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Research Article

A Limited Version of Crout Decomposition Method for Solving of Fuzzy Complex Linear Systems

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Abstract

In this paper, it is shown that the solution vector obtained by the classic Crout decomposition method is not an algebraic solution of a fuzzy complex linear system. Here, we propose a limited version of the mentioned method to obtain an algebraic solution of a fuzzy complex linear system (if it exists). Two numerical examples are presented to show ability and reliability of our method.

Keywords : Fuzzy complex number; Fuzzy complex linear system; Limited Crout decomposition method; Fuzzy number; Linear equations.

1 Introduction

Fuzzy complex system of linear equations are recently gaining more attention in the literature. These systems, plays a vital role in real life problems such as optimisation, current flow, economics and engineering [4]. However, few researchers have developed methods to solve fuzzy complex linear systems.

The concept of fuzzy complex number was first introduced by Buckley [6]. Qiu et al. [14, 15] investigated the sequence and series of fuzzy complex numbers and their convergence. Solution of fuzzy complex linear systems was described by Rahgooy et al. [16] and applied to circuit analysis problem. Jahantigh et al. [10] proposed a numerical method for solving fuzzy complex linear systems. In 2012, Behera and Chakraverty [3] proposed a new and simple centre and width based method for solving fuzzy real and complex system of linear equations. Also, in 2013, Majumdar [12] solved fuzzy complex linear systems by direct and iteration methods. Recently, Behera and Chakraverty [4, 5] introduced a new and simple method for solving general fuzzy complex linear systems where the elements of unknown variable vector and right hand side vector are considered as fuzzy complex number. Unfortunately, In 2017, the author [9] showed that there are two basic shortcomings in the method proposed in [4] and he presents the modified version of Behera and Chakraverty's method to avoid these shortcomings for solving a fuzzy complex linear system.

In this paper, we present a simple and interesting approach for algebraic solving a fuzzy complex linear system. Based on the proposed

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method, we first solve the fuzzy complex linear system by the classic Crout decomposition method and obtain the Crout's solution. It is shown that the Crout's solution does not satisfy all equations of the system and therefore it is not an algebraic solution. In the next step of our method, we obtain the algebraic solution of the fuzzy complex linear system (if it exists) by limiting of the Crout's solution. In fact, in this paper, we present a limited version of the classic Crout decomposition method so that the result is always an algebraic solution.

The outline of the paper is as follows. In Section 2 we present some basic definitions, remarks and lemmas. In Section 3, we define fuzzy complex linear system, algebraic solution and the Crout's solution. A limited version of the classic Crout decomposition method is presented in Section 4. The proposed method is applied to solve two numerical examples in Section 5. Conclusion is drawn in Section 6.

2 Preliminaries

Definition 2.1 *A fuzzy subset $\widetilde{x}$ of the real line $\mathbb{R}$, with membership function $\mu_{\widetilde{x}}$, is a fuzzy real number (or briefly fuzzy number) if*

(i) *$\widetilde{x}$ is normal, i.e.$\exists t_0 \in \mathbb{R}$ with $\mu_{\widetilde{x}}(t_0) = 1$,*

(ii) *$\widetilde{x}$ is a convex fuzzy set, i.e.,*

$$\mu_{\widetilde{x}}(\lambda s + (1-\lambda)t) \geqslant \min\{\mu_{\widetilde{x}}(s), \mu_{\widetilde{x}}(t)\},$$

for $s, t \in \mathbb{R}$ and $\lambda \in [0,1]$,

(iii) *$\mu_{\widetilde{x}}$ is upper semi-continuous on $\mathbb{R}$,*

(iv) *$\overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.*

In this paper, we denote the set of all fuzzy numbers by F_R. Obviously, $\mathbb{R} \subset F_R$, because we can define $\mathbb{R} = \{\chi_{\{t\}} : t \text{ is an usual real number}\}$ [2]. For $0 < \alpha \leqslant 1$, we define α-levels of fuzzy number $\widetilde{x}$ as $[\widetilde{x}]_\alpha = \{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) \geqslant \alpha\}$ and $[\widetilde{x}]_0 = \overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}$. Also, we define the support of fuzzy number $\widetilde{x}$ as

$$supp\,(\widetilde{x}) = [\widetilde{x}]_0 = \overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}}.$$

Then, from (i)-(iv) it follows that $[\widetilde{x}]_\alpha$ is a bounded closed interval for each $\alpha \in [0,1]$ [17]. In this paper, we denote the α-levels of fuzzy number $\widetilde{x}$ as $[\widetilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$, for each $\alpha \in [0,1]$. Sometimes it is important to know whether the given intervals $[\underline{x}(\alpha), \overline{x}(\alpha)]$, $0 \leqslant \alpha \leqslant 1$, are the α-levels of some fuzzy number in F_R. The following answer is presented in [11].

Lemma 2.1 *Let*

$$\{[\underline{x}(\alpha), \overline{x}(\alpha)] \;:\; 0 \leqslant \alpha \leqslant 1\},$$

be a given family of non-empty sets in $\mathbb{R}$. If

(i) $[\underline{x}(\alpha), \overline{x}(\alpha)]$ is a bounded closed interval, for each $\alpha \in [0,1]$,

(ii) $[\underline{x}(\alpha_1), \overline{x}(\alpha_1)] \supseteq [\underline{x}(\alpha_2), \overline{x}(\alpha_2)]$ for all $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$,

(iii) $[\lim_{k\to\infty} \underline{x}(\alpha_k), \lim_{k\to\infty} \overline{x}(\alpha_k)] = [\underline{x}(\alpha), \overline{x}(\alpha)]$, whenever $\{\alpha_k\}$ is a non-decreasing sequence in $[0,1]$ converging to α,

then the family $[\underline{x}(\alpha), \overline{x}(\alpha)]$ represents the α-levels of a fuzzy number $\widetilde{x}$ in F_R.
Conversely, if $[\underline{x}(\alpha), \overline{x}(\alpha)]$, $0 \leqslant \alpha \leqslant 1$, are the α-levels of a fuzzy number $\widetilde{x} \in F_R$, then the conditions (i)-(iii) are satisfied.

Remark 2.1 *[1, 8, 13] From Lemma 2.1 we conclude that if the family*

$$\{[\underline{x}(\alpha), \overline{x}(\alpha)] \;:\; 0 \leqslant \alpha \leqslant 1\},$$

are the α-levels of a fuzzy number, then:

1) The condition (i) implies the functions $\underline{x}$ and $\overline{x}$ are bounded over $[0,1]$ and $\underline{x}(\alpha) \leqslant \overline{x}(\alpha)$ for each $\alpha \in [0,1]$.

2) The condition (ii) implies the functions $\underline{x}$ and $\overline{x}$ are non-decreasing and non-increasing over $[0,1]$, respectively.

3) The condition (iii) implies the functions $\underline{x}$ and $\overline{x}$ are left-continuous over $[0,1]$.

For $\widetilde{x}, \widetilde{y} \in F_R$, and $\lambda \in \mathbb{R}$, α-levels of the sum $\widetilde{x} + \widetilde{y}$ and the product $\lambda \cdot \widetilde{x}$ are defined based on interval arithmetic as

$$\begin{aligned} [\widetilde{x} + \widetilde{y}]_\alpha &= [\widetilde{x}]_\alpha + [\widetilde{y}]_\alpha \\ &= \{s + t : s \in [\widetilde{x}]_\alpha,\; t \in [\widetilde{y}]_\alpha\} \\ &= [\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)], \end{aligned}$$

$$\begin{aligned}[\lambda \cdot \widetilde{x}]_\alpha &= \lambda \cdot [\widetilde{x}]_\alpha \\ &= \{\lambda t : t \in [\widetilde{x}]_\alpha\} \\ &= \begin{cases} [\lambda \underline{x}(\alpha), \lambda \overline{x}(\alpha)], & \lambda \geqslant 0, \\ [\lambda \overline{x}(\alpha), \lambda \underline{x}(\alpha)], & \lambda < 0. \end{cases}\end{aligned}$$

Now, we define a fuzzy complex number that can be found in [3, 4].

Definition 2.2 *An arbitrary fuzzy complex number $\widetilde{z}$ may be represented as $\widetilde{z} = \widetilde{p} + i\widetilde{q}$, where $\widetilde{p}$ and $\widetilde{q}$ are fuzzy real numbers, i.e. $\widetilde{p}, \widetilde{q} \in F_R$. Also, the set of all fuzzy complex numbers is denoted by F_C.*

The following definition can be obtained from [3, 4].

Definition 2.3 *We define α-levels of fuzzy comlex number $\widetilde{z}$ as*

$$\begin{aligned}[\widetilde{z}]_\alpha &= [\widetilde{p}]_\alpha + i[\widetilde{q}]_\alpha \\ &= [\underline{p}(\alpha), \overline{p}(\alpha)] + i[\underline{q}(\alpha), \overline{q}(\alpha)] \\ &:= [\underline{p}(\alpha) + i\,\underline{q}(\alpha), \overline{p}(\alpha) + i\,\overline{q}(\alpha)],\end{aligned}$$

thus, we contract that $[\widetilde{z}]_\alpha = [\underline{z}(\alpha), \overline{z}(\alpha)]$, where $\underline{z}(\alpha) = \underline{p}(\alpha) + i\,\underline{q}(\alpha)$ and $\overline{z}(\alpha) = \overline{p}(\alpha) + i\,\overline{q}(\alpha)$.

From Definition 2.3, we can present the following definition.

Definition 2.4 *For any two arbitrary fuzzy complex numbers $\widetilde{z}_1 = \widetilde{p}_1 + i\widetilde{q}_1$ and $\widetilde{z}_2 = \widetilde{p}_2 + \widetilde{q}_2$ and crisp complex number $(a+ib)$, α-levels of the sum $\widetilde{z}_1 + \widetilde{z}_2$ and the product $(a+ib)\cdot\widetilde{z}_1$ are defined based on interval arithmetic as follows,*

$$\begin{aligned}[\widetilde{z}_1 + \widetilde{z}_2]_\alpha &= ([\widetilde{p}_1]_\alpha + [\widetilde{p}_2]_\alpha) + i([\widetilde{q}_1]_\alpha + [\widetilde{q}_2]_\alpha) \\ &= [\underline{p_1}(\alpha) + \underline{p_2}(\alpha), \overline{p_1}(\alpha) + \overline{p_2}(\alpha)] \\ &\quad + i[\underline{q_1}(\alpha) + \underline{q_2}(\alpha), \overline{q_1}(\alpha) + \overline{q_2}(\alpha)] \\ &:= [(\underline{p_1} + \underline{p_2}) + i(\underline{q_1} + \underline{q_2}) \\ &\qquad , (\overline{p_1} + \overline{p_2}) + i(\overline{q_1} + \overline{q_2})],\end{aligned}$$

and

$$\begin{aligned}[(a+ib)\cdot\widetilde{z}_1]_\alpha &= (a+ib)\cdot([\widetilde{p}_1]_\alpha + i[\widetilde{q}_1]_\alpha) \\ &= (a[\widetilde{p}_1]_\alpha - b[\widetilde{q}_1]_\alpha) \\ &\quad + i(a[\widetilde{q}_1]_\alpha + b[\widetilde{p}_1]_\alpha).\end{aligned}$$

Definition 2.5 *We define α-center of the fuzzy complex number $\widetilde{z} = [\underline{z}(\alpha), \overline{z}(\alpha)]$ as follows:*

$$[\widetilde{z}]^c_\alpha = \frac{\overline{z}(\alpha) + \underline{z}(\alpha)}{2}, \qquad \alpha \in [0,1].$$

Definition 2.6 *We define α-radius of the fuzzy complex number $\widetilde{z} = [\underline{z}(\alpha), \overline{z}(\alpha)]$ as follows:*

$$[\widetilde{z}]^r_\alpha = \frac{\overline{z}(\alpha) - \underline{z}(\alpha)}{2}, \qquad \alpha \in [0,1].$$

Obviously, the α-center and α-radius of a fuzzy complex number is a crisp complex number in any level of $\alpha \in [0,1]$. In the next section, we define a fuzzy complex system of linear equations.

3 Fuzzy complex linear systems

Definition 3.1 *[4] The $n \times n$ linear system*

$$\begin{cases} c_{11}\,\widetilde{z}_1 + c_{12}\,\widetilde{z}_2 + \cdots + c_{1n}\,\widetilde{z}_n = \widetilde{w}_1, \\ c_{21}\,\widetilde{z}_1 + c_{22}\,\widetilde{z}_2 + \cdots + c_{2n}\,\widetilde{z}_n = \widetilde{w}_2, \\ \qquad\vdots \\ c_{n1}\,\widetilde{z}_1 + c_{n2}\,\widetilde{z}_2 + \cdots + c_{nn}\,\widetilde{z}_n = \widetilde{w}_n, \end{cases} \tag{3.1}$$

where the coefficient matrix $\boldsymbol{C} = (c_{kj})_{n\times n}$ is a crisp-valued complex $n \times n$ matrix and $\widetilde{w}_i$, $i = 1, 2, \cdots, n$, are fuzzy complex numbers, is called a fuzzy complex linear system.

We present the matrix form of the fuzzy complex linear system (3.1) as follows

$$\mathbf{C} \cdot \widetilde{Z} = \widetilde{W}, \tag{3.2}$$

where $\widetilde{Z} = (\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_n)^T$ and $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_n)^T$ are two column vectors of fuzzy complex numbers.

Also, if we write the elements of $\widetilde{Z}$ and $\widetilde{W}$ respectively as

$$\widetilde{z}_j = \widetilde{p}_j + i\,\widetilde{q}_j, \quad \widetilde{w}_j = \widetilde{u}_j + i\,\widetilde{v}_j,$$

for $j = 1, 2, ..., n$, then we have

$$\widetilde{Z} = \widetilde{P} + i\,\widetilde{Q}, \qquad \widetilde{W} = \widetilde{U} + i\,\widetilde{V},$$

where $\widetilde{P} = (\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n)^T, \widetilde{Q} = (\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_n)^T, \widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n)^T$ and $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \ldots, \widetilde{v}_n)^T$ are column vectors of fuzzy real numbers. Also, we conclude that

$$[\widetilde{Z}]_\alpha = [\underline{Z}(\alpha), \overline{Z}(\alpha)],$$

$$[\widetilde{W}]_\alpha = [\underline{W}(\alpha), \overline{W}(\alpha)],$$

where

$$\underline{Z}(\alpha) = \underline{P}(\alpha) + i\,\underline{Q}(\alpha),$$

$$\overline{Z}(\alpha) = \overline{P}(\alpha) + i\,\overline{Q}(\alpha),$$

$$\underline{W}(\alpha) = \underline{U}(\alpha) + i\,\underline{V}(\alpha),$$

$$\overline{W}(\alpha) = \overline{U}(\alpha) + i\,\overline{V}(\alpha).$$

In continuation, we define the algebraic solution of system (3.1).

Definition 3.2 *The fuzzy complex number vector* $\widetilde{Z}_A = (\widetilde{z}_{1_A}, \widetilde{z}_{2_A}, \ldots, \widetilde{z}_{n_A})^T$ *is called an "Algebraic solution" of the fuzzy complex linear system (3.1) if it satisfies all equations of Eq. (3.1) based on the arithmetic operations presented in Definition 2.4, or in other words*

$$\sum_{k=1}^{n} c_{kj} \cdot \widetilde{z}_{k_A} = \widetilde{w}_j, \qquad j = 1, 2, \ldots, n.$$

By the concept of α-levels of a fuzzy complex number, we can convert the fuzzy complex linear system (3.1) to a parametric interval complex linear system, as follows:

$$\begin{cases} c_{11}\,[\widetilde{z}_1]_\alpha + \cdots + c_{1n}\,[\widetilde{z}_n]_\alpha = [\widetilde{w}_1]_\alpha, \\ c_{21}\,[\widetilde{z}_1]_\alpha + \cdots + c_{2n}\,[\widetilde{z}_n]_\alpha = [\widetilde{w}_2]_\alpha, \\ \quad\vdots \\ c_{n1}\,[\widetilde{z}_1]_\alpha + \cdots + c_{nn}\,[\widetilde{z}_n]_\alpha = [\widetilde{w}_n]_\alpha, \end{cases} \tag{3.3}$$

or

$$\begin{cases} c_{11}\,[\underline{z_1}(\alpha), \overline{z_1}(\alpha)] + c_{12}\,[\underline{z_2}(\alpha), \overline{z_2}(\alpha)] + \cdots \\ \quad + c_{1n}\,[\underline{z_n}(\alpha), \overline{z_n}(\alpha)] = [\underline{w_1}(\alpha), \overline{w_1}(\alpha)], \\ c_{21}\,[\underline{z_1}(\alpha), \overline{z_1}(\alpha)] + c_{22}\,[\underline{z_2}(\alpha), \overline{z_2}(\alpha)] + \cdots \\ \quad + c_{2n}\,[\underline{z_n}(\alpha), \overline{z_n}(\alpha)] = [\underline{w_2}(\alpha), \overline{w_2}(\alpha)], \\ \quad\vdots \\ c_{n1}\,[\underline{z_1}(\alpha), \overline{z_1}(\alpha)] + c_{n2}\,[\underline{z_2}(\alpha), \overline{z_2}(\alpha)] + \cdots \\ \quad + c_{nn}\,[\underline{z_n}(\alpha), \overline{z_n}(\alpha)] = [\underline{w_n}(\alpha), \overline{w_n}(\alpha)], \end{cases} \tag{3.4}$$

where $\alpha \in [0,1]$. Therefore it is clear that the vector $\widetilde{Z}_A = (\widetilde{z}_{1_A}, \widetilde{z}_{2_A}, \ldots, \widetilde{z}_{n_A})^T$ is an algebraic solution of the equivalent systems (3.1)-(3.4) if for any $\alpha \in [0,1]$ we have

$$\sum_{k=1}^{n} c_{kj} \cdot [\underline{z_{k_A}}(\alpha), \overline{z_{k_A}}(\alpha)] = [\underline{w_j}(\alpha), \overline{w_j}(\alpha)],$$

for $j = 1, 2, \ldots, n$.

Unfortunately, there are few numerical procedures for obtaining the algebraic solution of a fuzzy complex linear system. In the continuation, we focus on the classic Crout decomposition method and show that the obtained solution via this method is not an algebraic solution of system (3.1). To this end, please consider the fuzzy complex linear system (3.2). Based on the classic Crout decomposition method, the coefficient matrix $\mathbf{C} = (c_{kj})_{n\times n}$ is decomposed into the product of the lower-triangular matrix $\mathbf{L} = (l_{kj})_{n\times n}$ and the upper-triangular matrix $\mathbf{U} = (u_{kj})_{n\times n}$, such that $u_{kj} = 1$ for $k = j$, i.e.

$$\mathbf{C} = \mathbf{L} \cdot \mathbf{U}, \tag{3.5}$$

where

$$\mathbf{L} = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It should be noted that the matrices $\mathbf{L}$ and $\mathbf{U}$ are complex crisp-valued. From Eqs. (3.2) and (3.5), we conclude

$$\mathbf{L} \cdot \mathbf{U} \cdot \widetilde{Z} = \widetilde{W}. \tag{3.6}$$

Therefore, by assuming that $\widetilde{Y} = \mathbf{U} \cdot \widetilde{Z}$, we will have $\mathbf{L} \cdot \widetilde{Y} = \widetilde{W}$.

In the first step, we obtain $\widetilde{Y}$ by forward substitution, since $\mathbf{L}$ is a lower-triangular matrix,

$$\widetilde{y}_1 = \widetilde{w}_1, \tag{3.7}$$

$$\widetilde{y}_k = \widetilde{w}_k - \sum_{j=1}^{k-1} l_{kj} \cdot \widetilde{y}_j, \qquad k = 2, 3, \ldots, n. \tag{3.8}$$

In the next step, we find Crout's solution $\widetilde{Z}$ from $\mathbf{U} \cdot \widetilde{Z} = \widetilde{Y}$ by backward substitution, since $\mathbf{U}$ is a upper-triangular matrix,

$$\widetilde{z}_n = \frac{1}{u_{nn}} \widetilde{y}_n, \tag{3.9}$$

$$\widetilde{z}_k = \frac{1}{u_{kk}} \left(\widetilde{y}_k - \sum_{j=k+1}^{n} u_{kj} \cdot \widetilde{z}_j \right), \tag{3.10}$$

where $k = n-1, n-2, \dots, 1$.

In this paper, the obtained solution via the above process is denoted by

$$\widetilde{Z}_C = (\widetilde{z}_{1_C}, \widetilde{z}_{2_C}, \dots, \widetilde{z}_{n_C})^T,$$

where $\widetilde{z}_{j_C} = \widetilde{p}_{j_C} + i\widetilde{q}_{j_C}$, $j = 1, 2, \dots, n$. Also, it should be noted that the arithmetic operations used in the Eqs. (3.7)-(3.10) are presented in Definition 2.4.

Theorem 3.1 *If for the system (3.1), the algebraic solution $\widetilde{Z}_A$ and Crout's solution $\widetilde{Z}_C$ both are available, then we have*

$$\widetilde{Z}_A \subseteq \widetilde{Z}_C.$$

It is sufficient to show

$$[\widetilde{z}_{k_A}]_\alpha \subseteq [\widetilde{z}_{k_C}]_\alpha, \quad \alpha \in [0,1],$$

where $k = 1, 2, \dots, n$. Now suppose that $\alpha \in [0,1]$ be arbitrary and fixed and also $x_k \in [\widetilde{z}_{k_A}]_\alpha$ for all $k = 1, 2, \dots, n$. Therefore, we conclude $X = (x_1, x_2, \dots, x_n)^T \in [\widetilde{Z}_A]_\alpha$. On the other hand, since $\widetilde{Z}_A$ is an algebraic solution, then $\mathbf{C} \cdot \widetilde{Z}_A = \widetilde{W}$ and consequently $\mathbf{C} \cdot [\widetilde{Z}_A]_\alpha = [\widetilde{W}]_\alpha$. Then we have

$$\exists W = (w_1, w_2, \dots, w_n)^T \in [\widetilde{W}]_\alpha ; \mathbf{C} \cdot X = W,$$

where $\mathbf{C}$ is a complex crisp matrix and X and W are two complex crisp vectors. By Eq. (3.5) and setting $\mathbf{U} \cdot X = Y'$, we will have $\mathbf{L} \cdot Y' = W$. Now, since $\mathbf{L}$ is a lower-triangular matrix, we obtain

$$y'_1 = w_1, \tag{3.11}$$

$$y'_k = w_k - \sum_{j=1}^{k-1} l_{kj} \cdot y'_j, \qquad k = 2, 3, \dots, n. \tag{3.12}$$

Since $W = (w_1, w_2, \dots, w_n)^T \in [\widetilde{W}]_\alpha$, then $Y' = (y'_1, y'_2, \dots, y'_n)^T \in [\widetilde{Y}]_\alpha$, where $\widetilde{Y}$ is defined based on the Eqs. (3.7) and (3.8).

On the other hand, since $\mathbf{U}$ is a upper-triangular matrix, we have

$$x_n = \frac{1}{u_{nn}} y'_n, \tag{3.13}$$

$$x_k = \frac{1}{u_{kk}} \left(y'_k - \sum_{j=k+1}^{n} u_{kj} \cdot x_j \right), \tag{3.14}$$

where $k = n-1, n-2, \dots, 1$. Now, since $Y' = (y'_1, y'_2, \dots, y'_n)^T \in [\widetilde{Y}]_\alpha$, then $X = (x_1, x_2, \dots, x_n)^T \in [\widetilde{Z}_C]_\alpha$, where $\widetilde{Z}_C$ is obtained by the Eqs. (3.9) and (3.10). Consequently $\widetilde{Z}_A \subseteq \widetilde{Z}_C$.

In the following theorem, we present another relation between the algebraic solution $\widetilde{Z}_A$ and Crout's solution $\widetilde{Z}_C$.

Theorem 3.2 *Suppose that for the fuzzy complex linear system (3.1), the solutions $\widetilde{Z}_A$ and $\widetilde{Z}_C$ both are available and also the coefficient matrix* $\mathbf{C}$ *be nonsingular. Then, based on Definition 2.5, we have*

$$[\widetilde{Z}_A]^c_\alpha = [\widetilde{Z}_C]^c_\alpha, \qquad \forall \alpha \in [0,1].$$

At first, since $\widetilde{Z}_A$ is the algebraic solution, then $\mathbf{C} \cdot \widetilde{Z}_A = \widetilde{W}$, that means

$$\sum_{k=1}^{n} c_{kj} \cdot [\underline{z_{k_A}}(\alpha), \overline{z_{k_A}}(\alpha)] = [\underline{w_j}(\alpha), \overline{w_j}(\alpha)],$$

and consequently

$$\sum_{k=1}^{n} c_{kj} \left(\underline{z_{k_A}}(\alpha) + \overline{z_{k_A}}(\alpha) \right) = \left(\underline{w_j}(\alpha) + \overline{w_j}(\alpha) \right),$$

where $j = 1, 2, \dots, n$, and $\alpha \in [0,1]$. This results $\mathbf{C} \cdot [\widetilde{Z}_A]^c_\alpha = [\widetilde{W}]^c_\alpha$ and since the α-center of a fuzzy complex number is a crisp complex number, therefore we conclude

$$[\widetilde{Z}_A]^c_\alpha = \mathbf{C}^{-1} \cdot [\widetilde{W}]^c_\alpha \tag{3.15}$$

On the other hand, by Eqs. (3.7) and (3.8), for any $\alpha \in [0,1]$ we have

$$[\widetilde{y}_1]^c_\alpha = [\widetilde{w}_1]^c_\alpha,$$

$$[\widetilde{y}_k]^c_\alpha = [\widetilde{w}_k]^c_\alpha - \sum_{j=1}^{k-1} l_{kj} \cdot [\widetilde{y}_j]^c_\alpha,$$

where $k = 2, 3, \dots, n$.

Since the α-center of a fuzzy complex number is a crisp complex number, therefore we can write the above equations as follows

$$[\widetilde{Y}]^c_\alpha = \mathbf{L}^{-1} \cdot [\widetilde{W}]^c_\alpha. \tag{3.16}$$

Similarly, by Eqs. (3.9) and (3.10), we have

$$[\widetilde{z}_{n_C}]^c_\alpha = \frac{1}{u_{nn}} [\widetilde{y}_n]^c_\alpha,$$

$$[\widetilde{z}_{k_C}]^c_\alpha = \frac{1}{u_{kk}}\left([\widetilde{y}_k]^c_\alpha - \sum_{j=k+1}^{n} u_{kj}\cdot[\widetilde{z}_{j_C}]^c_\alpha\right),$$

where $k = n-1, n-2, \ldots, 1$. Then

$$[\widetilde{Z}_C]^c_\alpha = \mathbf{U}^{-1}\cdot[\widetilde{Y}]^c_\alpha. \qquad (3.17)$$

From Eqs. (3.16) and (3.17) we conclude

$$[\widetilde{Z}_C]^c_\alpha = \mathbf{U}^{-1}\cdot\mathbf{L}^{-1}\cdot[\widetilde{W}]^c_\alpha. \qquad (3.18)$$

and since $\mathbf{C} = \mathbf{L}\cdot\mathbf{U}$, consequently

$$[\widetilde{Z}_C]^c_\alpha = \mathbf{C}^{-1}\cdot[\widetilde{W}]^c_\alpha \qquad (3.19)$$

From Eqs. (3.15) and (3.19), the proof is completed.

In the continuation, we represent a numerical example to illustrate two above theorems. All numerical computations are obtained by using of MATLAB software.

Example 3.1 *Consider the following* 3×3 *fuzzy complex linear system*

$$\begin{cases}(1+3i)\widetilde{z}_1 + (2-i)\widetilde{z}_2 + (1+i)\widetilde{z}_3 = \widetilde{w}_1,\\ (-1+i)\widetilde{z}_1 + (2-3i)\widetilde{z}_2 + (2+i)\widetilde{z}_3 = \widetilde{w}_2,\\ (1-i)\widetilde{z}_1 + (2-2i)\widetilde{z}_2 + (1-3i)\widetilde{z}_3 = \widetilde{w}_3,\end{cases}$$

where the fuzzy complex numbers $\widetilde{w}_1, \widetilde{w}_2$ and $\widetilde{w}_3$ are specified by their α-levels as follows:

$$\begin{aligned}[\widetilde{w}_1]_\alpha &= [-28+11\alpha,\, 32-12\alpha]\\ &\quad +i\,[-29+9\alpha,\, 26-18\alpha],\\ [\widetilde{w}_2]_\alpha &= [-21+14\alpha,\, 35-13\alpha]\\ &\quad +i\,[-34+10\alpha,\, 28-17\alpha],\\ [\widetilde{w}_3]_\alpha &= [-27+12\alpha,\, 39-14\alpha]\\ &\quad +i\,[-33+12\alpha,\, 19-14\alpha].\end{aligned}$$

The α-levels of unique algebraic solution of the above system is as

$$[\widetilde{Z}_A]_\alpha = \begin{pmatrix}[\widetilde{z}_{1_A}]_\alpha\\ [\widetilde{z}_{2_A}]_\alpha\\ [\widetilde{z}_{3_A}]_\alpha\end{pmatrix} = \begin{pmatrix}[-4+\alpha,\, 2-3\alpha] + i\,[-4+\alpha,\, 5-\alpha]\\ [-1+2\alpha,\, 4-\alpha] + i\,[-3+\alpha,\, 3-2\alpha]\\ [1+\alpha,\, 3-\alpha] + i\,[-4+\alpha,\, 5-\alpha]\end{pmatrix}.$$

Also, using the classic Crout decomposition method and Eqs. (3.7)-(3.10), the α-levels of the Crout's solution is obtained as

$$[\widetilde{Z}_C]_\alpha = \begin{pmatrix}[\widetilde{z}_{1_C}]_\alpha\\ [\widetilde{z}_{2_C}]_\alpha\\ [\widetilde{z}_{3_C}]_\alpha\end{pmatrix} = \begin{pmatrix}[-64.271+27.605\alpha,\, 62.271-29.605\alpha]\\ \quad +i\,[64.488+28.205\alpha,\, 65.488-28.205\alpha]\\ [-39.431+18.415\alpha,\, 42.431-17.415\alpha]\\ \quad +i\,[-39.515+17.415\alpha,\, 39.515-18.415\alpha]\\ [-31.169+15.122\alpha,\, 35.168-15.122\alpha]\\ \quad +i\,[-34.501+15.122\alpha,\, 35.501-15.122\alpha]\end{pmatrix}.$$

Obviously, the Crout's solution is not an algebraic solution and

$$[\widetilde{z}_{j_A}]_\alpha \subseteq [\widetilde{Z}_{j_C}]_\alpha,$$

for $j = 1, 2, 3$ and $\alpha \in [0, 1]$. This means that

$$\widetilde{Z}_A \subseteq \widetilde{Z}_C.$$

Also, it can be easily investigated that

$$\begin{aligned}[\widetilde{Z}_A]^c_\alpha &= [\widetilde{Z}_C]^c_\alpha\\ &= C^{-1}[\widetilde{W}]^c_\alpha\\ &= \begin{pmatrix}(-1-\alpha)+i(0.5)\\ (1.5+0.5\alpha)-i(0.5\alpha)\\ (2)+i(0.5\alpha)\end{pmatrix}.\end{aligned}$$

Therefore, in this example, the results of Theorems 3.1 and 3.2 are illustrated.

4 A limited version of Crout method

According to the Example 3.1, it is clear that the obtained solution by classic Crout decomposition method may not be an algebraic solution of the fuzzy complex linear system (3.1). In this section, we present a limited version of classic Crout method such that the obtained solution will be always an unique algebraic solution of system (3.1), if it exists. The proposed method is displayed based on Theorems 3.1 and 3.2. In the proposed method, as before, we convert a fuzzy complex linear system to an interval complex linear system by the concept of the α-levels of a fuzzy complex number. Then, we use the classic Crout method to obtain the Crout's solution $[\widetilde{Z}_C]_\alpha = ([\widetilde{z}_{1C}]_\alpha, [\widetilde{z}_{2C}]_\alpha, \ldots, [\widetilde{z}_{nC}]_\alpha)^T$ for the system (3.3). In the next step, we limit the Crout's solution $[\widetilde{Z}_C]_\alpha$ by some parameters such that the obtained new solution satisfies all equations of (3.3). Finally, if the obtained solution constructs the α-levels of a fuzzy number complex vector, then it is an unique algebraic solution of the fuzzy complex linear system (3.1). Otherwise, the system (3.1) does not has unique algebraic solution. In the proposed method, we define

$$[\widetilde{Z}_A]_\alpha = \begin{pmatrix} [\widetilde{z}_{1A}]_\alpha \\ [\widetilde{z}_{2A}]_\alpha \\ \vdots \\ [\widetilde{z}_{nA}]_\alpha \end{pmatrix}$$

$$= \begin{pmatrix} [\underline{z_{1C}}(\alpha) + \theta_1(\alpha)\,,\ \overline{z_{1C}}(\alpha) - \theta_1(\alpha)] \\ [\underline{z_{2C}}(\alpha) + \theta_2(\alpha)\,,\ \overline{z_{2C}}(\alpha) - \theta_2(\alpha)] \\ \vdots \\ [\underline{z_{nC}}(\alpha) + \theta_n(\alpha)\,,\ \overline{z_{nC}}(\alpha) - \theta_n(\alpha)] \end{pmatrix}, \tag{4.20}$$

where $\theta_j(\alpha)$, $j = 1, 2, \ldots, n$ are crisp complex functions in respect to α and satisfy the following conditions

$$0 \leqslant Real(\theta_j(\alpha)) \leqslant Real\left([\widetilde{z}_{jC}]^r\right), \tag{4.21}$$

$$0 \leqslant Imag(\theta_j(\alpha)) \leqslant Imag\left([z_{jD}]^r\right). \tag{4.22}$$

According to the Eq. (4.20), it is clear that the algebraic solution is obtained by limiting of the solution's Crout. By this reason, we call the functions θ_j, limiting functions. Also, it should be noted that the conditions (4.21) and (4.22) guarantee that the Eq. (4.20) be a complex interval vector, but they don't guarantee that the Eq. (4.20) constructs the α-levels of a fuzzy complex number. In fact, in the proposed method, we investigate that Eq. (4.20) constructs the α-levels of a fuzzy complex number in the final step.

If we set $\theta_j(\alpha) = \beta_j(\alpha) + i\,\gamma_j(\alpha)$ and $[\widetilde{z}_{jC}]_\alpha = [\widetilde{p}_{jC}]_\alpha + i[\widetilde{q}_{jC}]_\alpha$ for $j = 1, 2, \ldots, n$, then $\underline{z_{jC}}(\alpha) = \underline{p_{jC}}(\alpha) + i\,\underline{q_{jC}}(\alpha)$, $\overline{z_{jC}}(\alpha) = \overline{p_{jC}}(\alpha) + i\,\overline{q_{jC}}(\alpha)$ and the Eqs. (4.20)-(4.22) can be rewritten as

$$[\widetilde{Z}_A]_\alpha = \begin{pmatrix} [\underline{p_{1C}}(\alpha) + \beta_1(\alpha), \overline{p_{1C}}(\alpha) - \beta_1(\alpha)] \\ +i[\underline{q_{1C}}(\alpha) + \gamma_1(\alpha), \overline{q_{1C}}(\alpha) - \gamma_1(\alpha)] \\ \\ [\underline{p_{2C}}(\alpha) + \beta_2(\alpha), \overline{p_{2C}}(\alpha) - \beta_2(\alpha)] \\ +i[\underline{q_{2C}}(\alpha) + \gamma_2(\alpha), \overline{q_{2C}}(\alpha) - \gamma_2(\alpha)] \\ \vdots \\ [\underline{p_{nC}}(\alpha) + \beta_n(\alpha), \overline{p_{nC}}(\alpha) - \beta_n(\alpha)] \\ +i[\underline{q_{nC}}(\alpha) + \gamma_n(\alpha), \overline{q_{nC}}(\alpha) - \gamma_n(\alpha)] \end{pmatrix}, \tag{4.23}$$

also, the conditions (4.21) and (4.22) can be replaced, for $j = 1, 2, \ldots, n$, by

$$0 \leqslant \beta_j(\alpha) \leqslant [\widetilde{p}_{jC}]^r_\alpha, \tag{4.24}$$

$$0 \leqslant \gamma_j(\alpha) \leqslant [\widetilde{q}_{jC}]^r_\alpha. \tag{4.25}$$

After obtaining the solution's Crout, we determine $\beta_j(\alpha)$ and $\gamma_j(\alpha)$, for $j = 1, 2, \ldots, n$, such that the vector (4.23) be an algebraic solution for the system (3.3) or (3.4). In other words

$$\sum_{j=1}^{n} c_{kj} \Big([\underline{p_{jC}}(\alpha) + \beta_j(\alpha), \overline{p_{jC}}(\alpha) - \beta_j(\alpha)] + i\,[\underline{q_{jC}}(\alpha) + \gamma_j(\alpha), \overline{q_{jC}}(\alpha) - \gamma_j(\alpha)] \Big) = [\widetilde{w}_k]_\alpha,$$

for $k = 1, 2, \ldots, n$. By assuming that $[\widetilde{w}_k]_\alpha = [\underline{u_k}(\alpha), \overline{u_k}(\alpha)] + i\,[\underline{v_k}(\alpha), \overline{v_k}(\alpha)]$ and $c_{kj} = a_{kj} + i\,b_{kj}$, we conclude

$$\begin{aligned} \underline{u_k}(\alpha) &= \sum_{a_{kj} \geqslant 0} a_{kj} \left(\underline{p_{jC}}(\alpha) + \beta_j(\alpha) \right) \\ &+ \sum_{a_{kj} < 0} a_{kj} \left(\overline{p_{jC}}(\alpha) - \beta_j(\alpha) \right) \\ &- \sum_{b_{kj} < 0} b_{kj} \left(\underline{q_{jC}}(\alpha) + \gamma_j(\alpha) \right) \\ &- \sum_{b_{kj} \geqslant 0} b_{kj} \left(\overline{q_{jC}}(\alpha) - \gamma_j(\alpha) \right), \end{aligned}$$

$$\begin{aligned} \overline{u_k}(\alpha) &= \sum_{a_{kj} \geqslant 0} a_{kj} \left(\overline{p_{jC}}(\alpha) - \beta_j(\alpha) \right) \\ &+ \sum_{a_{kj} < 0} a_{kj} \left(\underline{p_{jC}}(\alpha) + \beta_j(\alpha) \right) \\ &- \sum_{b_{kj} < 0} b_{kj} \left(\overline{q_{jC}}(\alpha) - \gamma_j(\alpha) \right) \\ &- \sum_{b_{kj} \geqslant 0} b_{kj} \left(\underline{q_{jC}}(\alpha) + \gamma_j(\alpha) \right), \end{aligned}$$

$$\begin{aligned}\underline{v_k}(\alpha) &= \sum_{a_{kj}\geqslant 0} a_{kj}\left(\underline{q_{jC}}(\alpha)+\gamma_j(\alpha)\right)\\ &+ \sum_{a_{kj}<0} a_{kj}\left(\overline{q_{jC}}(\alpha)-\gamma_j(\alpha)\right)\\ &+ \sum_{b_{kj}\geqslant 0} b_{kj}\left(\underline{p_{jC}}(\alpha)+\beta_j(\alpha)\right)\\ &+ \sum_{b_{kj}<0} b_{kj}\left(\overline{p_{jC}}(\alpha)-\beta_j(\alpha)\right),\end{aligned}$$

$$\begin{aligned}\overline{v_k}(\alpha) &= \sum_{a_{kj}\geqslant 0} a_{kj}\left(\overline{q_{jC}}(\alpha)-\gamma_j(\alpha)\right)\\ &+ \sum_{a_{kj}<0} a_{kj}\left(\underline{q_{jC}}(\alpha)+\gamma_j(\alpha)\right)\\ &+ \sum_{b_{kj}\geqslant 0} b_{kj}\left(\overline{p_{jC}}(\alpha)-\beta_j(\alpha)\right)\\ &+ \sum_{b_{kj}<0} b_{kj}\left(\underline{p_{jC}}(\alpha)+\beta_j(\alpha)\right).\end{aligned}$$

By the above equations, we obtain

$$\begin{aligned}\overline{u_k}(\alpha)-\underline{u_k}(\alpha) &= \sum_{j=1}^{n}|a_{kj}|\left(\overline{p_{jC}}(\alpha)-\beta_j(\alpha)\right)\\ &- \sum_{j=1}^{n}|a_{kj}|\left(\underline{p_{jC}}(\alpha)+\beta_j(\alpha)\right)\\ &+ \sum_{j=1}^{n}|b_{kj}|\left(\overline{q_{jC}}(\alpha)-\gamma_j(\alpha)\right)\\ &- \sum_{j=1}^{n}|b_{kj}|\left(\underline{q_{jC}}(\alpha)+\gamma_j(\alpha)\right),\end{aligned}$$

$$\begin{aligned}\overline{v_k}(\alpha)-\underline{v_k}(\alpha) &= \sum_{j=1}^{n}|a_{kj}|\left(\overline{q_{jC}}(\alpha)-\gamma_j(\alpha)\right)\\ &- \sum_{j=1}^{n}|a_{kj}|\left(\underline{q_{jC}}(\alpha)+\gamma_j(\alpha)\right)\\ &+ \sum_{j=1}^{n}|b_{kj}|\left(\overline{p_{jC}}(\alpha)-\beta_j(\alpha)\right)\\ &- \sum_{j=1}^{n}|b_{kj}|\left(\underline{p_{jC}}(\alpha)+\beta_j(\alpha)\right).\end{aligned}$$

The above equations can be rewritten as follows

$$\begin{aligned}[\widetilde{u}_k]^r_\alpha &= \sum_{j=1}^{n}|a_{kj}|\cdot[\widetilde{p}_{jC}]^r_\alpha - \sum_{j=1}^{n}|a_{kj}|\,\beta_j(\alpha)\\ &+ \sum_{j=1}^{n}|b_{kj}|\cdot[\widetilde{q}_{jC}]^r_\alpha - \sum_{j=1}^{n}|b_{kj}|\,\gamma_j(\alpha),\end{aligned}$$

$$\begin{aligned}[\widetilde{v}_k]^r_\alpha &= \sum_{j=1}^{n}|a_{kj}|\cdot[\widetilde{q}_{jC}]^r_\alpha - \sum_{j=1}^{n}|a_{kj}|\,\gamma_j(\alpha)\\ &+ \sum_{j=1}^{n}|b_{kj}|\cdot[\widetilde{p}_{jC}]^r_\alpha - \sum_{j=1}^{n}|b_{kj}|\,\beta_j(\alpha),\end{aligned}$$

for $k = 1, 2, \ldots, n$. Hence, in the matrix form, we have

$$\begin{cases}[\widetilde{U}]^r_\alpha = |\mathbf{A}|\cdot[\widetilde{P}_C]^r_\alpha - |\mathbf{A}|\cdot\mathbf{fi}(\alpha)\\ \qquad +|\mathbf{B}|\cdot[\widetilde{Q}_C]^r_\alpha - |\mathbf{B}|\cdot\mathbf{fl}(\alpha),\\ [\widetilde{V}]^r_\alpha = |\mathbf{A}|\cdot[\widetilde{Q}_C]^r_\alpha - |\mathbf{A}|\cdot\mathbf{fl}(\alpha)\\ \qquad +|\mathbf{B}|\cdot[\widetilde{P}_C]^r_\alpha - |\mathbf{B}|\cdot\mathbf{fi}(\alpha),\end{cases} \tag{4.26}$$

where

$$\begin{gathered}[\widetilde{U}]^r_\alpha = ([\widetilde{u}_1]^r_\alpha, [\widetilde{u}_2]^r_\alpha, \ldots, [\widetilde{u}_n]^r_\alpha)^T,\\ [\widetilde{V}]^r_\alpha = ([\widetilde{v}_1]^r_\alpha, [\widetilde{v}_2]^r_\alpha, \ldots, [\widetilde{v}_n]^r_\alpha)^T,\\ [\widetilde{P}_C]^r_\alpha = ([\widetilde{p}_{1C}]^r_\alpha, [\widetilde{p}_{2C}]^r_\alpha, \ldots, [\widetilde{p}_{nC}]^r_\alpha)^T,\\ [\widetilde{Q}]^r_\alpha = ([\widetilde{q}_{1C}]^r_\alpha, [\widetilde{q}_{2C}]^r_\alpha, \ldots, [\widetilde{q}_{nC}]^r_\alpha)^T,\\ |\mathbf{A}| = (|a_{kj}|)_{n\times n},\ \ |\mathbf{B}| = (|b_{kj}|)_{n\times n},\\ \beta(\alpha) = (\beta_1(\alpha), \beta_2(\alpha), \ldots, \beta_n(\alpha))^T,\\ \gamma(\alpha) = (\gamma_1(\alpha), \gamma_2(\alpha), \ldots, \gamma_n(\alpha))^T.\end{gathered}$$

By Eq. (4.26) we conclude

$$\begin{cases}|\mathbf{A}|\cdot\mathbf{fi}(\alpha) + |\mathbf{B}|\cdot\mathbf{fl}(\alpha) = |\mathbf{A}|\cdot[\widetilde{P}_C]^r_\alpha\\ \qquad +|\mathbf{B}|\cdot[\widetilde{Q}_C]^r_\alpha\\ \qquad -[\widetilde{U}]^r_\alpha,\\ |\mathbf{B}|\cdot\mathbf{fi}(\alpha) + |\mathbf{A}|\cdot\mathbf{fl}(\alpha) = |\mathbf{A}|\cdot[\widetilde{Q}_C]^r_\alpha\\ \qquad +|\mathbf{B}|\cdot[\widetilde{P}_C]^r_\alpha\\ \qquad -[\widetilde{V}]^r_\alpha,\end{cases} \tag{4.27}$$

Also, if we set

$$K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

$$= \begin{pmatrix} |\mathbf{A}|\cdot[\widetilde{P}_C]^r_\alpha + |\mathbf{B}|\cdot[\widetilde{Q}_C]^r_\alpha - [\widetilde{U}]^r_\alpha \\ |\mathbf{A}|\cdot[\widetilde{Q}_C]^r_\alpha + |\mathbf{B}|\cdot[\widetilde{P}_C]^r_\alpha - [\widetilde{V}]^r_\alpha \end{pmatrix}, \quad (4.28)$$

then we have

$$\begin{pmatrix} |\mathbf{A}| & |\mathbf{B}| \\ |\mathbf{B}| & |\mathbf{A}| \end{pmatrix} \begin{pmatrix} \beta(\alpha) \\ \gamma(\alpha) \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \quad (4.29)$$

The values of vectors $\beta(\alpha)$ and $\gamma(\alpha)$ can be found by solving the parametric $2n \times 2n$ real linear system (4.29). If the obtained vectors $\beta(\alpha)$ and $\gamma(\alpha)$ satisfy the conditions (4.24) and (4.25) and also the Eq. (4.23) construct the α-levels of a fuzzy complex number vector, then Eq.(4.23) is an algebraic solution for the fuzzy complex linear system (3.1) in the α-levels form. Otherwise, the fuzzy complex linear system (3.1) does not has any algebraic solution. On the other hand, the parametric real linear system (4.29) has unique solution if and only if its coefficient matrix be nonsingular.

Theorem 4.1 *The coefficient matrix of the parametric real linear system (4.29) is nonsingular if and only if the matrices* $|\mathbf{A}|+|\mathbf{B}|$ *and* $|\mathbf{A}|-|\mathbf{B}|$ *are both nonsingular.*

The proof is the same as the proof of Theorem 1 in [7].

Based on the steps of our method, the following theorem is presented.

Theorem 4.2 *The fuzzy complex linear system (3.1) has unique algebraic solution if and only if all of the following conditions hold:*

1. The matrices $\mathbf{C}$, $|\mathbf{A}|+|\mathbf{B}|$ and $|\mathbf{A}|-|\mathbf{B}|$ be nonsingular,
2. The vectors $\beta(\alpha)$ and $\gamma(\alpha)$ satisfy the conditions (4.24) and (4.25),
3. The Eq. (4.23) construct the α-levels of a fuzzy complex number vector.

5 Numerical examples

In this section, we present two numerical example to illustrate the effectiveness of the proposed method.

Example 5.1 *Consider the following* 4×4 *fuzzy complex linear system*

$$\begin{cases} (3-i)\widetilde{z}_1 + (1+2i)\widetilde{z}_2 + (3+i)\widetilde{z}_3 \\ \qquad +(2+i)\widetilde{z}_4 = \widetilde{w}_1, \\ (1-2i)\widetilde{z}_1 + (-2+i)\widetilde{z}_2 + (1-3i)\widetilde{z}_3 \\ \qquad +(1-3i)\widetilde{z}_4 = \widetilde{w}_2, \\ (3-3i)\widetilde{z}_1 + (2+4i)\widetilde{z}_2 + (-1+i)\widetilde{z}_3 \\ \qquad +(3-2i)\widetilde{z}_4 = \widetilde{w}_3, \\ (1-3i)\widetilde{z}_1 + (2+2i)\widetilde{z}_2 + (3-2i)\widetilde{z}_3 \\ \qquad +(1+2i)\widetilde{z}_4 = \widetilde{w}_4, \end{cases} \quad (5.30)$$

where the α-levels of fuzzy complex numbers $\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3$ and $\widetilde{w}_4$ are as follows:

$$[\widetilde{w}_1]_\alpha = [-28+18\alpha,\, 18-26\alpha] \\ +i\,[-14+28\alpha,\, 43-22\alpha],$$

$$[\widetilde{w}_2]_\alpha = [-22+24\alpha,\, 36-26\alpha] \\ +i\,[-28+26\alpha,\, 17-17\alpha],$$

$$[\widetilde{w}_3]_\alpha = [-61+26\alpha,\, 7-34\alpha] \\ +i\,[-22+40\alpha,\, 56-29\alpha],$$

$$[\widetilde{w}_4]_\alpha = [-19+26\alpha,\, 37-25\alpha] \\ +i\,[-29+34\alpha,\, 36-25\alpha].$$

In the above system, we have

$$det(\mathbf{C}) = 106 + 51i,\ det(|\mathbf{A}|+|\mathbf{B}|) = 39,$$

$$det(|\mathbf{A}|-|\mathbf{B}|) = 17,$$

where $\mathbf{C}$ is the complex coefficient matrix of system and $\mathbf{A}$ and $\mathbf{B}$ are the real and imaginary parts of the matrix $\mathbf{C}$, respectively. Therefore, the systems (3.3) and (4.29) have unique solutions.

Now, based on the proposed method, we first solve the parametric interval complex linear system (3.3) by the classic Crout method. For this, by Eqs. (3.7)-(3.10) we obtain the α-levels of Crout's solution as follows:

$$[\widetilde{Z}_C]=\begin{pmatrix}[\widetilde{z}_{1_C}]_\alpha\\ [\widetilde{z}_{2_C}]_\alpha\\ [\widetilde{z}_{3_C}]_\alpha\\ [\widetilde{z}_{4_C}]_\alpha\end{pmatrix} = \begin{pmatrix}[-412.52+372.56\alpha, 410.52-374.56\alpha] \\ \quad +i[-411.46+374.50\alpha, 415.46-374.50\alpha] \\ [-183.41+165.58\alpha, 181.41-164.58\alpha] \\ \quad +i[-177.79+164.72\alpha, 184.79-164.72\alpha] \\ [-114.91+107.36\alpha, 121.91-107.36\alpha] \\ \quad +i[-117.25+108.47\alpha, 121.25-107.47\alpha] \\ [-93.61+82.12\alpha, 88.62-83.12\alpha] \\ \quad +i[-90.08+81.34\alpha, 89.08-81.34\alpha]\end{pmatrix}.$$

In the next step, we obtain the parametric vector K by Eq. (4.28) as follows:

$$K=\begin{pmatrix}K_1\\K_2\end{pmatrix}=\begin{pmatrix}2916.27-2644.34\alpha\\2591.56-2350.34\alpha\\4221.05-3829.09\alpha\\3215.29-2914.47\alpha\\2915.94-2643.49\alpha\\2596.36-2353.89\alpha\\4216.78-3824.02\alpha\\3209.27-2910.49\alpha\end{pmatrix},$$

Now, by solving the parametric real linear system (4.29), we obtain the vectors functions $\beta(\alpha)$ and $\gamma(\alpha)$ as follows:

$$\beta(\alpha)=\begin{pmatrix}\beta_1(\alpha)\\\beta_2(\alpha)\\\beta_3(\alpha)\\\beta_4(\alpha)\end{pmatrix}=\begin{pmatrix}409.52-371.56\alpha\\178.41-161.58\alpha\\117.91-106.36\alpha\\89.61-81.12\alpha\end{pmatrix},$$

and

$$\gamma(\alpha)=\begin{pmatrix}\gamma_1(\alpha)\\\gamma_2(\alpha)\\\gamma_3(\alpha)\\\gamma_4(\alpha)\end{pmatrix}=\begin{pmatrix}412.46-373.50\alpha\\179.79-163.72\alpha\\116.25-105.47\alpha\\88.08-80.34\alpha\end{pmatrix}.$$

Regarding to the results obtained in the above process, it can be easily investigated that the vector functions $\beta(\alpha)$ and $\gamma(\alpha)$ satisfy the conditions (4.24) and (4.25), respectively. Finally, by substituting the obtained values $[\widetilde{Z}_C]_\alpha$, $\beta(\alpha)$ and $\gamma(\alpha)$ in Eq. (4.23), we obtain the α-levels of unique algebraic solution of the fuzzy complex linear system (5.30) as follows:

$$[\widetilde{Z}_A]=\begin{pmatrix}[\widetilde{z}_{1_A}]_\alpha\\ [\widetilde{z}_{2_A}]_\alpha\\ [\widetilde{z}_{3_A}]_\alpha\\ [\widetilde{z}_{4_A}]_\alpha\end{pmatrix} = \begin{pmatrix}[-3+\alpha, 1-3\alpha] \\ \quad +i\,[1+\alpha, 3-\alpha] \\ [-5+4\alpha, 3-3\alpha] \\ \quad +i\,[2+\alpha, 5-\alpha] \\ [3+\alpha, 4-\alpha] \\ \quad +i\,[-1+3\alpha, 5-2\alpha] \\ [-4+\alpha, -1-2\alpha] \\ \quad +i\,[-2+\alpha, 1-\alpha]\end{pmatrix}.$$

Obviously, it can be showed that the obtained solution represents α-levels of a fuzzy complex number vector. Therefore, this algebraic solution is acceptable.

Example 5.2 *Consider the* 5×5 *fuzzy complex linear system*

$$\begin{cases}(2+2i)\widetilde{z}_1+(1+3i)\widetilde{z}_2+(3-2i)\widetilde{z}_3 \\ \quad +(4+i)\widetilde{z}_4+(1-3i)\widetilde{z}_5=\widetilde{w}_1, \\ (-2+i)\widetilde{z}_1+(-1-i)\widetilde{z}_2+(3+i)\widetilde{z}_3 \\ \quad +(-1+2i)\widetilde{z}_4+(2-4i)\widetilde{z}_5=\widetilde{w}_2, \\ (4-i)\widetilde{z}_1+(4-3i)\widetilde{z}_2+(1+i)\widetilde{z}_3 \\ \quad +(2-i)\widetilde{z}_4+(-2+i)\widetilde{z}_5=\widetilde{w}_3, \\ (3-2i)\widetilde{z}_1+(1+2i)\widetilde{z}_2+(2-2i)\widetilde{z}_3 \\ \quad +(1+i)\widetilde{z}_4+(3-3i)\widetilde{z}_5=\widetilde{w}_4, \\ (4-i)\widetilde{z}_1+(4-2i)\widetilde{z}_2+(3-4i)\widetilde{z}_3 \\ \quad +(1+2i)\widetilde{z}_4+(1-i)\widetilde{z}_5=\widetilde{w}_5,\end{cases} \tag{5.31}$$

where the α-levels of fuzzy complex numbers $\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3, \widetilde{w}_4$ and $\widetilde{w}_5$ are as follows:

$$[\widetilde{w}_1]_\alpha=[-56+40\alpha,\, 64-39\alpha] \\ \quad +i\,[-30+40\alpha,\, 81-36\alpha],$$

$$[\widetilde{w}_2]_\alpha=[-63+31\alpha,\, 41-28\alpha] \\ \quad +i\,[-43+29\alpha,\, 44-32\alpha],$$

$$[\widetilde{w}_3]_\alpha = [-33+47\alpha,\ 78-31\alpha] \\ +i\,[-47+28\alpha,\ 60-36\alpha],$$

$$[\widetilde{w}_4]_\alpha = [-46+38\alpha,\ 63-31\alpha] \\ +i\,[-29+33\alpha,\ 74-33\alpha],$$

$$[\widetilde{w}_5]_\alpha = [-46+53\alpha,\ 78-37\alpha] \\ +i\,[-21+34\alpha,\ 85-41\alpha].$$

In this example, we have

$$det(\mathbf{C}) = 799-3276i,\ det(|\mathbf{A}|+|\mathbf{B}|) = -615,$$

$$det(|\mathbf{A}|-|\mathbf{B}|) = -87.$$

Consequently, the systems (3.3) and (4.29) have unique solutions.

As before, we first obtain the α-levels of Crout's solution by Eqs. (3.7)-(3.10) as follows:

$$[\widetilde{Z}_C]=\begin{pmatrix}[\widetilde{z}_{1C}]_\alpha\\ [\widetilde{z}_{2C}]_\alpha\\ [\widetilde{z}_{3C}]_\alpha\\ [\widetilde{z}_{4C}]_\alpha\\ [\widetilde{z}_{5C}]_\alpha\end{pmatrix}=$$

$$\begin{pmatrix}[-15296.0+10080.3\alpha, 15298.0-10078.3\alpha] \\ +i[-15293.6+10079.0\alpha, 15298.6-10079.0\alpha] \\ \\ [-6276.1+4136.5\alpha, 6279.1-4134.5\alpha] \\ +i[-6273.9+4136.7\alpha, 6277.9-4135.7\alpha] \\ \\ [-1751.0+1152.2\alpha, 1747.0-1152.2\alpha] \\ +i[-1748.4+1152.2\alpha, 1749.4-1153.2\alpha] \\ \\ [-1187.6+784.1\alpha, 1194.6-785.1\alpha] \\ +i[-1190.5+784.0\alpha, 1191.5-785.0\alpha] \\ \\ [-564.6+372.1\alpha, 564.6-372.1\alpha] \\ +i[-563.8+372.8\alpha, 565.8-371.8\alpha]\end{pmatrix}.$$

In the next step, we obtain the parametric vector K by Eq. (4.28) as follows:

$$K=\begin{pmatrix}K_1\\K_2\end{pmatrix}=\begin{pmatrix}103190.6-67994.6\alpha\\ 072348.9-47675.8\alpha\\ 129131.8-85082.5\alpha\\ 108024.1-71181.0\alpha\\ 131029.2-86330.5\alpha\\ 103197.7-67994.7\alpha\\ 072356.1-47675.2\alpha\\ 129129.6-85089.5\alpha\\ 108028.0-71181.6\alpha\\ 131032.6-86338.1\alpha\end{pmatrix},$$

Now, by solving the parametric real linear system (4.29), we obtain the vectors functions $\beta(\alpha)$ and $\gamma(\alpha)$ as follows:

$$\beta(\alpha) = \begin{pmatrix}\beta_1(\alpha)\\ \beta_2(\alpha)\\ \beta_3(\alpha)\\ \beta_4(\alpha)\\ \beta_5(\alpha)\end{pmatrix} = \begin{pmatrix}15293.0-10076.3\alpha\\ 06275.1-04133.5\alpha\\ 01748.0-01151.2\alpha\\ 01188.6-00782.1\alpha\\ 00563.6-00371.1\alpha\end{pmatrix},$$

and

$$\gamma(\alpha) = \begin{pmatrix}\gamma_1(\alpha)\\ \gamma_2(\alpha)\\ \gamma_3(\alpha)\\ \gamma_4(\alpha)\\ \gamma_5(\alpha)\end{pmatrix} = \begin{pmatrix}15294.6-10078.0\alpha\\ 06272.9-04134.7\alpha\\ 01746.4-01150.2\alpha\\ 01187.5-00783.0\alpha\\ 00559.7-00370.8\alpha\end{pmatrix}.$$

It can be easily showed that the vector functions $\beta(\alpha)$ and $\gamma(\alpha)$ satisfy the conditions (4.24) and (4.25), respectively. Finally, by substituting the obtained values $[\widetilde{Z}_C]_\alpha$, $\beta(\alpha)$ and $\gamma(\alpha)$ in Eq. (4.23), we obtain the α-levels of unique algebraic solution of the fuzzy complex linear system (5.30) as follows:

$$[\widetilde{Z}_A]=\begin{pmatrix}[\widetilde{z}_{1A}]_\alpha\\ [\widetilde{z}_{2A}]_\alpha\\ [\widetilde{z}_{3A}]_\alpha\\ [\widetilde{z}_{4A}]_\alpha\\ [\widetilde{z}_{5A}]_\alpha\end{pmatrix}$$

$$= \begin{pmatrix}[-3+4\alpha, 5-2\alpha]+i\,[1+\alpha, 4-\alpha]\\ [-1+3\alpha, 4-\alpha]+i\,[-1+2\alpha, 5-\alpha]\\ [-3+\alpha, -1-\alpha]+i\,[-2+2\alpha, 3-3\alpha]\\ [1+2\alpha, 6-3\alpha]+i\,[-3+\alpha, 4-2\alpha]\\ [-1+\alpha, 1-\alpha]+i\,[-4+2\alpha, 6-\alpha]\end{pmatrix}.$$

Obviously, it can be showed that the obtained solution represents α-levels of a fuzzy complex number vector. Therefore, this algebraic solution is acceptable.

6 Conclusions

In this paper, we presented a limited version of the classic Crout decomposition method for algebraic solving of fuzzy complex linear systems. It is shown that unlike the original method, based on the limited version of method, we can obtain the algebraic solution of a fuzzy complex linear system, if it exists. It should be noted that the approach proposed in this paper, can be extended to other direct methods, for example Gaussian elimination method, Cramer method, LU (Cholesky and Doolittle) decomposition method, QR decomposition method and etc.

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