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Research Article

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Provide a New Model for Evaluating and Ranking DMUs With Ordinal Data

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Abstract

Classic Data Envelopment Analysis expect that the values of inputs and outputs are precisely determined. However, in most real-life issues, the values of some of inputs and outputs are imprecise. One of them is ordinal data. In this article, a new model is presented for evaluating decision making units with ordinal data. The general idea of this model is assigning real values to ordinal data with new approach. Furthermore, another new model for ranking efficient units is presented with the main idea of changes in controlled efficiency. Then, the results with results Cooper's model are compared. Therefore, the efficiency scores obtained from proposed model are more realistic and reasonable than the results obtained from the Cooper's model.

Keywords : Evaluation; Efficiency; Ranking; Data Envelopment Analysis; Exact Data; Imprecise Data; Ordinal Data.

1 Introduction

Data Envelopment Analysis (DEA) is a non-parametric methodology for measuring the efficiency of Decision Making Units (DMUs) that use multiple inputs to outputs configurations [1]. DEA expect that the values of inputs and outputs are precisely determined. However, in most real-life issues, the values of some of inputs and outputs are imprecise. because they either are not known or cannot be exactly measured [2]. Cook et al. extend DEA for units with ordinal data [3]. Education level with values of the elementary school education, high school graduate, college graduate are typical ordinal data. Cooper et al. introduced Imprecise Data Envelopment Analysis (IDEA) model using interval and ordinal data [4]. By applying scale transformation technique accompanied with variable they can successfully transform the nonlinear model to a linear one. Zhu simplifies this model to reduce the Computational load [5]. Afterwards this model was developed by Despotis and Smirlis to find the efficiency intervals for units with interval data [6]. Chiang Kao presented an approach to calculate the efficiency intervals in DEA with imprecise data [3]. More information about the IDEA can be found at Cooper et al. [7, 8], Zhu [9, 10], Park [11],

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Wang et al. [12], Cook et al. [13], Farzipoor [14, 15], Asoahe [16], Shokouhi [17], Toloo et al. [18, 19]. Our discussion in this paper is calculating the efficiency and ranking of units with ordinal data. The basic concept for measuring the efciency of a set of DMUs with ordinal data is still the most favorable condition [1]. Let the last output be ordinal, we tend to assign positive real values to ordinal numbers, taking other factors into account, which will be obtained the highest possible efficiency score for the DMU being evaluated. Then, all factors will become real and positive and therefore, all conventional DEA methods can be used to solve the problem. Similar to Cooper's model, we use BCC input model. To evaluate the superiority of the proposed model over the Cooper's model, a numerical example is presented in section 6 and the results are reviewed. Charnes et al. in 1985 introduced the first ranking model [20]. Anderson and Peterson proposed another model where efficiency scores of efficient units can be greater than 1 [21]. More information about the rankings can be found at Jahanshahloo et al. [22, 23], Pourmahmoud [24], Pourmahmoud et al. [25], Adler et al. [26], Doyle et al. [27]. In this article, we also presented a model for ranking efficient units. The general idea of this model is that the efficiency of evaluation units should be allowed to be reduced to a controlled size. This model is given in Section 4. In the next section, a model for finding the most appropriate amount of efficiency reduction is presented. Then, a numerical example is given and the results of applying the mentioned methods are reviewed.

2 Cooper's model

Consider n DMUs with m inputs and s outputs. assuming that the final output of the units is ordinal, Cooper et al. (1999) proposed a model for evaluating the DMUs under the above conditions. They assigned positive real values to each of the ordinal numbers that will obtain the highest possible efficiency score for the DMU being evaluated. $y_k, k = 1, 2, ..., n$ with the following features was the value assigned by them to ordinal number$k$. $y_k > y_{k+1}, k = 1, 2, ..., n-1$.Since scale of had no effect on efficiency score, they assumed $y_1 = 1$. To make mathematical calculations possible, they replaced strong inequality of k. $y_k > y_{k+1}, k = 1, 2, ..., n-1$with the weak inequality of:

$$y_k \geq y_{k+1} + \delta, k = 1, 2, ..., n-1$$

where δ is a positive small number that is large enough to differentiate the order of y_k and y_{k+1}. Then they proposed the following model:

$$\begin{aligned}
&E_o = max \textstyle\sum_{r=1}^{s-1} u_r y_{ro} + u_s y_{[o]} - u_o \\
&s.t. \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} = 1 \\
&\textstyle\sum_{r=1}^{s-1} u_r y_{rj} + u_s y_{[j]} - u_o - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n \\
&y_1 = 1, y_k \geq y_{k+1} + \delta, k = 1, 2, ..., n-1 \\
&u_r, v_i, y_k \geq \epsilon, i = 1, ..., m, r = 1, ..., s, k = 1, ..., n \\
&u_o, free. \qquad (2.1)
\end{aligned}$$

The subscript [j] illustrate the order of DMU_j iny_s. $u_s.y_{[j]}$ has caused model 2.1 to be non-linear. By using variable substitution of $w_j = u_s.y_{[j]}$, model 2.1 is transformed in to the following linear program:

$$\begin{aligned}
&E_o = max \textstyle\sum_{r=1}^{s-1} u_r y_{ro} + w_{[o]} - u_o \\
&s.t. \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} = 1 \\
&\textstyle\sum_{r=1}^{s-1} u_r y_{rj} + w_{[j]} - u_o - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n \\
&w_1 = u_s, w_{kk+1} + u_s\delta, k = 1, ..., n-1 \\
&u_r, v_i, w_k \geq \epsilon, i = 1, ..., m, r = 1, ..., s, k = 1, ..., n \\
&u_o, free. \qquad (2.2)
\end{aligned}$$

This model can measure efficiency of units with ordinal data, but efficiency scores obtained by this model are very close to each other and therefore are superficial.

3 The proposed model

In this section, a new model is proposed for evaluating the efficiency of n DMUs where the final output of DMUs is ordinal. The efficiency scores obtained by Cooper's model are very close to each other and therefore are superficial. To deal with this challenge, we propose a new model for assigning real values to ordinal numbers, taking other output factors into account, that will be obtained the highest possible efficiency score for the DMU being evaluated. Let $y_k, k = 1, 2, ..., n$, with the following features be the real and positive value assigned to ordinal number k .

$$y_k > y_{k+1}, \qquad k = 1, 2, ..., n-1$$

Since scale of had no effect on efficiency score, we assumed $y_1 = 1$.
Now for r'th output,$r = 1, 2, ..., s-1$,of DMU_j, $j = 2, 3, ..., n$, define:

$$\hat{y}_{rj} = \frac{y_{rj}}{y_{rmax}}, \quad j = 2, ..., n \tag{3.1}$$

where y_{rmax} is the largest value of r'th output of DMU_j, $j = 2, 3, ..., n$. And for i'th input,$i = 1, 2, ..., m$ of DMU_j, $j = 2, 3, ..., n$, define:

$$\hat{x}_{ij} = \frac{x_{ij}}{x_{imin}}, \quad j = 2, ..., n \tag{3.2}$$

Where x_{imin} is the smallest value of i'th input of DMU_j, $j = 2, 3, ..., n$.
For DMU_j, $j = 2, 3, ..., n$, Suppose δ_j is defined as follows:

$$\delta_j = \frac{\prod_{r=1}^{s-1} \hat{y}_{rj}}{\prod_{i=1}^{m} \hat{x}_{ij}}, \quad j = 2, ..., n \tag{3.3}$$

Clearly the value of $\delta_j, j = 2, 3, ..., n$,is less than or equal to one. Now we replaced strong inequality of $y_k > y_{k+1}, k = 1, 2, ..., n-1$, with the weak inequality of $y_{k+1} \leq \delta_{k+1} y_k, k = 1, 2, ..., n-1$, and propose the following model to measure the efficiency of DMU_o:

$$E_o = max \sum_{r=1}^{s-1} u_r y_{ro} + u_s y_{[o]} - u_o$$

$s.t.$

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s-1} u_r y_{rj} + u_s y_{[j]} - u_o - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n$$

$$y_{k+1} \leq \delta_{k+1} y_k, k = 1, 2, ..., n-1$$

$$u_r, v_i, y_k \geq \epsilon, i = 1, ..., m, r = 1, ..., s, k = 1, ..., n$$

$$u_o, free. \tag{3.4}$$

The subscript [j] illustrate the order of DMU_j iny_s. $u_s.y_{[j]}$ has caused model 3.4 to be non-linear. By using variable substitution of $w_j = u_s.y_{[j]}$, model 3.4 is transformed in to the following linear program:

$$E_o = max \sum_{r=1}^{s-1} u_r y_{ro} + w_{[o]} - u_o$$

$s.t.$

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s-1} u_r y_{rj} + w_{[j]} - u_o - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n$$

$$w_1 = u_s, w_{k+1} \leq \delta_{k+1} w_k, k = 1, 2, ..., n-1$$

$$u_r, v_i, w_k \geq \epsilon, i = 1, ..., m, r = 1, ..., s, k = 1, ..., n$$

$$u_o, free. \tag{3.5}$$

In addition to being able to identify efficient DMUs with ordinal data, this model also provides more appropriate and acceptable efficiency scores for inefficient units, which is the result of considering other outputs in assigning real values to ordinal numbers. this will be explained in section 5 with a numerical example.

4 A new model for ranking efficient units with ordinal data

Non-vertex efficient units in the usual ranking models have the same rank and are equal to 1. Here, we will use the idea that efficiency scores of units in evaluation should be allowed to be reduced to a controlled size. the efficiency score of these units is calculated from model (3.5). Now we suggest the following model:

$$R_o = max \sum_{r=1}^{s-1} u_r y_{ro} + w_{[o]} - u_o$$

$s.t.$

$\sum_{i=1}^{m} v_i x_{io} = 1$

$\sum_{r=1}^{s-1} u_r y_{rj} + w_{[j]} - u_o - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n \quad j \neq o$

$w_1 = u_s, w_{k+1} \leq \delta_{k+1} w_k, k = 1, 2, ..., n-1$

$\sum_{r=1}^{s-1} u_r y_{rj} + w_{[j]} - u_o - (E_j^* - \alpha) \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n \quad j \neq o$

$u_r, v_i, w_k \geq \epsilon, i = 1, ..., m, r = 1, ..., s, k = 1, ..., n$

$u_o, free. \qquad (4.1)$

Where E_j^* is efficiency score of DMU_j Which is calculated from model 3.5 and $0 \leq \alpha < 1$. This model allows us to rank units by creating an artificial border by the most appropriate .

Theorem 4.1. *There is an α in$[0,1)$ for which the model 4.1 is feasible.*

Proof. The values assigned to the ordinal data are positive, assuming that the other inputs and outputs are also positive, Let:
$v_m = \frac{1}{x_{mo}}, v_1 = ... = v_{m-1} = 0$ so $vx_o = 1$. Now let $u_1 = ... = u_{s-1} = 0$, In this case, $w_{[j]} - u_o \leq \frac{x_{mj}}{x_{mo}}$ for $j = 1, 2, ..., n, j \neq o$, must be established.
By selecting w_j in the form below:

$$w_j = min\{\frac{x_{mj}}{x_{mo}} | j \neq o\} + u_o \qquad (4.4)$$

The first set of constraint of model 2.1 is satisfied. With the above assumptions the second set of constraint imply that $\alpha + w_j - u_o \geq E_j^* + \frac{m_j}{x_{mo}}$ for $j = 1, 2, ..., n, j \neq o$. From 4.1, $\alpha \geq E_j^* + \{\frac{x_{mj}}{x_{mo}} - min\{x_{mj} x_{mo}\}\}$, Now let:

$$\alpha = \max_{j \neq o}\{\{E_j^*\} + \{\{\frac{x_{mj}}{x_{mo}}\} - min\{\frac{x_{mj}}{x_{mo}}\}\}\} (4.5)$$

With the above selection, it can be seen that the model 4.1 is feasible. □

Choose the most suitable α

Consider the following model:

$\min \alpha$

$s.t.$

$\sum_{i=1}^{m} v_i x_{io} = 1$

$\sum_{r=1}^{s-1} u_r y_{rj} + w_{[j]} - u_o - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n \quad j \neq o$

$w_1 = u_s, w_{k+1} \leq \delta_{k+1} w_k, k = 1, 2, ..., n-1$

$\sum_{r=1}^{s-1} u_r y_{rj} + w_{[j]} - u_o - (E_j^* - \alpha) \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n \quad j \neq o$

$u_r, v_i, w_k \geq \epsilon, i = 1, ..., m, r = 1, ..., s, k = 1, ..., n$

$u_o, free. \qquad (4.1)$

Where $w_j, j = 1, 2, ..., n$ calculated from model 3.5, by solving the above model for all efficient units and assuming that we have efficient units , in this case, the most suitable will be as follows :

$$\bar{\alpha} = \max\{\alpha_1^*, \alpha_2^*, ..., \alpha_k^*\} \qquad (4.6)$$

Where α_i^* is the optimal value of model 4.2. for efficient DMU_j Model 4.1 is feasible for all $\bar{\alpha}$ for efficient units.

5 Numerical Example

Consider six DMUs using two exact inputs x_1,x_2 to produce exact output y_1 and ordinal output y_2 with the data shown in Table 1. by applying models 2.2 and 3.5 for data in Table 1, the results of the Cooper's model in the second column of Table 2 and the results of the proposed model in the fourth column of this table are listed. For Cooper's model, the value of δ is assumed to be 0.00001.The efficiency scores obtained from Cooper's model, which are given in the second column of Table 2, shows that these scores are very exaggerated. The third column of Table 2 shows the values of δ_j calculated from 3.3.
It can be seen that these values are different for each DMU. The proposed model with the values obtained for δ_j is applied and the results

Table 1: Data of the six DMUs

DMU	Inputs		Outputs	
	x_1 (exact)	x_2 (exact)	y_1 (exact)	y_2 (exact)
A	100	0.75	900	1
B	200	0.85	1000	2
C	200	1	600	3
D	100	0.65	2000	4
E	150	1	1200	5
F	200	0.55	800	6

Table 2: Efficiencies of the six DMUs by cooper's and proposed Models

DMU	Efficiency scores obtained from Cooper' s model	Proposed model	
		δ_j	Efficiency scores
A	1	–	1
B	1	0.1618	0.4693
C	0.7496	0.0825	0.2109
D	1	0.8461	1
E	0.6659	0.2200	0.3999
F	1	0.2000	0.4727

Table 3: Ranking of results for example (model (3.5))

Efficient DMU's	Optimal objective value of model (7) for $\alpha = \bar{\alpha}$	Ranking with new model
A	1.390752	2
D	2.158651	1

are given in the fourth column of Table 2. In this model, the efficiency scores assigned to inefficient DMUs are more appropriate and seems real. This is the effect of considering other factors in assigning real values to ordinal numbers. Table 3 shows the results of ranking of the efficient units using Model 3.5. The maximum amount of $\alpha_i^*, i = 1, 2(0.08410237)$ was chosen as the most appropriate.

6 Conclusion

Measuring the efficiency of units that have specific type of data such as interval, ordinal, qualitative, etc. by conventional models of DEA, has always its own challenges. This leads the authors to address this issue. In this article, we discussed the efficiency scores of DMUs with ordinal data, such as the Cooper model. The model we proposed to obtain efficiency scores was based on the BCC model. We assigned a positive real number to each ordinal data using the model described in Section 3. Comparing our proposed model with the Coopers model, showed that, in addition to being able to identify efficient DMUs like the Cooper's model, proposed model also eliminated the deficiency of Cooper's model. Meaning that the efficiency scores calculated by proposed model for inefficient DMUs seems real. we also presented a model for ranking efficient units. The general idea of this model is that the efficiency of units in evaluation should be allowed to be reduced to a controlled size. This model can rank all of the efficient units. Future research can include other types of imprecise data.

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