). \tag{2.8}$$

Theorem 2.1 *Under the above assumptions, fractional-order system (2.8) can be written as[12]:*

$$\begin{cases} \frac{\partial\varphi(\omega,t)}{\partial t} &= -\omega^2\varphi(\omega,t) + F(t,x(t)), \\ X(t) &= \int_0^{\infty}\mu(\omega)\varphi(\omega,t)d\omega. \end{cases} \tag{2.9}$$

With $\mu(\omega) = \frac{2\sin(\pi\alpha)}{\pi}\omega^{1-2\alpha}$, $\alpha \in (0,1)$ *and* $\varphi(\omega,t) = \int_0^t e^{-\omega^2(t-\tau)}F(x,\tau)\,d(\tau)$.

Theorem 2.2 *(The Extended Second Method of Lyapunov) [16] : For* $t_0 = 0$, *the fractional-order system (2.5) is Mittag-Leffler stable at the equilibrium point* $\bar{x} = 0$ *if there exists a continuous function* $V(t,x(t))$ *satisfies:*

$$\begin{aligned} &\alpha_1 \|x(t)\|^a \leq V(t,(t)) \leq \alpha_2 \|x(t)\|^{ab}, \\ &{}_0D_t^{\beta}V(t,x(t)) \leq -\alpha_3 \|x(t)\|^{ab} \\ &(holding\,almost\,everywhere) \end{aligned} \tag{2.10}$$

Where $V(t,x(t)) : [0,\infty)\times D \to R$ *satisfies locally Lipschitz condition on* x, $\dot{V}(t,x(t))$ *is piecewise continuous, and* $\lim_{\tau\to t}\dot{V}(\tau,x(\tau))$ *exists for any* $t \in [0,\infty)$; $D \subset R^n$ *is a domain containing the origin and* $V(t,x(t)) = \lim_{\tau\to t}\dot{V}(\tau,x(\tau))$; $t \geq 0$, $\beta \in (0,1)$, $\alpha_1, \alpha_2, \alpha_3$, *a and b are arbitrary positive constants. If the assumptions hold globally on* R^n, *then* $\bar{x} = 0$ *is globally Mittag-Leffler stable.*

Theorem 2.3 *If* $x(t)$ *denotes a continuously differentiable function, the following inequality hold almost everywhere [16].*

$${}_0D_t^{\alpha}|x(t)| \leq sgn(x(t))_0D_t^{\alpha}x(t),\; 0 < \alpha < 1 \tag{2.11}$$

Where $x(t) = \lim_{\tau\to t}x(\tau)$.

3 Controller design

One of the ways to reduce the chattering is to use a continuous approximation for the discontinuous sliding mode controller. For this purpose, in this paper, to minimizing the chattering of the system in the design of the sliding surface as well as the controller, we use the continuous function $\tanh(x_i(t))$ instead of using the sign function [18]. For the fractional system (2.5), we define the sliding surface as follows:

$$S_i(t) = |x_i(t)| + \gamma_i x_i(t) + k_i D^{-\alpha}|x_i(t)|^{\mu}\tanh(x_i(t)). \tag{3.12}$$

Where $k_i > 0$, $\gamma_i > 1$, $0 < \mu < 1$.
Based on the SMC approach, the following equations are met when the system works on the sliding mode

$$\begin{aligned} S_i(t) = 0 \Rightarrow & D^\alpha S_i(t) = 0, \\ & \Rightarrow |x_i(t)| + \gamma_i x_i(t) \\ & \quad + k_i D^{-\alpha} |x_i(t)|^\mu \tanh(x_i(t)) = 0, \end{aligned}$$

Therefor dynamics of sliding surface (3.12) is obtained, by operating D^α on (3.12) as follows

$$\begin{aligned} \gamma_i x_i(t) = & - |x_i(t)| \\ & - k_i D^{-\alpha} |x_i(t)|^\mu \tanh(x_i(t)), \end{aligned}$$

then

$$\begin{aligned} \gamma_i D^\alpha x_i(t) = & - sgn(x_i(t)) D^\alpha x_i(t) \\ & - k_i |x_i(t)|^\mu \tanh(x_i(t)), \end{aligned}$$

Therefore,

$$\begin{aligned} & D^\alpha x_i(t)(\gamma_i + sgn(x_i(t))) \\ & \quad = -k_i |x_i(t)|^\mu \tanh(x_i(t)), \end{aligned}$$

Therefore,

$$D^\alpha x_i(t) = \frac{-k_i |x_i(t)|^\mu \tanh(x_i(t))}{\gamma_i + sgn(x_i(t))}. \tag{3.13}$$

Theorem 3.1 *If the sliding surface is chosen to form (3.12), then the sliding dynamics (3.13) will be stable and the state trajectories converge to zero.*

Proof: According to the theorem 1, the relation (3.13) can be converted into the frequency distributed model as follows:

$$\begin{cases} \frac{\partial \varphi(x,t)}{\partial t} = -\omega^2 \varphi(\omega,t) - \frac{k_i |x_i(t)|^\mu \tanh(x_i(t))}{\gamma_i + sgn(x_i(t))}, \\ x(t) = \int_0^\infty \mu(\omega) \varphi(\omega,t) \, d\omega. \end{cases} \tag{3.14}$$

To prove the stability of the sliding surface, using the direct method of the Lyapunov theory, it is sufficient to select a positive-definite function as Lyapunov function, and show that its derivative is a negative-definite function.We select the Lyapunov function as follows:

$$V(t) = \frac{1}{2} \int_0^\infty \mu(\omega) \varphi^2(\omega,t) \, d\omega. \tag{3.15}$$

Clearly, the above function is positive-definite. By deriving this function we have:

$$\begin{aligned} \dot{V}(t) = & \int_0^\infty \mu(\omega)\varphi(\omega,t) \frac{\partial \varphi(\omega,t)}{\partial t} d\omega, \\ = & - \int_0^\infty \mu(\omega)\varphi(\omega,t) \Big[\omega^2 \varphi(\omega,t) \\ & + \frac{k_i |x_i(t)|^\mu \tanh(x_i(t))}{\gamma_i + sgn(x_i(t))}\Big] d\omega \\ = & - \int_0^\infty \Big[\mu(\omega)\omega^2 \varphi^2(\omega,t) \\ & + \mu(\omega)\varphi(\omega,t) \frac{k_i |x_i(t)|^\mu \tanh(x_i(t))}{\gamma_i + sgn(x_i(t))}\Big] d\omega \\ = & - \int_0^\infty \mu(\omega)\omega^2 \varphi^2(\omega,t) d\omega \\ & - \frac{k_i |x_i(t)|^\mu \tanh(x_i(t))}{\gamma_i + sgn(x_i(t))} x_i(t). \end{aligned} \tag{3.16}$$

Note that $\tanh(x_i(t))$ for all $x_i(t) > 0$ is positive and for all $x_i(t) < 0$ have a negative value; then the expression $x_i(t) \tanh(x_i(t))$ will always be positive and, consequently, the derivative of the Lyapunov function is negative-definite.
Therefore, proof is complete.

Now, we design the control input as shown below and show that the system (2.5) with the designed control input converges to the sliding surface (3.12).

$$\begin{aligned} u_i(t) = -\Big(& f_i(t, x_i(t)) + \lambda_i \tanh(x_i(t)) + \rho_i \\ & + \frac{\tanh(x_i(t))(k_i |x_i(t)|^\mu + \beta_i |S_i(t)|^q)}{\gamma_i + sgn(x_i(t))}\Big) \end{aligned} \tag{3.17}$$

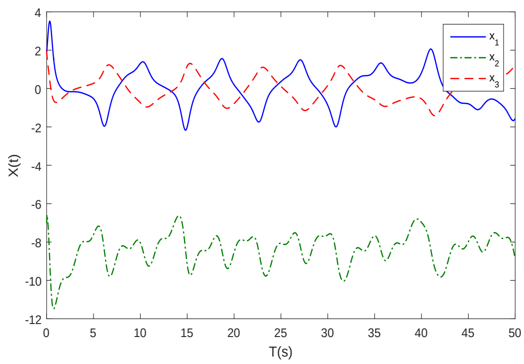

Figure 1: Uncontrolled fractional-order BLDCM system (4.21)

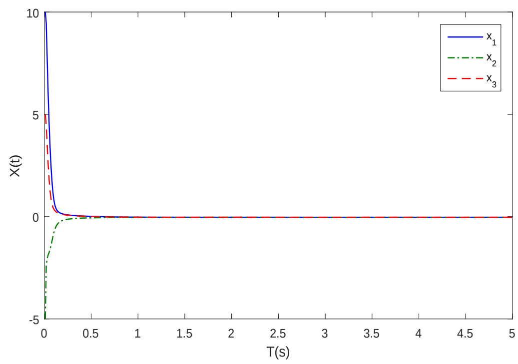

Figure 2: Controlled BLDCM system (4.21)

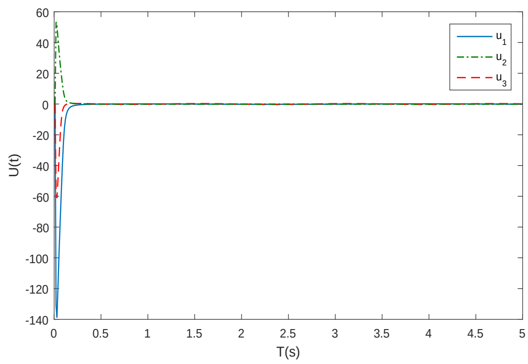

Figure 4: Controller signals (4.22) used for BLDCM system (4.21)

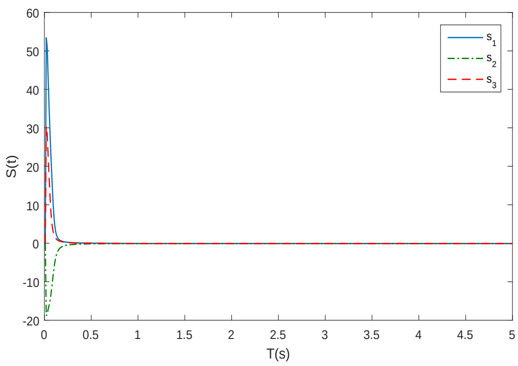

Figure 3: Stability of sliding surface (4.22)

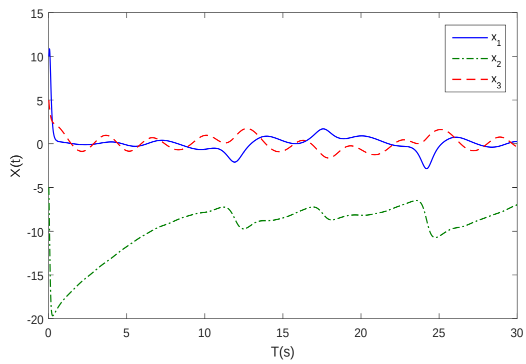

Figure 5: Uncontrolled FO economic system (4.25)

Theorem 3.2 *Consider the fractional system (2.5) with the condition (2.6) and the sliding surface (3.12). If the system is controlled by the controller (3.17), then the system state trajectories converge to the zero asymptotically.*

Proof: We define the Lyapunov function as

$$V(t, x(t)) = \|S(t)\|_1 = \sum_{i=1}^{n} |S_i(t)|. \qquad (3.18)$$

Now by taking fractional derivation from both side of above equality, we have

$$\begin{aligned} D^{\alpha}V(t, x(t)) &= D^{\alpha} \sum_{i=1}^{n} |S_i(t)| \\ &= \sum_{i=1}^{n} D^{\alpha} |S_i(t)|. \end{aligned} \qquad (3.19)$$

Using theorem 2.3, one gets

$$\begin{aligned} & D^{\alpha}V(t, x(t)) \left(\sum_{i=1}^{n} [sgn(S_i(t)) \, (sgn(x_i(t)) \right. \\ & \times D^{\alpha}x_i(t) + \gamma_i D^{\alpha}x_i(t) \\ & + k_i \, |x_i(t)|^{\mu} \tanh(x_i(t)))] \,, \\ & = sgn(S_i(t)) \, (D^{\alpha}x_i(t) \, (\gamma_i + sgn(x_i(t))) \\ & + k_i \, |x_i(t)|^{\mu} \tanh(x_i(t))) \,, \\ & = sgn(S_i(t)) \, [(f_i(t, x_i(t)) + \Delta f_i(t, x_i(t)) \\ & + d_i(t) + u_i(t) + k_i \, |x_i(t)|^{\mu} \tanh(x_i(t)))] \,, \end{aligned} \qquad (3.20)$$

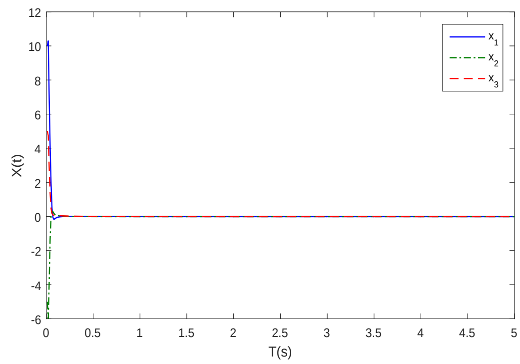

Figure 6: Controlled FO economic system (4.25)

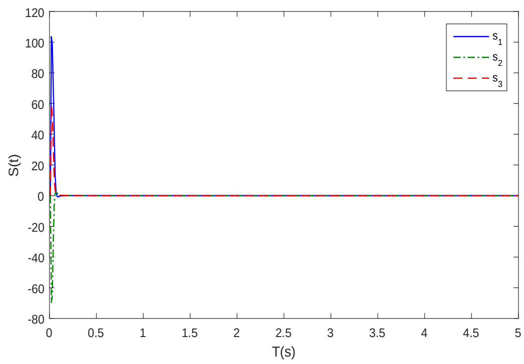

Figure 7: sliding surface of economic system (4.26)

Therefore,

$$
\begin{aligned}
&D^{\alpha}V(t,x(t)) \leq sgn(S_i(t))[(f_i(t,x_i(t)) + \rho_i \\
&- f_i(t,x_i(t)) - \lambda_i \tanh(x_i(t))b, \\
&- \frac{\tanh(x_i(t))(k_i\,|x_i(t)|^{\mu} + \beta_i\,|S_i(t)|^{q})}{\gamma_i + sgn(x_i(t))} - \rho_i) \\
&+ k_i\,|x_i(t)|^{\mu}\tanh(x_i(t))], \\
&= -\lambda_i \tanh(x_i(t))\,(\gamma_i + sgn(x_i(t))) \\
&- \beta_i\,|S_i(t)|^{q}\tanh(S_i(t)) - \lambda_i \tanh(x_i(t)) \\
&- \gamma_i.\lambda_i \tanh(x_i(t)) sgn(S_i(t)), \\
&= -\lambda_i \tanh(x_i(t))\,(1 + \gamma_i sgn(S_i(t))) < 0.
\end{aligned}
$$

Thus the proof is finished.

4 Numerical simulation

In this section, two practical examples are presented to illustrate the effectiveness of the proposed control method. The first example relates to control of fractional order Brushless DC Motor system (BLDCM) and the second example is the

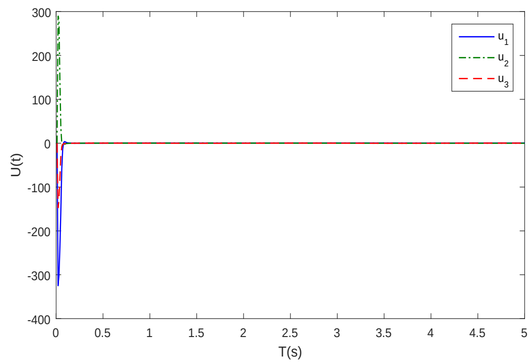

Figure 8: Controller signals (4.27), operated for Fo economic system (4.25)

control of fractional order economic system. Numerical simulations are carrieds out with MATLAB software.

Example 4.1 *Differential equations of the chaotic Brushless DC Motor system (BLDCM) are given as follows:*

$$
\begin{cases}
D^{\alpha}x_1(t) = & -0.875x_1(t) + x_2(t)x_3(t) \\
& +\Delta f_1(X\,,\,t) + d_1(t) + u_1 \\
D^{\alpha}x_2(t) = & -x_2(t) + 55x_3(t) - x_1(t)x_3(t) \\
& +\Delta f_2(X\,,\,t) + d_2(t) + u_2 \\
D^{\alpha}x_3(t) = & 4(x_2(t) - x_3(t)) \\
& +\Delta f_3(X\,,\,t) + d_3(t) + u_3
\end{cases}
\tag{4.21}
$$

In which the system's uncertainties and external disturbances have been introduced as follows:

$$
\begin{cases}
\Delta f_1 & (X\,,\,t) + d_1(t) = \\
& 0.2\cos(3t)x_1(t) - 0.15\sin(2t), \\
\Delta f_2 & (X\,,\,t) + d_2(t) = \\
& 0.25\sin(4t)x_2(t) - 0.2\sin(3t), \\
\Delta f_3 & (X\,,\,t) + d_3(t) = \\
& 0.3\sin(2t)x_3(t) - 0.25\cos(4t).
\end{cases}
$$

This system for $0.96 < \alpha \leq 1$ is chaotic. So, we consider $\alpha = 0.98$ and the initial states $x_1(0)$, $x_2(0)$ and $x_3(0)$ to be 10, -5 and 5, respectively. In Figure 1, the chaotic nature of the uncontrolled system is clearly visible.

In the design of the sliding surface, the parameters are selected as $\mu = [0.7, 0.7, 0.7]$, $\gamma = [2, 2, 2]$ and $k =$

[0.1, 0.1, 0.1].

$$S_i(t) = |x_i(t)| + 2x_i(t) + 0.1D^{-\alpha}|x_i(t)|^{0.7}\tanh(x_i(t)),\ i = 1, 2, 3. \tag{4.22}$$

Controller parameters are selected as $\rho = [0.2, 0.35, 0.35]^T$, $\lambda = [1.5, 1.5, 1.5]^T$, $q = [0.6, 0.6, 0.6]$. Then the controller's formula will be as follows.

$$\begin{aligned} u_1(t) = & -(-0.875x_1(t) + x_2(t)x_3(t) \\ & + \tfrac{1.5\tanh(x_1(t))\left(|x_1(t)|^{0.7}+|S_1(t)|^{0.6}\right)}{2+sgn(x_1(t))} \\ & +0.35 + 1.5\tanh(x_1(t))), \\ u_2(t) = & -(-x_2(t) + 55x_3(t) - x_1(t)x_3(t) \\ & + \tfrac{1.5\tanh(x_2(t))\left(|x_2(t)|^{0.7}+|S_2(t)|^{0.6}\right)}{2+sgn(x_2(t))} \\ & +0.45 + 1.5\tanh(x_2(t))), \\ u_3(t) = & -(4(x_2(t) - x_3(t)) \\ & + \tfrac{1.5\tanh(x_3(t))\left(|x_3(t)|^{0.7}+|S_3(t)|^{0.6}\right)}{2+sgn(x_3(t))} \\ & +0.55 + 1.5\tanh(x_3(t))) \end{aligned} . \tag{4.23}$$

The stability of the FOBLDCM system (4.21), using the proposed control input(4.23), is shown in Fig. 2.
Fig.3 shows the sliding surface (4.22) to control FOBLDCM system. It is seen that the sliding surface achieve to zero. The control signals (4.23) are shown in Fig 4. Clearly, the control efforts are feasible in practice, without damaging chattering actions.

Example 4.2 *Now we introduce a fractional-order economic system as follows:*

$$\begin{cases} D^{\alpha}x_1 = & x_3 + (x_2 + 9)x_1 \\ & +\Delta f_1(X, t) + d_1(t) + u_1, \\ D^{\alpha}x_2 = & -0.1x_2 - x_1^2 \\ & +\Delta f_2(X, t) + d_2(t) + u_2, \\ D^{\alpha}x_3 = & -x_1 - x_3 \\ & +\Delta f_3(X, t) + d_3(t) + u_3 \end{cases} \tag{4.24}$$

The three state variables, represent the rate of profit, the required capital and the price index, respectively. Figure 5 shows the instability of the uncontrolled system. In this system $\alpha = 0.98$ has been selected. In addition, we have introduced the system's uncertainties and external disturbances as follows:

$$\begin{cases} \Delta f_1 & (X, t) + d_1(t) = \\ & 0.1\sin(2t)x_1 + 0.1\cos(3t), \\ \Delta f_2 & (X, t) + d_2(t) = \\ & 0.2\sin(3t)x_3 - 0.15\sin(t), \\ \Delta f_3 & (X, t) + d_3(t) = \\ & -0.15\cos(2t)x_2 - 0.2\sin(3t). \end{cases} \tag{4.25}$$

In the design of the sliding surface, the parameters are selected as $\mu = [0.7, 0.7, 0.7]$, $\gamma = [2, 2, 2]$ and $k = [0.1, 0.1, 0.1]$.

$$S_i(t) = |x_i(t)| + 2x_i(t) + 0.1D^{-\alpha}|x_i(t)|^{0.7}\tanh(x_i(t)),\ i = 1, 2, 3. \tag{4.26}$$

To use the proposed controller to control the FO economic system (4.24), we select the controller parameters $\rho = [0.2, 0.35, 0.35]^T$, $\lambda = [1.5, 1.5, 1.5]^T$, $q = [0.6, 0.6, 0.6]$. In addition, the initial states are selected as $x_1(0) = 10$, $x_2(0) = -5$, $x_3(0) = 5$.

$$\begin{aligned} u_1(t) = & -(x_3 + (x_2 + 9)x_1 \\ & + \tfrac{1.5\tanh(x_1(t))\left(|x_1(t)|^{0.7}+|S_1(t)|^{0.6}\right)}{2+sgn(x_1(t))} \\ & +0.2 + 1.5\tanh(x_1(t))), \\ u_2(t) = & -(-0.1x_2 - x_1^2 \\ & + \tfrac{1.5\tanh(x_2(t))\left(|x_2(t)|^{0.7}+|S_2(t)|^{0.6}\right)}{2+sgn(x_2(t))} \\ & +0.35 + 1.5\tanh(x_2(t))), \\ u_3(t) = & -(-x_1 - x_3 \\ & + \tfrac{1.5\tanh(x_3(t))\left(|x_3(t)|^{0.7}+|S_3(t)|^{0.6}\right)}{2+sgn(x_3(t))} \\ & +0.35 + 1.5\tanh(x_3(t))) \end{aligned} . \tag{4.27}$$

Fig. 6 shows the state trajectories of this system after applying the controller. Obviously, all the state variables of the system converge to the equilibrium point and the system is stable. Fig. 7 shows the sliding surface (4.26) to control the economic system. Fig. 8 also shows the behavior of the controller (4.27). It is also seen that the controller components converge to zero at a very short time. So the controller will spend a little energy. So, the proposed method is consistent and can be applied in practice.

5 Conclusion

In this paper, a nonlinear controller method is proposed for the stabilization of chaotic fractional-order systems. The theoretical and analytical results in this paper are based on the direct method of the Lyapunov stability theory for fractional systems and with utilizing of frequency distributed model. Fast convergence to the point of equilibrium, stability and high resistance against the external disturbances and uncertainties of the system and to minimize chattering are the main features of this method. In addition, numerical examples to stabilize chaotic systems indicate the applicability of this method in practice.

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