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Research Article

On The Fractional Minimal Cost Flow Problem of a Belief Degree Based Uncertain Network

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Abstract

A fractional minimal cost flow problem under linear type belief degree based uncertainty is studied for the first time. This type of uncertainty is useful when no historical information of an uncertain event is available. The problem is crisped using an uncertain chance-constrained programming approach and its non-linear objective function is linearized by a variable changing approach. An illustrative example is solved to prove the efficiency of the proposed formulation.

Keywords : Uncertainty theory; Belief degree; Fractional minimal cost flow problem; Chance-constrained programming.

1 Introduction

FRactional minimal cost flow problem (FMCFP) is defined on a network of nodes and their associated arcs. Each node has to supply other nodes or being supplied by other nodes. It is also possible for a node to be supplied by some nodes and also supply some other nodes simultaneously. The amount that each node supplies the other nodes or is supplied by other nodes is restricted by supply and demand values given for the nodes. On the other hand, each arc has a unit flow cost (arc cost) which is defined according to the concepts of the network under study. This cost may have the units, e.g. time, length, money, etc. In the classic version of FMCFP, the costs of the arcs are represented by real numbers. Therefore, the FMCFP aims to find the amount of flow in each arc which minimizes the sum of total arc costs respecting to the demand and supply values of the nodes.

The FMCFP has been focused in few studies of the literature (see [11, 6, 20], developed the dual formulation of FMCFP and a special simplex algorithm to obtain its optimal solution. Sherali [15] suggested a dual formulation based algorithm for linear fractional programming problems in network based problems. The study shows that the algorithm of Xu et al. [20] is exactly the same as the method of Gilmore and Gomory [4]. As mentioned by Sherali [15], the obtained duality results can also be obtained by using the transformation of Charnes and Cooper [1] to convert the linear fractional problem to a linear problem. Fakhri and Ghatee [3] studied a typical FMCFP as the fractional multi-commodity flow problem. They defined the dual of this problem based on its linear programming representation, where some other duality properties were derived. Singh and Yadav [17] introduced an approach to tackle triangular intuitionistic fuzzy linear fractional programming problem by converting the problem to

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a multi-objective problem and then to a single-objective one.
Optimization problems including the transportation problem and its fractional form, can be even more interesting and realistic when tackled in uncertain environments with uncertain values for parameters and even variables. For coping with such problems with uncertain parameter, in existence of its historical information, any approach based on fuzzy theory (see [5, 13, 2, 12, 18, 14, 10], etc.), stochastic programming (see [19], etc.), probability theory, etc. can be applied. For the cases that no historical information for an uncertain event exists, uncertainty theory based on belief degree has been introduced by Liu [7]. This uncertainty theory can be explained by a simple example. Consider a box of 100 balls including blue and red balls when there is no information on the number of the blue and red balls. To select a ball from the box, no probability for determining its color exists and there is no priority for the colors blue and red. In such case, the belief degree of selecting for example blue ball is calculated to cope with this uncertain event. Some basic concepts of the belief degree based uncertainty theory will be explained in Section 2 where a complete study of this topic can be found in Liu [7].
In the real-world applications of the FMCFP, in most of cases the values given for the costs of arcs cannot be an exact value as those may be fuzzy numbers, or of given intervals, or even stochastic values. When there is no historical data for this aim, the uncertainty theory based on belief degree can be useful for modelling the uncertain FMCFP. In this study the FMCFP in existence of belief degree based uncertain parameters (UFMCFP) is studied for the first time. The UFMCFP is crisped applying some techniques of uncertainty theory that will be presented in Section 2 and the variable changing technique of Charnes and Cooper [1] is applied to linearize its fractional terms. The approach is then evaluated by a numerical example.
Other sections of the paper are organized as follows: In Section 2 some initial definitions of the belief degree based uncertainty theory is presented. The FMCFP and UFMCFP formulations are presented by Section [3]. A numerical example used to show the efficiency of the proposed uncertain formulation is presented by Section [4]. The study is concluded by Section [5].

2 Uncertainty theory

In this section some preliminaries of uncertainty theory are introduced. The definitions and theorems are taken from [7, 8, 10].

Definition 2.1 *Considering Γ as a non-empty set and L as a $\sigma-$algebra on the set,$M\{\Lambda\}$ is defined as the belief degree function of occurring the uncertain event Λ (where $\Lambda \in L$). Therefore, the function M should follow the following conditions,*

(i) $M\{\Lambda\} = 1$.

(ii) $M\{\Lambda\} + M\{\Lambda'\} = 1$ *where Λ' is the complement for Λ.*

(iii) *Considering a countable sequence $\{\Lambda_i\}$ where $i = 1, 2, ..., \infty$, the inequality $M\{\cup_{i=1}^{\infty}\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}$ is held.*

(iv) *Considering infinite uncertainty spaces where the k-th one is shown by $\{\Gamma_k, L_k, M_k\}$, the uncertain function $M\{\sqcap_{k=1}^{\infty}\Lambda_k\} = \wedge_{k=1}^{\infty} M_k\{\Lambda_k\}$ is defined, where Λ_k is an uncertain event from L_k.*

Definition 2.2 *Uncertain variable ξ is a function of the uncertainty space $\{\Gamma, L, M\}$. For any Borel set B of the real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an uncertain event.*

Definition 2.3 *Uncertainty distribution Φ is defined as $\Phi(x) = M\{\xi \leq x\}$ where $x \in \mathbb{R}$.*

Definition 2.4 *Expected value of ξ is defined as $E[\xi] = \int_0^{\infty} M\{\xi \geq r\}dr - \int_{-\infty}^{0} M\{\xi \leq r\}dr$ with the condition that one of the integrals be finite.*

Definition 2.5 *The uncertainty distribution $\Phi(x)$ is a regular uncertainty distribution where it is continuous and strictly increasing respecting to x, with the following conditions,*

(i) $0 < \Phi(x) < 1$.

(ii) $\lim_{x\to-\infty} \Phi(x) = 0$.

(iii) $\lim_{x\to\infty} \Phi(x) = 1$.

As for the regular uncertainty distribution $\Phi(x), 0 < \Phi(x) < 1$ is held, its inverse function $\Phi^{-1}(\alpha)$ is defined on the interval(0,1).

Definition 2.6 *The uncertain variable ξ from the uncertainty space $\{\Gamma, L, M\}$, is positive if, $M\{\xi \leq 0\} = 0$.*

Definition 2.7 *The uncertain variables $\xi_1, \xi_2, ..., \xi_n$ from the Borel sets $B_1, B_2, ..., B_n$ are independent if,*

$$M\{\prod_{i=1}^{n}(\xi_i \in B_i)\} = \Lambda_{i=1}^{n} M\{\xi_i \in B_i\}$$

Theorem 2.1 *Assume that $\xi_1, \xi_2, ..., \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, ..., \Phi_n$, respectively. If the function $f(x_1, ..., x_n)$ is strictly increasing on $x_1, x_2, ..., x_m$ and strictly decreasing on $x_{m+1}, ..., x_n$ then the uncertain variable $\xi = f(x_1, ..., x_n)$ has the inverse uncertainty distribution,*

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), ..., \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), ..., \Phi_n^{-1}(1-\alpha)\right)$$

Definition 2.8 *The expected value of the uncertain variable ξ with regular uncertainty distribution $\Phi(x)$ is calculated as $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$.*

Theorem 2.2 *Assume that $\xi_1, \xi_2, ..., \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, ..., \Phi_n$, respectively. If the function $f(x_1, ..., x_n)$ is strictly increasing on $x_1, x_2, ..., x_m$ and strictly decreasing on $x_{m+1}, ..., x_n$ then the uncertain variable $\xi = f(x_1, ..., x_n)$ has the expected value of,*

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), ..., \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), ..., \Phi_n^{-1}(1-\alpha)\right)d\alpha$$

Theorem 2.3 *Assuming ξ and η as independent uncertain variables, then for any real numbers a and b,*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Definition 2.9 *A linear type uncertain variable ξ represented by $L(a, b)$ has the following uncertainty distribution where $a < b$ are real numbers.*

$$\Phi(x) = \begin{cases} 0 & x < a, \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

3 Deterministic and uncertain fractional minimal cost flow problem

In order to construct the mathematical model of the FMCFP, we assume $G = (V, E)$ as a connected and directed graph with the set of vertices $V = (v_1, , v_m)$ and the set of arcs E. Each vertex $v_i \in V$ has a value of b_i. This value shows the difference between the amount sent from this vertex and the amount received by this vertex. It is supposed that $\sum_{i=1}^{m} b_i = 0$ which is a basic condition for feasibility. Parameters c_{ij} and d_{ij} are the coefficient for cost of the arc $(i, j) \in E$. Moreover, the value u_{ij} is considered for the upper bound of the capacity of arc $(i, j) \in E$. Also, α and β are given constants. Here, the variable X_{ij} is flow on arc (i, j). Therefore, the fractional minimum cost flow problem is modeled using the following formulation [20].

$$\text{Min } f(X) = \frac{\sum_{(i,j)\in E} c_{ij}X_{ij} + \alpha}{\sum_{(i,j)\in E} d_{ij}X_{ij} + \beta} \tag{3.1}$$

s.t.

$$\sum_{(i,j)\in E} X_{ij} - \sum_{(j,i)\in E} X_{ji} = b_i, \; i = 1, ..., m \tag{3.2}$$

$$0 \leq X_{ij} \leq u_{ij} \qquad (i, j) \in E \tag{3.3}$$

In the formulation 3.1-3.3, the constraint set (3.2) guarantees that net flow that is sent out of each node is equal to its capacity. Furthermore, the condition $\sum_{(i,j)\in E} d_{ij}X_{ij} + \beta > 0$ is considered for feasibility of the problem.

An uncertain environment for optimization problems usually considers uncertain parameters for such problems. The uncertainty type of such parameters usually can be determined by historical information about the parameters' values. For example historical data about a parameter like demand can help decision maker to estimate the uncertain value of demand by fuzzy theory, stochastic theory, interval theory, etc. Another uncertainty type can be considered for the uncertain parameters of optimization problems when no historical data exists. In this situation decision maker asks the opinion of an expert for determining the parameters' uncertain value. Therefore, the uncertain values of the parameters are determined according to the belief degree of the expert. The belief degree based uncertain values can be shown by zigzag, normal, and linear type uncertainty variables (see [7, 8]).

To extend the FMCFP to its uncertain form say uncertain fractional minimal cost flow problem (UFMCFP), the parameters c_{ij}, d_{ij}, and u_{ij} are assumed to be of linear independent uncertain variables shown by ξ_{ij}, η_{ij}, and $\tilde{u}_{ij}$ respectively. Therefore the UFMCFP is formulated by the following model.

$$\text{Min } f(X;\xi,\eta) = \frac{\sum_{(i,j)\in E} \xi_{ij} X_{ij} + \alpha}{\sum_{(i,j)\in E} \eta_{ij} X_{ij} + \beta} \quad (3.4)$$

$s.t.$

$$\sum_{(i,j)\in E} X_{ij} - \sum_{(j,i)\in E} X_{ji} = b_i, \ i = 1, ..., m \quad (3.5)$$

$$0 \le X_{ij} \le \tilde{u}_{ij} \qquad (i,j) \in E \quad (3.6)$$

Notably, the objective function value $f(X;\xi,\eta)$ is an uncertain variable.

In order to cope with the uncertainty of the UFMCFP, first it is converted to a deterministic form, and then it is solved. For the conversion purpose, two main criteria of expected value and critical value of the uncertain variables can be considered (see [7, 8]). According to these criteria, three different deterministic forms of the UFMCFP as expected value model, expected value and chance-constrained model, and chance-constrained model can be obtained (see [7, 8]). In this paper the chance-constrained model is used to obtain the crisp form of the UFMCFP. Using the chance-constrained model, the UFMCFP is converted to a deterministic form considering the following issues,

A chance-constrained technique is considered to crisp the uncertain constraints. Therefore, the belief degree based function of each constraint should be greater than a confidence level which is determined in advance from the interval $(0, 1]$

A new objective function say $\bar{f}$ is introduced where a new constraint is defined such that the main uncertain objective function $f(X;\xi,\eta)$ be less than or equal to $\bar{f}$. Then, a chance-constrained technique is considered to crisp the new uncertain constraint. Therefore, the belief degree based function of this constraint should be greater than a confidence level which is determined in advance from the interval $(0, 1]$. So, the ULFTP first is converted to the following form,

$$\text{Min } \bar{f} \quad (3.7)$$

$s.t.$

$$M\left\{\frac{\sum_{(i,j)\in E} \xi_{ij} X_{ij} + \alpha}{\sum_{(i,j)\in E} \eta_{ij} X_{ij} + \beta} \le \bar{f}\right\} \ge \gamma \quad (3.8)$$

$$\sum_{(i,j)\in E} X_{ij} - \sum_{(j,i)\in E} X_{ji} = b_i i = 1, ..., m, \quad (3.9)$$

$$M\left\{X_{ij} \le \tilde{u}_{ij}\right\} \ge \lambda_{ij} \qquad (i,j) \in E, \quad (3.10)$$

$$X_{ij} \ge 0, \quad (3.11)$$

where, the confidence levels γ and λ are determined in advance.

In order to find the crisp form of the UFMCFP represented by the formulation 3.7-3.11, the following theorem is presented.

Theorem 3.1 *Assume that the independent uncertain variables ξ_{ij}, η_{ij}, and $\tilde{u}_{ij}$ have the regular uncertainty distributions Φ_{ij}, Ψ_{ij} and Ω_{ij}respectively. If ξ_{ij}, η_{ij} be positive uncertain variables, the uncertain model 3.7-3.11 is equivalent to the following deterministic model.*

$$\text{Min} \frac{\sum_{(i,j)\in E} \Phi_{ij}^{-1}(\gamma) X_{ij} + \alpha}{\sum_{(i,j)\in E} \Psi_{ij}^{-1}(1-\gamma) X_{ij} + \beta} \quad (3.12)$$

$s.t.$

$$\sum_{(i,j)\in E} X_{ij} - \sum_{(j,i)\in E} X_{ji} = b_i i = 1, ..., m, \quad (3.13)$$

$$X_{ij} \le \Omega_{ij}^{-1}(1-\lambda_{ij}) \qquad (i,j) \in E, \quad (3.14)$$

$$X_{ij} \ge 0, \quad (3.15)$$

Proof. In order to prove the equivalency of constraint 3.8 and objective function 3.12, using the concept of Definition 2.3, the following conversion is done where Υ is the uncertainty distribution of $\frac{\sum_{(i,j)\in E} \xi_{ij} X_{ij} + \alpha}{\sum_{(i,j)\in E} \eta_{ij} X_{ij} + \beta}$

$$M\left\{\frac{\sum_{(i,j)\in E} \xi_{ij} X_{ij} + \alpha}{\sum_{(i,j)\in E} \eta_{ij} X_{ij} + \beta} \le \bar{f}\right\} \ge \gamma$$
$$\Longleftrightarrow \Upsilon(\bar{f}) \ge \gamma$$

Now, as the term $\frac{\sum_{(i,j)\in E} \xi_{ij} X_{ij} + \alpha}{\sum_{(i,j)\in E} \eta_{ij} X_{ij} + \beta}$ is strictly increasing according to ξ_{ij} and strictly decreasing according to η_{ij}, the following conversion is done according to Theorem 3.1.

$$\Upsilon(\bar{f}) \geq \gamma \Longleftrightarrow \bar{f} \geq \Upsilon^{-1}(\gamma) \Longleftrightarrow$$
$$\bar{f} \geq \frac{\sum_{(i,j)\in E} \Phi_{ij}^{-1}(\gamma) X_{ij} + \alpha}{\sum_{(i,j)\in E} \Psi_{ij}^{-1}(1-\gamma) X_{ij} + \beta}$$

To prove the equivalency of the constraints 3.10 and 3.14, using Definition 2.3 and Theorem 2.1 the following conversion is done,

A fuzzy number is a function $u : \Re \longrightarrow [0,1]$ satisfying the following properties:

$$M\left\{X_{ij} \leq \tilde{u}_{ij}\right\} \geq \lambda_{ij} \Longleftrightarrow 1 - M\left\{X_{ij} > \tilde{u}_{ij}\right\}$$
$$\geq \lambda_{ij} \Longleftrightarrow 1 - \Omega_{ij}(X_{ij}) \geq \lambda_{ij} \Longleftrightarrow \Omega_{ij}^{-1}(X_{ij})$$
$$\leq 1 - \lambda_{ij} \Longleftrightarrow X_{ij} \leq \Omega_{ij}^{-1}(1-\lambda_{ij})$$

Therefore, the theorem is proved.

The crisp form of the UFMCFP which is represented by the formulation 3.12-3.15, is a nonlinear model because of its fractional term appeared in the objective function 3.12. This nonlinearity is linearized by the approach of Charnes and Cooper [1]. For this aim the variable changings $\sum_{(i,j)\in E} \Psi_{ij}^{-1}(1-\gamma)X_{ij} + \beta = \frac{1}{T}$ and $X_{ij}T = Y_{ij}$ is introduced. Then, the following linearized version for the crisp form of the UFMCFP is obtained.

$$\text{Min} \sum_{(i,j)\in E} \Phi_{ij}^{-1}(\gamma) Y_{ij} + \alpha \qquad (16) \qquad (3.16)$$

s.t.

$$\sum_{(i,j)\in E} \Psi_{ij}^{-1}(1-\gamma) Y_{ij} + \beta T = 1 \qquad (3.17)$$

$$\sum_{(i,j)\in E} Y_{ij} - \sum_{(j,i)\in E} Y_{ji} = b_i T, \; i = 1, ..., m \quad (3.18)$$

$$Y_{ij} \leq T\Omega_{ij}^{-1}(1-\lambda_{ij}) \qquad (i,j)\in E \qquad (3.19)$$

$$Y_{ij}, T \geq 0, \qquad (3.20)$$

4 An illustrative example

In order to study the performance of the proposed UFMCFP, a simple example is solved in this section. For this aim a network of seven nodes with their given capacities is considered and represented by Figure 1. The data related to the arcs of the network is represented by Table 1, where $\alpha = \beta = 0$.

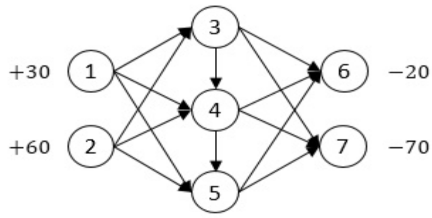

Figure 1: Schematic representation of the network of the example.

According to the data of the example and the linear formulation of the crisp version of the UFMCFP represented by the formulation 3.16-3.19, the expanded formulation of the example is shown as follow,

$$\begin{aligned} \text{Min } & Y_{13}\Phi_{13}^{-1}(\gamma) + Y_{14}\Phi_{14}^{-1}(\gamma) + Y_{15}\Phi_{15}^{-1}(\gamma) + \\ & Y_{23}\Phi_{23}^{-1}(\gamma) + Y_{24}\Phi_{24}^{-1}(\gamma) + Y_{25}\Phi_{25}^{-1}(\gamma) + \\ & Y_{34}\Phi_{34}^{-1}(\gamma) + Y_{36}\Phi_{36}^{-1}(\gamma) + Y_{37}\Phi_{37}^{-1}(\gamma) + \\ & Y_{45}\Phi_{45}^{-1}(\gamma) + Y_{46}\Phi_{46}^{-1}(\gamma) + Y_{47}\Phi_{47}^{-1}(\gamma) + \\ & Y_{56}\Phi_{56}^{-1}(\gamma) + Y_{57}\Phi_{57}^{-1}(\gamma) \end{aligned} \qquad (4.21)$$

s.t.

$$\begin{aligned} & Y_{13}\Phi_{13}^{-1}(1-\gamma) + Y_{14}\Phi_{14}^{-1}(1-\gamma) + \\ & Y_{15}\Phi_{15}^{-1}(1-\gamma) + Y_{23}\Phi_{23}^{-1}(1-\gamma) + \\ & Y_{24}\Phi_{24}^{-1}(1-\gamma) + Y_{25}\Phi_{25}^{-1}(1-\gamma) + \\ & Y_{34}\Phi_{34}^{-1}(1-\gamma) + Y_{36}\Phi_{36}^{-1}(1-\gamma) + \\ & Y_{37}\Phi_{37}^{-1}(1-\gamma) + Y_{45}\Phi_{45}^{-1}(1-\gamma) + \\ & Y_{46}\Phi_{46}^{-1}(1-\gamma) + Y_{47}\Phi_{47}^{-1}(1-\gamma) + \\ & Y_{56}\Phi_{56}^{-1}(1-\gamma) + Y_{57}\Phi_{57}^{-1}(1-\gamma) = 1 \end{aligned}$$

$$\begin{aligned} Y_{13} + Y_{14} + Y_{15} &= 30T \\ Y_{23} + Y_{24} + Y_{25} &= 60T \\ Y_{34} + Y_{36} + Y_{37} - Y_{13} - Y_{23} &= 0 \\ Y_{45} + Y_{46} + Y_{47} - Y_{14} - Y_{24} &= 0 \\ Y_{56} + Y_{57} - Y_{15} - Y_{25} &= 0 \\ -Y_{36} - Y_{46} - Y_{56} &= -20T \\ -Y_{37} - Y_{47} - Y_{57} &= -70T \end{aligned}$$

$$\begin{aligned} Y_{13} &\leq T\Omega_{13}^{-1}(1-\lambda_{13}) \\ Y_{14} &\leq T\Omega_{14}^{-1}(1-\lambda_{14}) \\ Y_{15} &\leq T\Omega_{15}^{-1}(1-\lambda_{15}) \\ Y_{23} &\leq T\Omega_{23}^{-1}(1-\lambda_{23}) \\ Y_{24} &\leq T\Omega_{24}^{-1}(1-\lambda_{24}) \\ Y_{25} &\leq T\Omega_{25}^{-1}(1-\lambda_{25}) \end{aligned}$$

Table 1: The arc related data of the example.

Arc index	Tail-node(i)	Head-node (j)	ξ_{ij}	η_{ij}	$\tilde{u}_{ij}$
1	1	3	$L(5,7)$	$L(2,3)$	$L(20,30)$
2	1	4	$L(4,5)$	$L(3,4)$	$L(50,90)$
3	1	5	$L(5,7)$	$L(6,8)$	$L(50,80)$
4	2	3	$L(10,12)$	$L(5,7)$	$L(50,80)$
5	2	4	$L(8,9)$	$L(8,9)$	$L(50,80)$
6	2	5	$L(3,6)$	$L(5,6)$	$L(25,50)$
7	3	4	$L(2,4)$	$L(9,10)$	$L(60,90)$
8	3	6	$L(8,10)$	$L(7,8)$	$L(50,90)$
9	3	7	$L(2,5)$	$L(1,3)$	$L(60,85)$
10	4	5	$L(3,10)$	$L(2,3)$	$L(20,50)$
11	4	6	$L(8,9)$	$L(4,6)$	$L(30,60)$
12	4	7	$L(5,7)$	$L(8,9)$	$L(40,55)$
13	5	6	$L(5,8)$	$L(2,4)$	$L(10,30)$
14	5	7	$L(2,5)$	$L(4,5)$	$L(55,90)$

Table 2: The results obtained for the example

Exp.	λ_{ij}	γ	X_{13}	X_{14}	X_{15}	X_{14}	X_{24}	X_{25}	X_{34}
1	0	0	30	0	0	10	0	50	40
2	0	0.35	30	0	0	10	0	50	40
3	0	0.65	30	0	0	25	0	35	55
4	0	1	30	0	0	0	45	15	30
5	0.5	0	25	0	5	22.5	0	37.5	47.5
6	0.5	0.35	25	0	5	22.5	0	37.5	47.5
7	0.5	0.65	25	0	5	22.5	0	37.5	47.5
8	0.5	1	25	0	5	0	42.5	17.5	25
9	1	0	20	0	10	35	0	25	55
10	1	0.35	20	0	10	35	0	25	55
11	1	0.65	20	0	10	35	0	25	55
12	1	1	20	0	10	0	40	20	20

Table 2. (Continue)

Exp	X_{36}	X_{37}	X_{45}	X_{46}	X_{47}	X_{56}	X_{57}	O.F
1	0	0	0	0	40	20	30	0.589
2	0	0	0	0	40	20	30	0.759
3	0	0	0	0	55	20	15	0.915
4	0	0	0	20	55	0	15	1.078
5	0	0	0	0	47.5	20	22.5	0.614
6	0	0	0	0	47.5	20	22.5	0.773
7	0	0	0	0	47.5	20	22.5	0.925
8	0	0	0	20	47.5	0	22.5	1.097
9	0	0	5	10	40	10	30	0.644
10	0	0	0	15	40	5	30	0.795
11	0	0	0	15	40	5	30	0.939
12	0	0	0	20	40	0	30	1.117

$$Y_{34} \leq T\Omega_{34}^{-1}(1-\lambda_{34})$$
$$Y_{36} \leq T\Omega_{36}^{-1}(1-\lambda_{36})$$
$$Y_{37} \leq T\Omega_{37}^{-1}(1-\lambda_{37})$$
$$Y_{45} \leq T\Omega_{45}^{-1}(1-\lambda_{45})$$
$$Y_{46} \leq T\Omega_{46}^{-1}(1-\lambda_{46})$$
$$Y_{47} \leq T\Omega_{47}^{-1}(1-\lambda_{47})$$
$$Y_{56} \leq T\Omega_{56}^{-1}(1-\lambda_{56})$$
$$Y_{57} \leq T\Omega_{57}^{-1}(1-\lambda_{57})$$
$$Y_{13}, Y_{14}, Y_{15}, Y_{23}, Y_{24}, Y_{25}, Y_{34}, Y_{36}, Y_{37}, Y_{45}, Y_{46}, Y_{47}, Y_{56}, Y_{57}, T \geq 0$$

The model 4.21 was coded in GAMS solver and was run on a PC with an Intel Pentium Dual 2 GHz processor and 1024 MB RAM. In order to run the experiments the constants $\Phi_{ij}^{-1}(\gamma), \Psi_{ij}^{-1}(1-\gamma)$and $\Omega_{ij}^{-1}(1-\lambda_{ij})$ are easily calculated from the values of Table 2 and the definitions of Section 2, prior to running the experiments. Notably, the confidence levels are selected from the sets $\gamma \in \{0, 0.35, 0.65, 1\}$ and $\lambda_{ij} \in \{0, 0.5, 1\}$. In an experiment a similar value from the set $\{0, 0.5, 1\}$ is fixed for all λ_{ij} values of the model 4.21. Therefore, 12 combinations of the confidence levels are considered where each of them construct an experiment, giving totally 12 experiments. The results obtained by the experiments are reported in Table 2. In order to analyze the sensitivity of the formulation 4.21 over the confidence level values, marginal mean of the objective function values (MMOFV) is taken for each confidence level value of γ and λ_{ij}. The obtained MMOFVs are plotted for γ and λ_{ij} separately shown by Figure 2 and Figure 3. According to the trend of the graph of Figure 2, it can be concluded that the objective function value is generally increased by increasing the value of confidence level λ_{ij}. This increase is because of the nature of linear uncertainty function considered for the data of the example. A similar trend with the same reason happens when the value of confidence level γ is increased as shown by the graph of Figure 3.

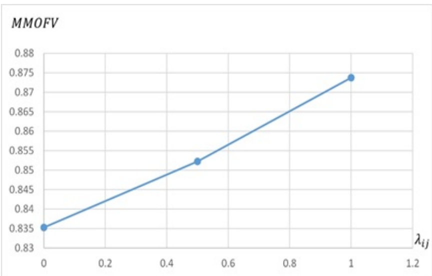

Figure 2: Marginal mean of the objective function values (MMOFV) over λ_{ij} values.

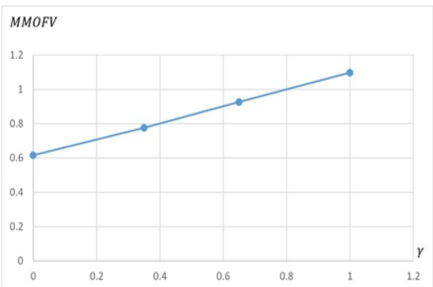

Figure 3: Marginal mean of the objective function values (MMOFV) over γ values.

5 Conclusion

A fractional minimal cost flow problem under belief degree based uncertainty was studied in this paper for the first time. This type of uncertainty is useful for the cases that no historical information of an uncertain event exists. Linear uncertainty distribution was used to show the uncertainty of objective function parameters and node capacity values. The uncertain fractional minimal cost flow problem was converted to a crisp form using a chance-constrained approach and its non-linear objective function was linearized by a variable changing approach. An illustrative example was solved to prove the efficiency of the proposed formulation. The sensitivity analysis illustrated the high dependency of the objective function value to the change of the confidence level values of the chance constraints.

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