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Research Article

Enlargements of Abstract Monotone Operators Determind by Representing Functions

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Abstract

In this paper, we study a new enlargement of abstract sub-differential for any IPH function. We know that $\epsilon-$abstract sub-differential of any IPH function is an enlargement of its abstract sub-differential and any point from the graph of $\epsilon-$abstract sub-differential can be approximated by a point from the graph of abstract sub-differential. This nice property, apart from its theoretical importance, gives also the possibility to use the enlargement of abstract sub-differential in finding approximate solutions of inclusions determined by abstract sub-differentials. We define a new enlargement and observe, in the case abstract sub-differential, the relation between this new enlargement and the $\epsilon-$abstract sub-differential.

Keywords : Abstract Sub-Differential; Abstract Monotone Operator; Enlargement; $\epsilon-$Abstract Sub-Differential.

1 Introduction

Several approaches to the theory of monotone operators have established links between maximal monotone operators and convex functions (See [[5], [6], [7], [9], [13], [15], [18], [28], [29]]). The richness of the theory of monotone operators has given rise to a great number of works and the simplification of proof and theory that has resulted from the use of convex analysis techniques justifies an interest in these links. Recently many authors have explored the use of convex representative function in the study of monotone operators, e.g., [[5], [6], [7], [13], [15], [18]]. Indeed, let $\mathfrak{X}$ be a real Banach space and $\mathfrak{X}^*$ be the dual space of $\mathfrak{X}$. Denote by $\langle .,. \rangle$ the duality product between $\mathfrak{X}$ and $\mathfrak{X}^*$. Rockafellar in [20] proved that sub-differentials of proper lower semi-continuous convex functions on X are maximal monotone. In general, maximal monotone operators are not sub-differentials of convex functions. Krauss in [9] managed to represent maximal monotone operators by sub-differentials of saddle functions on $\mathfrak{X}\times\mathfrak{X}$. After that, Fitzpatrick

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[7] proved that the family

$$H(A) := \{h : \mathfrak{X} \times \mathfrak{X}^* \longrightarrow (-\infty, \infty] : \\ h \text{ is a lower serni- continuous convex function,} \\ \text{for every } (x, x^*) \in \mathfrak{X} \times \mathfrak{X}^*, h(x, x^*) \geq \langle x, x^* \rangle, \\ (x, x^*) \in A \Longleftrightarrow h(x, x^*) = \langle x, x^* \rangle\}.$$

is non-empty where A is an arbitrary maximal monotone subset of $\mathfrak{X} \times \mathfrak{X}^*$. He defined the function $\varphi_A : \mathfrak{X} \times \mathfrak{X}^* \longrightarrow (-\infty, \infty]$ by

$$\varphi_A(x, x^*) = \sup_{(x,x^*)\in A}(x - y, y^* - x^*) + \langle x, x^* \rangle \quad \forall (x, x^*) \in \mathfrak{X} \times \mathfrak{X}^*.$$

and showed that $\varphi_A \in H(A)$. It is worth noting that φ_A is called the Fitzpatrick function and moreover φ_A represents A, that is, $\varphi_A \in H(A)$. In a recent paper, Martinez-Leqaz and Thera [15] rediscovered the Fitzpatrick function associated to maximal monotone operators and characterized the family

$$\{\varphi_A : \ A \in \mathfrak{X} \times \mathfrak{X}^* \text{ is maximal monotone}\}.$$

In [5] Burachik and Svaiter also rediscovered Fitzpatrick functions and studied the whole family of lower semi-continuous convex functions associated with a given maximal monotone operator A, that is, those functions $h \in H(A)$. Recently, Martinez- Legaz and Svaiter [13] extended the representation of maximal monotone operators by lower semi-continuous convex functions to a larger class of monotone operators. Roughly speaking the study of monotone operators is reduced to the study of the convexification of the coupling function, restricted to the monotone set. However, convexity is sometimes a restrictive assumption, and therefore the problem arises how to generalize the theory of monotone operators via abstract convexity. The theory of Fenchel's conjugation and subdifferentials plays a central role in convex analysis. Fenchels theorem on the second conjugate and duaity for sum of two convex functions, and the Fenchel-Rockafellar's theorem on the sum of the subdifferentails have substantially influenced the development of convex analysis and is applications in various ways. For instance, Fenchels duality theorem, which states an equality between the minimization of a sum of two convex functions and the maximization of the sum of concave functions, using conjugates, is fundamental to the study of convex optimization.

In 1970, Moreau [16] observed the Fenchel's conjucation and the second conjugation theorem can be established in a very general setting, using two arbitrary sets and arbitrary copling functions. The second conjugation theorem in this setting, known as fenchel moreau theorem, has given rise to the rich theory of abstract convexity (See [[17], [21], [23], [27]]). Morever, extensions of Fenchel's duality theorem and Fenchel-Rockafellar's theorem, which have played key roles in the application of convex analysis, have been presented for abstract convex functions in [8]. In fact, Generalized Fenchel's conjugation theorem [[8], Corollay 5.2] is fundamental to our study of abstract monotonicity.

Abstract convexity has found many applications in study of problem of mathematical analysis and optimization. Also, it has found interesting applications to the theory of inequalities. Abstract convexity opens the way for extending some main ideas and results from classical convex analysis to much more general classes of functions, mappings and sets. It is well-known that every convex, proper and lower semi-continuous function is the upper envelope of a set of affine functions. Therefore, affine functions play a crucial role in classical convex analysis. In abstract convexity, the role of the set of affine functions is taken by alternative set H of functions, and their upper envelopes constitute the set of abstract convex functions Different choices of the set H generate variants of the classical concepts, and have shown important applications in global optimization (See [22], [23], [24], [25]). Moreover, if family of functions is abstract convex for a specific choice of H, then we can use some key ideas of convex analysis in order to gain new insight on these functions. On the other hand, by using an alternative set for affine functions, we indentify those facts in classical convex analysis which depend on the specific properties of affine functions. Abstract convexity has mainly been used for the study of point-to-point functions. Examples of its use in the analysis of multifunctions can be found in works of Levin [10, 11], who focused in the study of abstract cyclical monotonicity, and also, Penot [18] by using a framework of generalized convexity showed the existence of a convex representation of a maximal monotone operator by a convex function which is invariant with respect to the Fenchel conjugacy. Recently, Burachik and Rubinov [2] studied semi-continuity properties of abstract monotone operators. In this paper, we study a new anlaryement of abstract for any IPH function. We define a new enlargement and observe, in the case abstract sub-differential, the relation between this new enlargement and the $\epsilon-$abstract sub-differential.

2 Preliminaries and notations

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two sets. Recall (See [2]) that a set valued mapping (multifunction) from $\mathfrak{X}$ to $\mathfrak{Y}$ is a mapping $F:\mathfrak{X}\longrightarrow 2^{\mathfrak{Y}}$, where $2^{\mathfrak{Y}}$ represents the collection of all subsets of $\mathfrak{Y}$. we define the domain and graph of F by

$$domF := \{x\in\mathfrak{X} : F(x)\neq 0\}.$$

and

$$G(F) := \{(x, y)\in\mathfrak{X}\times\mathfrak{Y} : y\in F(x)\}.$$

Respectively. Let $\mathfrak{X}$ be a set and $\mathfrak{L}$ be a set of real valued fuctions $l:\mathfrak{X}\longrightarrow\mathbb{R}$, which will be called abstract liner. For each $l\in L$ and $c\in\mathbb{R}$ consider the shift $h_{l,c}$ of l on the constant c, $h_{l,c}(x) = l(x) - c$. The function $h_{l,c}$ is called $L-$affine. Recall (See [16]) that the set L is called a set of abstract linear functions if $h_{l,c}\in L$ for all $l\in L$ and all $c\in\mathbb{R}\setminus\{0\}$. The set of all $L-$affine functions will be denoted by H_L If L is a set of abstract linear functions, then $h_{l,c} = h_{l0,c0}$ if and only if $l = l_0$ and $c = c_0$. If L is a set of abstract linear functions, then the mapping $(l, c)\longrightarrow h_{l,c}$ is a one-to-one correspondence. In this case, we identify $h_{l,c}$ with (l, c), in other words, we consider an element $(l, c)\in L\times\mathbb{R}$ as a function defined on $\mathfrak{X}$ by $x\longrightarrow l(x) - c$. A function $f:\mathfrak{X}\longrightarrow(-\infty,\infty]$ is called proper if $domf\neq 0$ where $domef$ is defined by $domf = \{x\in\mathfrak{X} : f(x) < \infty\}$. Let $F(\mathfrak{X})$ be the set of all function $f:\mathfrak{X}\longrightarrow(-\infty,\infty]$ and function $-\infty$. Recall (See [17]) that a function $f\in F(\mathfrak{X})$ is called $H-$convex ($H = L$, or $H = H_L$) if

$$f(x) = \sup\{h(x) : h\in\sup P(f, H)\}.$$

where $\sup P(f, H) = \{h\in H : h\leq f\}$ is called the support set of the function$f$, and $h\leq f$ if and only $h(x)\leq f(x)$ for every $x\in\mathfrak{X}$.

Example 2.1 *Let $\mathfrak{X}$ be a locally convex Hausdorff topological vector space. Let L be the set of all real valued continuous liner functional defined on $\mathfrak{X}$. Then, $f:\mathfrak{X}\longrightarrow(-\infty,\infty]$ is an $L-$convex function if and only if f is lower semi-continuous and sublinear also, f is an H_L-convex function if and only if f is lower semi-continuous and convex. Now, we consider the coupling function $\langle .,.\rangle:\mathfrak{X}\times L\longrightarrow\mathbb{R}$ is defined by $\langle x, l\rangle = l(x)$ for all $x\in\mathfrak{X}$ and $l\in L$. For a function $f\in F(\mathfrak{X})$, define the Fenchel-Moreau L-conjugate f^*_L of (See[[16], [21]]) by $f^*_L(l) = \sup_{x\in\mathfrak{X}}(l(x)-f(x))$, $l\in L$. The function $f^{**}_{L,\mathfrak{X}} = (f^*_L)_{\mathfrak{X}^*}$ is called the second conjugate (or biconjugate) of f , and by definition we have $f^{**}_{L,\mathfrak{X}}(x) = \sup_{l\in L}(l(x) - f^*(l))$. The following property of the conjugate function follows directly from the definition Fenchel-Youngs inequality: for a proper function $f\in F(\mathfrak{X})$ one has,*

$$\forall x\in\mathfrak{X}, \forall l\in L \quad f(x) + f^*_L(x)\geq l(x).$$

Let $f:\mathfrak{X}\longrightarrow(-\infty,\infty]$ be a function and $x_0\in domf$. Recall (See [[16], [21]]) that an element $l\in L$ is called an $L-$subgradient of f at x_0 if

$$\forall x\in\mathfrak{X} \quad f(x)\geq f(x_0) + l(x) - l(x_0).$$

The set $\partial_L f(x_0)$ of all $L-$subgradinents of f at x_0 is called $L-$subdifferential of f at x_0. The sub-differential $\partial_L f(x_0)$ is non-empty (See [[16], [21]]) if and only if $x_0\in domf$ and $f(x_0) = \max\{h(x_0) : h\in supp(f, H_L)\}$

In the following we gather some results which will be used later.

Lemma 2.1 *(See [[21], Theorem 7.1]) Let $f\in F(\mathfrak{X})$. Then $f = f^{**}_{L,\mathfrak{X}}$ if and only if f is an H_L-convex function.*

Lemma 2.2 *(See [[21], Proposition 7.7]) Let $x_0\in\mathfrak{X}$, $f\in F(\mathfrak{X})$ and $l_0\in L$. Then the following assertions are equiualent:*

1. *$f(x_0) + f^*_L(l_0) = l_0(x_0)$ (Fenchel-Young's equality)*

2. $l_0 \in \partial_L f(x_0)$.

In the sequel, let $\mathfrak{X}$ be a topological vector space. We assume that $\mathfrak{X}$ is equipped with a closed convex pointed cone $S \subset \mathfrak{X}$ (the latter means that $S \cap (-S) = \{0\}$. We say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$. An extended real valued function $f : \mathfrak{X} \longrightarrow [-\infty, \infty]$ is called positively homogenous (of degree one) if $f(\lambda x) = \lambda f(x)$ for all $x \in \mathfrak{X}$ and all $\lambda > 0$. The function f is called increasing if $x \geq y \Rightarrow f(x) \geq f(y)$. Now, consider the function $l : \mathfrak{X} \times \mathfrak{X} \longrightarrow [0, \infty]$ defined by $l(x, y) = \max\{\lambda \geq 0 : \lambda y \leq x\}$. The function l has the following properties (See [[14], [16]]). In fact, for every $x, y, x', y' \in \mathfrak{X}$ and every $\nu > 0$, one has,

$$l(\nu x, y) = \nu l(x, y) \tag{2.1}$$

$$l(x, \nu y) = \frac{1}{\nu} l(x, y) \tag{2.2}$$

$$l(x, y) = \infty \Longrightarrow y \in -S \tag{2.3}$$

$$l(x, x) = 1 \Longleftrightarrow x \notin -S \tag{2.4}$$

$$x \in S, y \in -S \Longrightarrow l(x, y) = +\infty \tag{2.5}$$

$$x \leq x' \Longrightarrow l(x, y) \leq l(x', y) \tag{2.6}$$

$$y \leq y' \Longrightarrow l(x, y) \geq l(x, y') \tag{2.7}$$

Define $L_S = \{l_y : y \in \mathfrak{X} \setminus (-S)\}$, where $l_y(x) = l(x, y)$ for all $x, y \in \mathfrak{X}$. Note that l_y is an increasing positively homogeneous (IPH) function for each $y \in \mathfrak{X}$. Therefore, L_S is a set of non-negative increasing positively homogeneous (IPH) functions defined on $\mathfrak{X}$. The following results for non-negative IPH functions have been proved in [13].

Lemma 2.3 *Let $f : \mathfrak{X} \longrightarrow [0, \infty]$ be a function. Then the following assertions are equiualent:*

1. *f is IPH.*

2. *$\forall \lambda > 0 \quad \lambda y \leq x \Longrightarrow f(x) \geq \lambda f(y)$.*

3. *$f(x) \geq l_y(x) f(y)$ for all $x, y \in \mathfrak{X}$ with the conuention $+\infty \times 0 = 0$.*

Lemma 2.4 *Let $f : \mathfrak{X} \longrightarrow [0, \infty]$ be an IPH function and $f(x) \neq 0, +\infty$. Then*

$$\partial_{LS} f(x) = \{l_y \in L_S : l_y(x) = f(x), f(y) = 1\}.$$

3 Abstract Monotone Operators

Assume that $\mathfrak{X}$ is a set and L is a set of real valued functions $l : \mathfrak{X} \longrightarrow \mathbb{R}$, Which is called abstract linear with the coupling function $\langle ., . \rangle : \mathfrak{X} \times L \longrightarrow \mathbb{R}$ defined by $\langle x, l \rangle = l(x)$ for all $x \in \mathfrak{X}$ and $l \in L$. In the following, we present some definitions and properties of abstract monotone operators (See [2]).

Definition 3.1 *A set value mapping $T : \mathfrak{X} \longrightarrow 2^L$ is called $L-$monotone operator (or, abstract monotone operator) if*

$$l(x) - l(x') - l'(x) + l'(x') \geq 0 \tag{3.8}$$

for all $x, x' \in \mathfrak{X}$, $l \in Tx$ and $l' \in Tx'$.

If $\mathfrak{X}$ is a Banach space with the dual space $\mathfrak{X}^*$ and $L = \mathfrak{X}^*$ Then T is called monotone opearator in the classical case.

Definition 3.2 *A set valued mapping $T : \mathfrak{X} \longrightarrow 2^L$ is called maximal Lmonotone operator (or, maximal abstract monotone operator) if T is $L-$monotone and $T = T'$ for any $L-$monotone operator $T' : \mathfrak{X} \longrightarrow 2^L$ such that $G(T) \subset G(T')$.*

Definition 3.3 *A subset S of $\mathfrak{X} \times L$ is called $L-$monotone (or, abstract monotone) if*

$$\forall (x, l), (x', l') \in S \;\; l(x) - l(x') - l'(x) + l'(x') \geq 0.$$

Definition 3.4 *A subset S of $\mathfrak{X} \times L$ is called maximal $L-$monotone (or, maximal abstract monotone) if S is $L-$monotone and $S = S'$ for any $L-$monotone set S' such that $S \subset S'$.*

Definition 3.5 *Let $T : \mathfrak{X} \longrightarrow 2^L$ be a set valued mapping. Correspondence to the mapping T define the $L-$fitzpatrick function (or, abstract Fitzpatrick function) $\varphi_T : \mathfrak{X} \times L \longrightarrow \bar{\mathbb{R}}$ by $\varphi_T(x, l) = \sup_{l' \in Tx', x' \in \mathfrak{X}} [l(x') + l'(x) - l'(x') - l(x)] + l(x)$ For all $x \in \mathfrak{X}, l \in L$*

Lemma 3.1 *Let $T : \mathfrak{X} \longrightarrow 2^L$ be a maximal $L-$monotone operator. Then $\varphi_T(x,l) \geq l(x)$ for any $x \in \mathfrak{X}$ and $l \in L$ With equality holds if and only if $l \in Tx$.*

Proof. since T is maximal $L-$monotone operator, it follows that

$$\sup_{l' \in Tx', x' \in \mathfrak{X}}[l(x') + l'(x) - l'(x') - l(x)] \geq 0$$

Now we obtain $\varphi_T(x,l) \geq l(x)$. Since T is maximal $L-$monotone,

$$\sup_{l' \in Tx', x' \in \mathfrak{X}}[l(x') + l'(x) - l'(x') - l(x)] = 0$$

If and only if $l \in Tx$. So $\varphi_T(x,l) = l(x)$ if and only if $l \in Tx$ and hence the proof is complete.

Theorem 3.1 *Let $f : \mathfrak{X} \longrightarrow [0,\infty]$ be an IPH function and $f(x) \neq 0, \infty$. Then $\partial_{LS}(f)$ is a maximal L_S-monotone operator.*

4 Main Results

Enlargements

As it was seen from the definition of $\epsilon-$abestract subdifferential, for any IPH function and any $\epsilon > 0$, the $\epsilon-$absestract subdifferential $\partial_{L_\epsilon}(f)$ is an enlargement of the abstract subdifferential $\partial_L(f)$,i.e., $\partial_L f(x) \subset \partial_{L_\epsilon} f(x)$. for maximal .on the other hand, this enlargement is not for enough from the initial operator, the well-known Brondsted Rockafellar theorem [1], asserts that any point for the graph of $\partial_\epsilon(f)$ can be approximated (depending $\mathfrak{X}$) with a point from the graph ∂f. This nice property, a part from its theoretical importance, give also the possibility to use the enlargement of sub differential in finding solutions of inclusions determined by sub differentials. Motivated by the above. The search of possible enlargements of an arbitrary abstract (maximal) monotone operator has been done during in recent years. attempts for such notions, and pried by various properties of variational inequalities, could be found in [[12], [19], [21]]. Given a abstract monotone $T : \mathfrak{X} \longrightarrow 2^L$, $\epsilon > 0$ and $(x', l') \in Gr(T)$. Let

$$T^\epsilon(x) = \{l \in L : l'(x') + l(x) - l'(x) - l(x') \geq -\epsilon\}.$$

In case $L = \mathfrak{X}^*$,this definition was given in [14] but the notion was not studied. Indepently, this concept has been studied, first in finite dimensions in [2] with application to approximate solutions of variational inequalities. And then in Hilbert space in [3], with application to finding a zero of maximal monoton operator, An approach with families of enlargements was futher investiqated in [26] in the case of abstract subdifferential,i.e., $T = \partial f$ for a IPH function, one easily sees that $\partial_{L_\epsilon} f \subset T^\epsilon$ for every $\epsilon > 0$. First, for given a (proper) convex function $g : \mathfrak{X} \times L \longrightarrow \mathbb{R} \cup \{+\infty\}$, let us define the following operator $T_g : \mathfrak{X} \longrightarrow 2^L$, $T_g(x) = \{l \in L : (l,x) \in \partial g(x,l)\}$. The so-defined operator T_g is abstract monotone [[7], Proposition 2.2]. Further, for a given abstract monotone operator $T : \mathfrak{X} \longrightarrow 2^L$ Let us define the following function $\varphi_T : \mathfrak{X} \times L \longrightarrow \mathbb{R} \cup \{+\infty\}$ by $\varphi_T(x,l) = \sup\{l(x') + l'(x) - l'(x') : (x', l') \in Gr(T)\}$.

Theorem 4.1 *(See [7]) $T : \mathfrak{X} \longrightarrow 2^L$ be abstract monotone operator with $DomT \neq \phi$. Then*

1. *for any one has $T(x) \subset T_{\varphi_T}(x)$. If T is abstract maximal monotone $T = T_\varphi$.*
2. *If T is abstract maximal monotone, then $\varphi_T(x,l) \geq l(x)$ for every $(x,l) \in \mathfrak{X} \times L$ and $\varphi_T(x,l) = l(x)$ if and only if $(x,l) \in Gr(T)$ moreover, φ_T is the minimal convex function on $\mathfrak{X} \times L$ with these two properties.*

The above representation of a given abstract monotone operator by abstract subdifferentials of convex function in $\mathfrak{X} \times L$ is the transformation of the representation of the abstract monoton operators by abstract sub-differentials of saddle functions provided by krauss [9]. The Fitizpatrick approach was also studied in [6]. Now, let us use the usual $\epsilon-$abstract sub-differntials of the function φ_T in order to define enlargement of a given abstract monotone operator $T : \mathfrak{X} \longrightarrow 2^L$ for which we will always assume that $Dom(T) \neq \phi$. For $\epsilon > 0$, Let $T_\epsilon(x) = \{l \in L : (l,x) \in \partial_\epsilon \varphi_T(x)\}$. Because of Theorem 4.1, this operator needs an enlargements of T, i.e., $T(x) \subset T_\epsilon(x)$ for any $x \in \mathfrak{X}$ Moreover, at the case $L = \mathfrak{X}^*$, it can be easily by verified that $T_{\epsilon(x)}$ is convex and since $\varphi_T(x,.)$ is lower semi continuous for the w^*-closed in $\mathfrak{X}^*$.

Proposition 4.1 *Let $T : \mathfrak{X} \longrightarrow 2^L$ be abstract maximal monotone then $T_\epsilon \subset T^\epsilon$.*

Proof. Let $\epsilon > 0$ and $l \in T_\epsilon(x)$ for $x \in \mathfrak{X}$. Then , by definition, for every $(x', l') \in \mathfrak{X} \times L$, we have

Proposition 4.2 *Let $T : \mathfrak{X} \longrightarrow 2^L$ be abstract maximal monotone then $T_\epsilon \subset T^\epsilon$.*

Proof. Let $\epsilon > 0$ and $l \in T_\epsilon(x)$ for $x \in \mathfrak{X}$. then , by definition, for every $(x', l') \in \mathfrak{X} \times L$ we have $\varphi_T(x', l') - \varphi_T(x, l) \geq \langle (x' - x, l' - l), (l, x) \rangle - \epsilon$. Since $\varphi_T(x, l) \geq l(x)$ and for $(x', l') \in Gr(T)$ one by $\varphi_T(x', l') = l'(x')$ for every $(x', l') \in Gr(T)$, the latter inequality gives $\langle (x' - x, l' - l), (l, x) \rangle \geq -\epsilon$. So $l \in T^\epsilon(x)$.
Further, we wish to investigate the particular case of abstract sub-differentals.. First we observe the following simple estimation.

Lemma 4.1 *Let $T : \mathfrak{X} \longrightarrow 2^L$ be abstract monotone, $\epsilon > 0$ and $l \in T^\epsilon(x)$. Then $\varphi_T(x, l) \leq l(x) + \epsilon$.*

Proof. The proof comes directly from the definitions since $l \in T^\epsilon(x)$ for every $(x', l') \in Gr(T)$ we have $\langle x' - x, l' - l \rangle \geq -\epsilon$ which for every $(x', l') \in Gr(T)$,gives

$$l(x) + \epsilon \geq l(x') + l'(x) - l'(x') \tag{4.9}$$

$$l(x)+\epsilon \geq \sup_{(x',l')\in Gr(T)} \{l(x')+l'(x)-l'(x')\}. \tag{4.10}$$

$$\varphi_T(x, l) \leq l(x) + \epsilon \tag{4.11}$$

Theorem 4.2 *Let $T = \partial f$ for some IPH function $f : \mathfrak{X} \longrightarrow \mathbb{R} \cup \{+\infty\}$ then for any $\epsilon > 0$, $x \in domf$, we have $T_\epsilon(x) \subset \partial_{L_\epsilon} f(x)$.*

Proof. Take $x \in domf$ and $l \in T_\epsilon(x)$. For every $(x', l') \in \mathfrak{X} \times L$, $\varphi_T(x', l') - \varphi_T(x, l) \geq \langle (x' - x, l' - l), (l, x) \rangle - \epsilon$. As above, using $\varphi_T(x, l) \geq l(x)$, for every $(x', l') \in \mathfrak{X} \times L$, $\varphi_T(x', l') - l'(x) \geq l(x') - l(x) - \epsilon$. Let us take an arbitrary $x' \in \mathfrak{X}$, $\delta > 0$. take a $l'_\delta \in \partial_\delta f(x)$. By the last inequality we have $\varphi_T(x', l'_\delta) - l'_\delta(x) \geq l(x') - l(x) - \epsilon$. From $f^*(l) = \sup_{x\in\mathfrak{X}}\{l(x) - f(x)\}$, by the theorm 4.1, since $l'_\delta \in \partial_\delta f(x)$, we know that $f(x') - f^*(l'_\delta) \leq l'_\delta(x) + \delta$ which together with the previous inequality give $f(x') - f(x) + \delta \geq l(x') - l(x) - \epsilon$. Passing to the limit for δ we get $f(x') - f(x) \geq l(x') - l(x) - \epsilon$ and since x' was arbitrary. We conclude that $l \in \partial_\delta f(x)$.

5 Conclusion

Abstract convexity has mainly been used for the study of point-to-point functions. Examples of its use in the analysis of multifunctions can be found in works. By using a framework of generalized convexity showed the existence of a convex representation of a maximal monotone operator by a convex function which is invariant with respect to the Fenchel conjugacy. Recently, Burachik and Rubinov [2] studied semi-continuity properties of abstract monotone operators. In this paper, we studied a new enlargement of abstract for any IPH function. We defined a new enlargement and observe, in the case abstract sub-differential, the relation between this new enlargement and the $\epsilon-$abstract sub-differential.

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