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Research Article

An Effective Numerical Technique for Solving Second Order Linear Two-Point Boundary Value Problems with Deviating Argument

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Abstract

Based on reproducing kernel theory, an effective numerical technique is proposed for solving second order linear two-point boundary value problems with deviating argument. In this method, reproducing kernels with Chebyshev polynomial form are used (C-RKM). The convergence and an error estimation of the method are discussed. The efficiency and the accuracy of the method is demonstrated on some numerical examples.

Keywords : Two-point boundary value problem; Second order boundary value problem; Deviating argument; Polynomial reproducing kernel; Chebyshev polynomials; Chebyshev reproducing kernel method.

1 Introduction

Boundary value problems (BVPs) associated with different kinds of differential equations play important rolls to modelling a wide variety of nature phenomena [1, 2, 3]. Differential equations with deviating argument have many applications such as study of problems in automatic control theory, problems of self-oscillating systems, long-term planning theory in economics, problems in rocket motion, a series of problems in biological science, and many other areas of science and technology. Many authors have been studied the question of existence and uniqueness of a solution for this type of differential equations, see e.g. [4, 5] and references cited therein. Finite differences method [4], collocation methods [10, 15], Richardson extrapolation [6], shooting techniques [7, 11], projection method [16], successive and Pad approximations [5, 12] and successive interpolations [8] are some of the existing numerical methods for boundary value problems of differential equations with deviating argument. In this paper, based on reproducing kernel with polynomial form [13, 14], we propose an effective numerical technique for solving the following second order two-point boundary value problems with a deviating argument:

$$\begin{cases} x''(t) + p(t)x(t) + q(t)x(\phi(t)) = f(t), \\ 0 \leq t \leq T, \\ x(0) = \alpha, \\ x(T) = \beta, \end{cases} \tag{1.1}$$

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Table 1: Numerical results for Example 4.1

τ_j	$x(\tau_j)$	$\lvert r_9(\tau_j)\rvert$	$\lvert R_9(\tau_j)\rvert$
0.07853981634	1.0815433471031854	1.30590E-14	1.50629E-11
0.15707963268	1.1687714705415926	1.64035E-14	2.15166E-11
0.23561944901	1.2612036262339692	4.20775E-14	1.85277E-11
0.31415926535	1.3583650163803942	2.08167E-14	5.11430E-12
0.39269908169	1.4597897167486004	2.50910E-14	2.05089E-11
0.47123889804	1.5650235492518021	2.48690E-14	5.52214E-12
0.54977871438	1.6736268821076297	4.59632E-14	2.00872E-11
0.62831853072	1.7851773403142603	1.92069E-14	2.45562E-11
0.70685834706	1.8992724097327580	1.67644E-14	1.82232E-11
0.78539816340	2.0155319187205900	0	2.10839E-10

where $T > 0, \alpha, \beta \in \mathbb{R}$, and $\phi : [0, T] \to \mathbb{R}$ is such that $0 \le \phi(t) \le T, \forall t \in [0, T]$. We suppose that ϕ is Lipschitzian, $p, q \in C^2(0, a)$ and $f \in L^2_w[0, T]$ are sufficiently regular given functions such that Eq. (1.1) satisfies the existence and uniqueness of the solution. Without loss of generality, we can assume that the boundary conditions in Eq. (1.1) are homogeneous. In this paper, based on reproducing kernel theory, reproducing kernels with polynomial form will be constructed and a computational method is described in order to obtain the accurate numerical solution with polynomial form of the Eq. (1.1) in the reproducing kernel spaces spanned by the Chebyshev basis polynomials.

2 Construction of reproducing kernels with Chebyshev polynomials

For $T > 0$, denote by $P_n[0, T]$ the set of all polynomials on $[0, T]$ with real coefficients and degree less than or equal to n, namely,

$$P_n[0, T] := span\{1, t, t^2, ..., t^n\} \tag{2.2}$$

equipped with the following inner product

$$\langle x, y\rangle_{P_n} = \int_0^T \omega(t)x(t)y(t)dt, \quad \forall x, y \in P_n[0, T], \tag{2.3}$$

and the norm

$$||x||_{P_n} = \sqrt{\langle x, x\rangle_{P_n}}, \quad \forall x \in P_n[0, T], \tag{2.4}$$

where ω is a positive weight function.
For $T > 0$ and positive weight function $\omega(t) = \frac{T}{2\sqrt{Tt-t^2}}$, the sequence of shifted Chebyshev polynomials of the first kind in t are defined on $[0, T]$ and can be determined as follows:

$$T^*_i(t) = 2(2T^{-1}t - 1)T^*_{i-1}(t) - T^*_{i-2}(t), \quad i \ge 2, \tag{2.5}$$

with $T^*_0(t) = 1, \quad T^*_1(t) = 2T^{-1}t - 1.$
The orthogonality condition is

$$\langle T^*_i, T^*_j\rangle = \int_0^T \omega(t)T^*_i(t)T^*_j(t)dt = \begin{cases} 0, & i \ne j, \\ \frac{T\pi}{2}, & i = j = 0, \\ \frac{T\pi}{4}, & i = j \ne 0. \end{cases} \tag{2.6}$$

Theorem 2.1 *[13, 14] $P_n[0, T]$ is a reproducing kernel space and its reproducing kernel is*

$$R_n(t, s) = \sum_{i=0}^{n} e_i(t)e_i(s), \tag{2.7}$$

where $e_i(t) = \begin{cases} \frac{2}{T\pi}T^*_i(t), & i = 0, \\ \frac{4}{T\pi}T^*_i(t), & i \ne 0. \end{cases}$

Note that the functions in space $P_n[0, T]$ do not satisfy the boundary conditions of Eq. (1.1). So we define a closed subspace of $P_n[0, T]$ by imposing required homogeneous boundary conditions on it.

Definition 2.1 *Let*

$$P^0_n[0, T] = \{x | x \in P_n[0, T], x(0) = x(T) = 0\}. \tag{2.8}$$

Lemma 2.1 *[9] For $2 \leq i \leq n$, the functions defined by*

$$\varphi_i(t) = \begin{cases} T_i^*(t) - 1, & i \quad \text{is even}, \\ T_i^*(t) - 2T^{-1}t + 1, & i \quad \text{is odd}, \end{cases} \tag{2.9}$$

have the property

$$\varphi_i(0) = \varphi_i(T) = 0, \tag{2.10}$$

and for the function space $P_n^0[0,T]$, the basis functions defined by Eq. (2.9) are complete.

The orthonormal basis $\{\bar{\varphi}_i\}_{i=2}^n$ of $P_n^0[0,T]$ can be deduced from Gram-Schmidt orthogonalization process using $\{\varphi_i\}_{i=2}^n$. In [13] the following closed form of $\bar{\varphi}_i$ for $i = 2, 3, \dots, n$ is obtained,

$$\bar{\varphi}_i(t) = \kappa_i \begin{cases} T_i^*(t) - \alpha_i U_i^*(t), & i \quad \text{even}, \\ \\ T_i^*(t) - \alpha_i V_i^*(t), & i \quad \text{odd}, \end{cases} \tag{2.11}$$

where $\kappa_i = 2\sqrt{\frac{(i-1)}{\pi(i+1)T}}, \alpha_i = \frac{2}{i-1}, U_i^*(t) = \sum_{k=1}^{\frac{i-2}{2}} T_{2k}^*(t) - \frac{1}{2}$ and $V_i^*(t) = \sum_{k=1}^{\frac{i-1}{2}} T_{2k-1}^*(t)$.

Similar to the proof of Theorem 2.1, we can prove that the function space $P_n^0[0,T]$ is a reproducing kernel space and its reproducing kernel is,

$$K_n(t,s) = \sum_{i=2}^{n} \bar{\varphi}_i(t)\bar{\varphi}_i(s). \tag{2.12}$$

3 C-RKM for Eq. (1.1)

Let Eq. (1.1) can be transformed into the following operator form:

$$\begin{cases} \mathbb{L}x(t) = f(t), \quad 0 \leq t \leq T, \\ x(0) = x(T) = 0, \end{cases} \tag{3.13}$$

where

$$\mathbb{L}x(t) = x''(t) + p(t)x(t) + q(t)x(\phi(t)), \tag{3.14}$$

is a linear operator. We shall give the approximate solution of Eq. (3.13) in space $P_n^0[0,T]$ based on reproducing kernel theory. Let $\{t_i\}_{i=0}^{n-2}$ be $(n-1)$-distinct nodes in interval $[0,T]$. put $\psi_i(t) = \mathbb{L}_s K_n(t,s)|_{s=t_i}, \quad (i = 0, 1, \dots, n-2)$, where the subscript s in the operator $\mathbb{L}$ indicates that the operator $\mathbb{L}$ applies to the function s. $\{\psi_i\}_{i=0}^{n-2}$ is a basis for $P_n^0[0,T]$ [13, 14]. Gram-Schmidt orthogonalization of $\{\psi_i\}_{i=0}^{n-2}$ yields $\{\bar{\psi}_i\}_{i=0}^{n-2}$ which is an orthonormal basis for $P_n^0[0,T]$,

$$\bar{\psi}_i = \sum_{k=0}^{i} \beta_{ik}\psi_k, \quad (\beta_{ii} > 0, i = 0, 1, \dots, n-2). \tag{3.15}$$

So, x_n as an approximation solution of Eq. (3.13) in space $P_n^0[0,T]$ can be represented the following form:

$$x_n(t) = \sum_{i=0}^{n-2} \langle x, \bar{\psi}_i \rangle_{P_n} \bar{\psi}_i(t), \tag{3.16}$$

where x is the exact solution.

Theorem 3.1 *If x_n is the approximation solution of Eq. (3.13) in $P_n^0[0,T]$ then it can be represented the following form:*

$$x_n(t) = \sum_{i=0}^{n-2} \sum_{k=0}^{i} \beta_{ik} f(t_k) \bar{\psi}_i(t). \tag{3.17}$$

According to Eqs. (3.15) and (3.16), one obtains

$$\begin{aligned} \langle x, \bar{\psi}_i \rangle_{P_n} &= \sum_{k=0}^{i} \beta_{ik} \langle x(.), \mathbb{L}_s K_n(.,s)|_{s=t_k} \rangle_{P_n} \\ &= \sum_{k=0}^{i} \beta_{ik} \mathbb{L}_s \langle x(.), K_n(.,s) \rangle_{P_n}|_{s=t_k}. \end{aligned} \tag{3.18}$$

Based on reproducing kernel theory and the fact that x is the exact solution, we have

$$\langle x, \bar{\psi}_i \rangle_{P_n} = \sum_{k=0}^{i} \beta_{ik} \mathbb{L}_s x(s)|_{s=t_k} = \sum_{k=0}^{i} \beta_{ik} f(t_k). \tag{3.19}$$

Hence,

$$\begin{aligned} x_n(t) &= \sum_{i=0}^{n-2} \langle x, \bar{\psi}_i \rangle_{P_n} \bar{\psi}_i(t) \\ &= \sum_{i=0}^{n-2} \sum_{k=0}^{i} \beta_{ik} f(t_k) \bar{\psi}_i(t). \end{aligned} \tag{3.20}$$

To analyze the convergence and the error estimation in $L_2^0[0,T] = \{x \mid x \in L_w^2[0,T], u(0) = u(T) = 0\}$, we define the error and the residual functions as

$$r_n(t) = x(t) - x_n(t), \tag{3.21}$$

$$R_n(t) = f(t) - \mathbb{L}x_n(t) = \mathbb{L}r_n(t), \qquad (3.22)$$

where $t \in [0,T]$, $x \in L_2^0[0,T]$ be the exact solution of Eq. (1.1) and x_n is given by Eq. (3.17).

Theorem 3.2 *Let $x \in L_2^0[0,T]$ be the exact solution of Eq. (1.1), $x_n \in P_n^0[0,T]$ in Eq. (3.17) be the approximation of x and for $t \in [0,T]$, $r_n(t) = x(t) - x_n(t)$ then*

$$||r_n||_{{}^{o}L_w^2} \longrightarrow 0, \quad n \longrightarrow \infty.$$

From Lemma 2.1 and Eq. (2.11), it follows that

$$x(t) = \sum_{i=2}^{\infty} \Big\langle x, \bar{\varphi}_i \Big\rangle_{{}^{o}L_w^2} \bar{\varphi}_i(t), \qquad (3.23)$$

and for any integer n, $i = 0, 1, \ldots, n-2$,

$$\Big\langle \bar{\varphi}_i, \psi_i \Big\rangle_{{}^{o}L_w^2} = 0, \quad j = n+1, n+2, \ldots. \qquad (3.24)$$

Let

$$L_2^0[0,T] = P_n^0[0,T] \oplus P_n^{0\perp}[0,T], \qquad (3.25)$$

where $P_n^{0\perp}[0,T] = \overline{Span\{\bar{\varphi}_i\}_{i=n+1}^{\infty}}$. So

$$r_n \in P_n^{0\perp}[0,T], \qquad (3.26)$$

and we have

$$\begin{aligned} ||r_n||^2_{{}^{o}L_w^2} &= || \sum_{i=n+1}^{\infty} \Big\langle r_n, \bar{\varphi}_i \Big\rangle_{{}^{o}L_w^2} \bar{\varphi}_i ||^2_{{}^{o}L_w^2} \\ &= \sum_{i=n+1}^{\infty} (\Big\langle r_n, \bar{\varphi}_i \Big\rangle_{{}^{o}L_w^2})^2. \end{aligned}$$

Thus

$$||r_n||_{{}^{o}L_w^2} \longrightarrow 0, \quad n \longrightarrow \infty. \qquad (3.27)$$

Theorem 3.3 *For $n \geq 2$ and $T > 0$, Let $t_0 < t_1 < \cdots < t_{n-2}$ be any $(n-1)$-distinct nodes in $(0,T)$ such that $\lim_{n\to\infty} t_0 = 0$, $\lim_{n\to\infty} t_{n-2} = T$. If p, q, ϕ and $f \in C^{n-1}[0,T]$ then there exists a positive constant c such that*

$$||r_n||_{L_w^2} \leq c\sqrt{\frac{T\pi}{2}}\hbar_n^{n-1}, \qquad (3.28)$$

where $\hbar_n = \max_{0\leq i\leq n-3}\{|t_{i+1} - t_i|\}$.

At the first, we prove that

$$R_n(t_j) = 0, \quad j = 0, 1, \ldots, n-2. \qquad (3.29)$$

We rewrite Eq. (3.17) as

$$x_n(t) = \sum_{i=0}^{n-2} A_i \bar{\psi}_i(t), \qquad (3.30)$$

where $A_i = \sum_{k=0}^{i} \beta_{ik} f(t_k)$. By the following reproducing property of $K_n(t,s)$, we have

$$\begin{aligned} (\mathbb{L}x_n)(t_k) &= \sum_{i=0}^{n-2} A_i \mathbb{L}_t \bar{\psi}_i(t)|_{t=t_k} \\ &= \sum_{i=0}^{n-2} A_i \mathbb{L}_t \Big\langle \bar{\psi}_i(s), K_n(t,s) \Big\rangle|_{t=t_k} \\ &= \sum_{i=0}^{n-2} A_i \Big\langle \bar{\psi}_i(s), \mathbb{L}_t K_n(t,s) \Big\rangle|_{t=t_k} \\ &= \sum_{i=0}^{n-2} A_i \Big\langle \bar{\psi}_i(s), \mathbb{L}_t K_n(t,s)|_{t=t_k} \Big\rangle \\ &= \sum_{i=0}^{n-2} A_i \Big\langle \bar{\psi}_i(s), \psi_k(s) \Big\rangle \end{aligned} \qquad (3.31)$$

Note here that

$$\begin{aligned} \sum_{k=0}^{j} \beta_{jk} (\mathbb{L}x_n)(t_k) &= \sum_{i=0}^{n-2} A_i \Big\langle \bar{\psi}_i, \sum_{k=0}^{j} \beta_{jk}\psi_k \Big\rangle \\ &= \sum_{i=0}^{n-2} A_i \Big\langle \bar{\psi}_i, \bar{\psi}_j \Big\rangle \\ &= A_j. \end{aligned} \qquad (3.32)$$

In the above equation, putting $j = 0$. So, we have $(\mathbb{L}x_n(t_0)) = f(t_0)$. Similarly, taking $j = 1, 2, \ldots, n-2$ in Eq. (3.32), we obtain

$$(\mathbb{L}x_n)(t_j) = f(t_j), \quad j = 1, \ldots, n-2. \qquad (3.33)$$

From the proof of theorem (3.6) in [13], there exists a positive constant d such that

$$||R_n||_\infty = \max_{t\in[0,T]} |R_n(t)| \leq d\hbar_n^{n-1}. \qquad (3.34)$$

According to the Eq. (2.6), we have

$$||R_n||_{L_w^2} = \sqrt{\int_0^T \omega(t)|R_n(t)|^2 dt} \leq d\sqrt{\frac{T\pi}{2}}\hbar_n^{n-1}. \qquad (3.35)$$

Nothing that

$$r_n = \mathbb{L}^{-1} R_n, \tag{3.36}$$

then there exists a constant c such that

$$||r_n||_{L^2_w} = ||\mathbb{L}^{-1} R_n||_{L^2_w} \le ||\mathbb{L}^{-1}||_{L^2_w} . ||R_n||_{L^2_w} \tag{3.37}$$

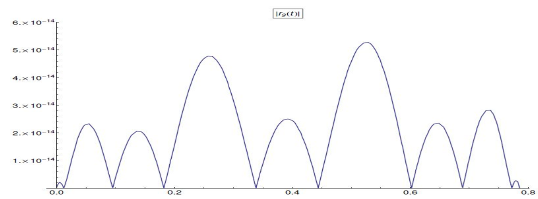
$$\le c\sqrt{\frac{T\pi}{2}} \hbar_n^{n-1}.$$

4 Numerical examples

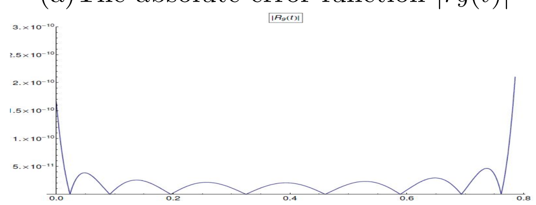
Example 4.1 *[8] Consider the following boundary value problem for* $t \in [0, \frac{\pi}{4}]$,

$$\begin{cases} x''(t) + 2\cos(\frac{t}{2}).x(\frac{t}{2}) = 1 + 2(1 + \frac{t^2}{8})\cos(\frac{t}{2}), \\ x(0) = 1, \quad x(\frac{\pi}{4}) = 1 + \frac{\sqrt{2}}{2} + \frac{\pi^2}{32}. \end{cases} \tag{4.38}$$

Here $T = \frac{\pi}{4}, \phi(t) = \lambda t$ with $\lambda = \frac{1}{2}$. The exact solution is $x(t) = \frac{t^2}{2} + \sin(t) + 1, t \in [0, \frac{\pi}{4}]$. Apply C-RKM with $n = 9, t_i = \frac{\pi}{8}(\cos(\frac{(i+1)\pi}{n}) + 1), \quad i = 0, 1, \ldots, n-2$. For $\tau_j = \frac{\pi j}{40}, \quad j = 1, 2, \ldots, 10$, the results are in Table 1, where the absolute errors $|r_n(\tau_j)|$ and the absolute residual values $|R_n(\tau_j)|$ reveal the accuracy of the method in the third and forth columns, respectively. The absolute error function $|r_n(t)|$ and the absolute residual function $|R_n(t)|$ are shown in Figure 1.



(a)The absolute error function $|r_9(t)|$



(b)The absolute residual function $|R_9(x)|$

Figure 1: The absolute error and residual functions of Example 4.1

Example 4.2 *For the following boundary value problem*

$$\begin{cases} x''(t) - e^t.x(t) + x(\sin(t)) = f(t), \\ t \in [0, 1], \\ x(0) = 0, \quad x(1) = 0. \end{cases} \tag{4.39}$$

the exact solution is $x(t) = e^t - e^{t^2}$, where $f(t) = e^t - e^{2t} + e^{t+t^2} + e^{\sin(t)} - e^{\sin^2(t)} - 2e^{t^2}(1 + 2t^2)$. Here $T = 1, \phi(t) = \sin(t)$. Apply C-RKM with $n = 2, 3, \ldots, 9, t_i = \frac{1}{2}(\cos(\frac{(i+1)\pi}{n}) + 1), \quad i = 0, 1, \ldots, n-2$. Figure 2 gives the order of error for Example 4.2, where the number of nodal points covers the range from 1 to 8. This figure goes in agreement with the results of convergence and error analysis.

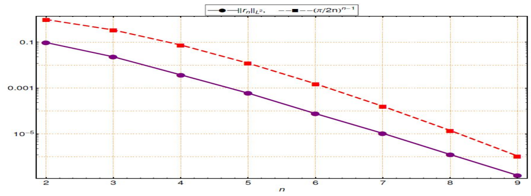


Figure 2: The order error of $x_n(t), n = 2, 3, \ldots, 9$ for the Example 4.2

5 Conclusion

The aim of this paper is to propose an effective reproducing kernel numerical technique for two-point boundary value problems associated to second order differential equations with deviating argument. This method uses reproducing kernels with polynomial form. The main advantage of the the paper consist in Theorem (3.2) and (3.3) that demonstrate the convergence, lower computational cost and high accuracy of the method.

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