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Research Article

B-spline Method for Solving Fredholm Integral Equations of the First Kind

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Abstract

Ill-posed problems arise as inverse problems in many areas of science and engineering. The first type Fredholm integral equation is a special kind of the inverse problem. A large number of inverse problems of mathematical physics have been translated into the first type Fredholm integral equation to solve the problem of computing. Fredholm integral equation of the first kind is one of the inverse problems that arise in many areas of science, physical models such as radiography, stereology, spectroscopy, cosmic radiation and engineering fields such as image processing and electromagnetic. In this paper, we use the collocation method for to find an approximate solution of the problem by cubic B-spline basis. The proposed method as a basic function led matrix systems, including band matrices and smoothness and capability to handle low calculative costly. The absolute errors in the solution are compared to existing methods to verify the accuracy and convergent nature of proposed method.

Keywords : Ill-posed problems; Fredholm Integral Equations; Cubic B-spline; Collocation; Regularization.

1 Introduction

Ill-posed problems arise as inverse problems in many areas of science and engineering [5], [11]. Most inverted problems are ill-posed, so that solving discrete systems of such problems with the large condition number have a lot of difficulties. We focus on linear inverse problems that can be formulated in the following very general form:

$$\int_{\Delta} input \times system \, d\Delta = output.$$

that Δ is the domain that can be defined in the system input. In this formulation, the direct problem is to compute the output, given, the input and the mathematical description of the system. The goal of the inverse problem is to determine either the input or the systems that give rise to the measurement on the output [14]. For example, in astronomical image deblurring the input is the night sky; the blurring systems consist of the telescope and the atmosphere, and the output is the recorded blurred image. The goal is to reconstruct the input, i.e., the unblurred image, given a mathematical description of the blurring effects of the telescope and the atmosphere. The classical example of a linear ill-posed problem is a Fredholm integral equation of the first kind (FK1), which always be written in the generic

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form:

$$(K\phi)(x) = \int_a^b k(x,s)\phi(s)ds = f(x), a \leq x \leq b. \quad (1.1)$$

To classify integral equations, we denote the unknown function by $\phi(x)$, the kernel of the equation by $k(x,s)$, and the free term, which is assumed known, by $f(x)$ [2], [3]. We introduce the integral operator K defined by $K\phi = f$. A FK1 is of the form 1.1 where the functions $\phi(x)$ and $f(x)$ are assumed to belong to the class $L_2[a,b]$, and the kernel $k(x,s)$ is assumed to be continuous on the square $S = \{(x,s) : x, s \in [a,b] \times [a,b]\}$, or such that $k(x,s)$ is square-integrable on S. since the integral operator K with a non-degenerate and continuous kernel $k(x,s)$ is a compact operator with non-closed range in $L_2[a,b]$, and hence it is not continuously invertible. Various methods are applied in order to solve 1.1, Wavelets-Galerkin method [7], Preconditioned technique [8], Haar wavelet [9], Legendre multi wavelets [15], Chebyshev wavelet [1].

The layout of the paper is as follows. Section 2 is devoted to the cubic B-spline collocation method and properties of the basis functions that are necessary for the formulation of the discrete system. In Section 3, the method of regularization is used to approximate the solution of 1.1. As a result, a set of algebraic equations is achieved and the solution of the considered problem is introduced. In Section 4, error and convergence consideration of regularization method and theorems are proven and is shown that the procedure converges to the solution. In the last section, numerical examples are given to show accuracy, validity and applicability of the numerical technique.

2 Cubic B-spline collocation method

B-splines are a set of special spline functions that be used to construct the piece wise polynomial by computing the appropriate linear combination [12], [4]. These functions have their computational advantage from the fact that any B-spline basis function of order m is nonzero over at almost m adjacent intervals and zero otherwise, and since they have compact support, numerical methods in which B-spline functions are used as a basic function led matrix systems, including band matrices. Through smoothness and capability to handle local phenomena, B-spline basis functions offer distinct advantages in comparison to other basis functions. Such systems can be handled and solved with low computational cost. Cubic B-spline function (CBS) has already been used as the basis functions to solve many physical models.

In this instance attention, we focused on this method: first with regularization technique, we use cubic B-spline collocation (CBSC) and then use of the Newton-Cotes (NC) rule for approximating integral.

Definition 2.1 *Let* $\Delta = \{a = x_0 < x_1 < ... < x_N = b\}$ *be the partition in* $[a,b]$, *where* $h = (b-a)/(N+1), s_i = a + ih, i = 0,1,...,N+2$. *We introduce spline space* $S_3(\Delta) = \Gamma \in C^2[a,b];\ \Gamma|_{[s_i,s_{i+1}]} \in P_3;\ i = 0,1,...,N+2$, *where* P_3 *is the class of cubic polynomials, by introducing adjacent knots* $s_{-1} < s_0 < s_1 < ... < s_n < s_{N+1}$.

Table 1: The values of B-spline

s	s_{i-2}	s_{i-1}	s_i	s_{i+1}	s_{i+2}
$B_i(s)$	0	1	4	1	0

We define the CBS for $i = -1, 0, ..., N+1$ *by the following relation as:*

$$B_i(s) = \frac{1}{h^3}$$

$$\begin{cases} (s-s_{i-2})^3 & s \in [s_{i-2}, s_{i-1}] \\ h^3 + 3h^2(s-s_{i-1}) + 3h(s-s_{i-1})^2 - \\ 3(s-s_{i-1})^3 & s \in [s_{i-1}, s_i] \\ h^3 + 3h^2(s_{i+1}-s) + 3h(s_{i+1}-s)^2 - \\ 3(s_{i+1}-s)^3 & s \in [s_i, s_{i+1}] \\ (s_{i+2}-s)^3 & s \in [s_{i+1}, s_{i+2}] \\ 0 & else \end{cases} \quad (2.2)$$

Our numerical treatment for solving 1.1 using the collocation method with CBS is to find an approximate solution $\phi(s)$ to exact solution in the form :

$$\phi(s) = \sum_{i=-1}^{N+1} c_i \varphi_i(s), \quad (2.3)$$

the values of $\varphi_i(s) = B_i(s)$ may be tabulated as in Table (1) and we have $\phi(s) = c_{i-1} + 4c_i + c_{i+1}$ and also we can approximate the integrand by Newton-Cots type methods.

For numerical solving of 1.1 we should choose a finite dimensional family of functions that the exact solution may be estimated by them. Methods that use in this strategic are called projection methods, because the exact solution of equation is projected into the space with the finite dimensions. One of the most famous for these methods, is collocation method. We choose a sequence of finite dimensional subspaces $X_n \subset L^2(\mathbb{R})$ for $n \geq 1$, with X_n having dimension d_n. Assume thatX_n has a basis of form $\varphi_1, \varphi_2, ..., \varphi_d$ with $d \equiv d_n$ for notational simplicity and φ_n is a function belongs to X_n, so that we write it as $\phi(s) \approx \phi_n(s) = \sum_{i=1}^{d} c_i\varphi_i(s)$. By substitution into 1.1 we have:

$$r_n(x) = \int_a^b k(x,s)\phi_n(s)ds - f(x) = \int_a^b k(x,s)\sum_{i=1}^{d} c_i\varphi_i(s)ds - f(x), \tag{2.4}$$

where r_n is called the residual in the approximation of the equation when using $\phi \approx \phi_n$, in the operator form we have $r_n = k\phi_n - f$. Now to determine unknown coefficients $\{c_i\}_{i=1}^{d}$ we impose the following requirements:

$$r_n(x_i) = 0, \qquad i = 1, 2, ..., d,$$

where x_i are the collocation node points. These coefficients are determined uniquely if and only if $\varphi_i(x)$ are being independent. In this paper, we use CBS basis, so that $\phi_n(s) = \sum_{i=1}^{d} c_i\varphi_i(s)$ is uniquely determined.

3 The Method of Regularization

Assuming that a solution exists to the linear ill-posed problem FK1 which can always be written in the generic form $K\phi = f$. We modify 1.1 and consider [16]:

$$\int_a^b k(x,s)\phi(s)ds + \gamma\phi(x) = f(x), \tag{3.5}$$

where γ is known as the regularization parameter and 3.5 is an FK2 whose solution, denoted by $\phi_\gamma(x)$, can be found. These equations may be written as:

$$\phi(K + \gamma I) = f,$$

substituting $\phi_\gamma(x)$ for $\phi(x)$ in 1.1, we get:

$$\int_a^b k(x,s)\phi_\gamma(s)ds = f_\gamma(x). \tag{3.6}$$

If $\|f(x) - f_\gamma(x)\| \leq \delta$ where δ is a preassigned quantity representing the tolerance of error, then the function $\phi_\gamma(x)$ is considered an acceptable approximate solution to 1.1 that we prove it in error consideration section. Thus, if the kernel $k(x,s)$ is discontinuous along a curve $s = g(x)$ and the discontinuity is finite, then an FK1 can be changed to an FK2, as follows:

$$f(x) = \int_a^b k(x,s)\phi(s)ds = \int_a^{g(x)} k(x,s)\phi(s)ds + \int_{g(x)}^b k(x,s)\phi(s)ds, \tag{3.7}$$

and differentiating both sides with respect to x, we get:

$$f'(x) = \int_a^b \frac{\partial k(x,s)}{\partial x}\phi(s)ds + k(x,g(x)_-)\phi(g(x))g'(x) - k(x,g(x)_+)\phi(g(x))g'(x) = \int_a^b \frac{\partial k(x,s)}{\partial x}\phi(s)ds + S(x)\phi(g(x))g'(x), \tag{3.8}$$

where $k(x,g(x)_-)$ and $k(x,g(x)_+)$are defined everywhere in (a,b), and the difference $S(x) = k(x,g(x)_-) - k(x,g(x)_+)$. On dividing the above equation by $S(x)g'(x)$ and replacing x by $x = g^{-1}(y)$, we obtain a FK2. Now, the following theorem can be stated [6].

Theorem 3.1 *(i) If $k(x,s)$ is bounded in the domain $\Omega = [a,b] \times [a,b]$ and continuous except on the curve $s = g(x)$, where $g(x)$ has a nonzero continuous derivative in $[a,b]$, with $g(a) = a$, and $g(b) = b$*
(ii) $S(x) \in C[a,b]$
(iii) $k_x(x,s)$ is real and exists in Ω
(iv) $f(x)$ and $f'(x)$ are continuous in $[a,b]$
then, if the quantity

$$\left|\frac{k_x(x,s)}{S(x)g'(x)}\right|_{x=g^{-1}(y)}$$

does not vanish in Ω, then FK1 can be changed into the following FK2:

$$\int_a^b \left| \frac{k_x(x,s)}{S(x)g'(x)} \right|_{x=g^{-1}(y)} \phi(s)ds + \phi(y) = \left| \frac{f'(x)}{S(x)g'(x)} \right|_{x=g^{-1}(y)} \quad (3.9)$$

If $k(x,s)$ is continuous in Ω, but if $\frac{\partial^n k(x,s)}{\partial x^n}$ for some n has a finite discontinuity at $s = g(x)$, then the theorem can be generalized.

By using CBS method we can approximate the solution by the bases $\varphi_i(s) = B_i(s)$ in 3.5 then we obtain the vector c_i , and also we suppose $c_{-2} = c_{n+2} = 0$ by substituting c_i in 3.5 we can obtain an approximate solution for 1.1 :

$$h \sum_{j=-1}^{n+1} \left(w_j \sum_{i=-2}^{n+2} c_i B_i(s_j)\, k(x_j, s_i) \right) + \gamma \sum_{i=-2}^{n+2} c_i B_i(s_j) = \sum_{i=-2}^{n+2} c_i B_i(s_j). \quad (3.10)$$

4 Error and Convergence Consideration

Theorem 4.1 *Suppose $\kappa : L_2(0,1) \to L_2(0,1)$ be the linear operator defined by*

$$\kappa\phi = \int_0^1 k(x,s)\phi(s)ds, \quad (4.11)$$

where $K(x,s) \in L_2[(0,1) \times (0,1)]$. If ϕ and ϕ_j be the exact and approximate solutions of 3.5 obtained by our method, respectively, then

$$\|\phi(x) - \phi_j(x)\| \le MC2^{-jm} \left\| \phi^{(m)} \right\|, \quad (4.12)$$

where M and C are appropriate constants.
Proof. See [2].

Theorem 4.2 *In 3.5 if γ regularization parametr and $\phi_\gamma(x)$ is approximate solution and in 3.6 we have:*

$$\|f(x) - f_\gamma(x)\| \le \delta$$

where δ is a known bound on the measurement error.

Proof. In 3.5, the operator $K+\gamma I$ has a bounded inverse, that is, the problem of solving the equation

$$(k + \gamma I)\phi_\gamma = f,$$

is well-posed. This FK2 equation has a unique solution

$$\phi_\gamma = (k + \gamma I)^{-1} f. \quad (4.13)$$

From this we see that $\gamma\phi_\gamma = f - k\phi_\gamma$ and we may, in passing to the limit as

$$\lim_{\gamma \to 0} \left\| \phi_\gamma - k^\Lambda f \right\|^2 = 0,$$

that $k^\Lambda = (k + \gamma I)^{-1}$, the vectors $\{\phi_\gamma\}$ are therefore genuine approximations to k^Λ in the sense that $\phi_\gamma \to k^\Lambda f$ as $\gamma \longrightarrow 0$.

Moreover, since for each $\gamma > 0$ the operator k^Λ is bounded, we see that the approximation ϕ_γ depends continuously on f, for each $\gamma > 0$.

We saw that the function f is typically a measured or observed quantity and hence in practice the true f is not available to us. The best we can hope for is some estimate f^δ of f satisfying:

$$\left\| f - f^\delta \right\| \le \delta, \quad (4.14)$$

where δ is a known bound on the measurement error. Instead of forming 4.13 with the true f, we must make do with the available data f^δ and form the approximations

$$\phi_\gamma^\delta = (k + \gamma I)^{-1} f^\delta. \quad (4.15)$$

Now, we know that the approximations ϕ_γ using data f converge to the minimum norm least squares solution k^Λ; it is therefore reasonable to compare ϕ_γ^δ to ϕ_γ :

$$\phi_\gamma^\delta - \phi_\gamma = (k + \gamma I)^{-1}(f^\delta - f)$$
$$\left\| \phi_\gamma^\delta - \phi_\gamma \right\|^2 = \left\langle (k + \gamma I)^{-1}(f^\delta - f), (k + \gamma I)^{-1}(f^\delta - f) \right\rangle$$

and

$$\left\| (k + \gamma I)^{-1} \right\| \le 1/\gamma,$$

then with 4.15 obtain

$$\left\| \phi_\gamma^\delta - \phi_\gamma \right\| \le \delta/\sqrt{\gamma}. \quad (4.16)$$

Table 2: Absolute Error for Example 5.1

s_i	$\gamma = 0.01$	$\gamma = 0.001$	method of [1]
0.0	2.3E-2	4.4E-3	2.0E-9
0.1	3.4E-3	2.2E-4	5.3E-5
0.2	1.3E-3	5.9E-4	1.3E-4
0.3	3.2E-3	4.0E-5	2.2E-4
0.4	4.8E-3	7.1E-6	3.0E-4
0.5	4.3E-4	1.1E-6	5.9E-4
0.6	8.5E-4	5.2E-6	3.8E-4
0.7	1.9E-4	9.0E-5	5.5E-4
0.8	5.6E-4	7.8E-5	7.2E-4
0.9	7.1E-4	6.6E-6	8.9E-4

Table 3: Absolute Error for Example 5.2

s_i	$\gamma = 0.01$	$\gamma = 0.001$	method of [10]
0.0	5.1E-2	3.1E-2	1.1E-3
0.1	1.3E-2	4.3E-3	5.0E-4
0.2	6.4E-3	3.1E-3	3.3E-4
0.3	5.6E-4	2.7E-4	4.6E-4
0.4	4.6E-4	2.3E-4	5.9E-4
0.5	3.1E-4	1.4E-4	2.2E-3
0.6	3.7E-4	1.7E-4	8.3E-4
0.7	1.2E-4	4.1E-5	7.8E-4
0.8	1.6E-4	2.2E-5	4.8E-4
0.9	2.7E-5	1.3E-4	8.5E-4

Table 4: Absolute Error for Example 5.3

s_i	$\gamma = 0.01$	$\gamma = 0.001$	method of [13]
0.0	4.9E-3	3.4E-2	3.8E-1
0.1	2.5E-3	1.8E-3	1.5E-1
0.2	3.0E-3	7.1E-4	4.2E-1
0.3	4.3E-4	8.3E-4	1.1E-1
0.4	7.4E-4	2.1E-4	6.2E-2
0.5	8.7E-4	3.5E-5	3.3E-2
0.6	2.3E-4	7.9E-4	2.3E-2
0.7	4.5E-4	3.7E-5	2.8E-2
0.8	2.5E-4	2.0E-5	8.9E-3
0.9	1.0E-4	4.7E-5	1.5E-2

Choosing a suitable regularization parameter, based on the error in the data, then becomes the heart of the matter. we say that a choice $\gamma = \gamma(\delta)$

$$\gamma(\delta) \to 0 \quad and \quad \phi^{\delta}_{\gamma(\delta)} \to k^{\Lambda} f \quad as \quad \delta \to 0.$$

Since

$$\left\| \phi^{\delta}_{\gamma(\delta)} - k^{\Lambda} f \right\| \leq \left\| \phi^{\delta}_{\gamma(\delta)} - \phi_{\gamma(\delta)} \right\| + \left\| \phi_{\gamma(\delta)} - k^{\Lambda} f \right\| \quad \leq \delta / \sqrt{\gamma(\delta)} + \left\| \phi_{\gamma(\delta)} - k^{\Lambda} f \right\|,$$

and since we have shown that $\phi_{\gamma(\delta)} \to k^{\Lambda} f$ as $\gamma(\delta) \to 0$, we see that the condition

$$\delta^2 / \gamma(\delta) \to 0 \quad as \quad \delta \to 0,$$

is sufficient to ensure that method gives a regular algorithm for 4.16. With minimizing an augmented least squares function $t(x) = \sum_{j=0}^{m} a_j t_j(x)$ that $t_j(x)$ known functions and a_j are unknown coefficients then

$$\Phi_\gamma(a_0, a_1, ..., a_m) = \sum_{i=0}^{n}((\sum_{j=0}^{m} a_j t_j(x_i)) - \phi_i)^2$$
$$= \left\|kt - f^\delta\right\|^2 + \gamma\|t\|^2. \tag{4.17}$$

If we consider linear function $t(x) = a_0 + a_1 x$ any minimizer of 4.17 must satisfy

$$\frac{d}{dx}\left\{\|k(a_0 + a_1 x) - f^\delta\|^2 + \gamma\|a_0 + a_1 x\|^2\right\}\Big|_{x=0} = 0. \tag{4.18}$$

Expressing the squared norms in terms of the inner product and expanding the quadratic forms this is equivalent to

$$\left\langle ka_0 - f^\delta, ka_1\right\rangle + \gamma\left\langle a_0, a_1\right\rangle = 0,$$

or

$$\left\langle (k + \gamma I)a_0 - f^\delta, a_1\right\rangle = 0$$

$$(k + \gamma I)a_0 = f^\delta.$$

We therefore see that the unique minimizer of the augmented least squares functional 4.15 is

$$\phi_\gamma^\delta = (k + \gamma I)^{-1} f^\delta = (k + \gamma I)^{-1}(k + \gamma I)a_0.$$

For the solution of $k\phi = f$ choose the regularization parameter so that the size of the residual $r(\gamma) = \left\|k\phi_\gamma^\delta - f^\delta\right\|$ is the same as the error level in the data and the vector ϕ of minimum norm satisfying the requirement

$$\left\|k\phi - f^\delta\right\| \leq \delta.$$

If P is the orthogonal projector of Hilbert space , we can write:

$$P(k + \gamma I)\phi_\gamma^\delta = Pf^\delta$$
$$P(\gamma\phi_\gamma^\delta) + P(k\phi_\gamma^\delta) = Pf^\delta$$
$$k\phi_\gamma^\delta = f^\delta - \gamma\phi_\gamma^\delta$$
$$\left\|k\phi_\gamma^\delta - f^\delta\right\| = \left\|\gamma\phi_\gamma^\delta\right\| = \left\|Pf^\delta - Pk\phi_\gamma^\delta\right\|,$$

if $\gamma \longrightarrow 0$ given:

$$\lim_{\gamma\to 0^+} r(\gamma) = \left\|Pf^\delta - Pk\phi_\gamma^\delta\right\| = \left\|Pf^\delta - Pf\right\|$$
$$\leq \left\|f^\delta - f\right\| \leq \delta,$$

and

$$\lim_{\gamma\to\infty} r(\gamma) = \left\|f^\delta\right\| > \delta,$$

the choice $\gamma(\delta)$ as given leads to a regular scheme for approximating $k^\Lambda f$, that is

$$f_{\gamma(\delta)}^\delta \to k^\Lambda f \quad as \quad \delta \to 0.$$

Theorem 4.3 *Error analysis for spline interpolant $B(s)$ to a given function f defined on an interval $[a, b]$ follows from $\Delta = \{a = x_0 < x_1 < ... < x_N = b\}$ be the partition in $[a, b]$, where $h = (b - a)/(N + 1), s_i = a + ih, i = 0, 1, ..., N + 2$, if $f \in C^4[a, b]$ then*

$$\left\|D^j(f - B_i(s))\right\|_2 \leq \mu_j \left\|B_i^{(4)}(s)\right\|_2 h^{4-j-1/2},$$
$$j = 0, .., 3 \quad , \quad \mu_j = 6/j!\,, \tag{4.19}$$

and D^j is the j- th derivative $j = 0, 1, 2, ...$ then method to be convergent if the step length h tends to zero.

Proof. See [4].

5 Examples

In this section, the theoretical results of the previous sections are used for some numerical examples. The numerical experiments are implemented in Maple 15 software. The programs are executed on a PC with 2.00 GHz Intel Core 2 dual processor with 2 GB RAM. In illustrative examples, to show the accuracy and efficiency of the described method, we present three numerical examples then we compare the results of our method with the results from some other methods.

Example 5.1 *Consider the Fk1 integral equation with exact solution $\phi(x) = x$.*

$$\int_0^1 \sin(xs)\,\phi(s)\,ds = \frac{sinx - x\,cosx}{x^2}.$$

Table 2 shows, a comparison between the absolute errors of the proposed method together with method in [1].

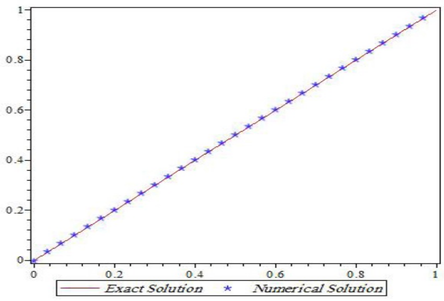

Figure 1: Exact and approximate solutions for Example 5.1 ($\gamma = 0.01$).

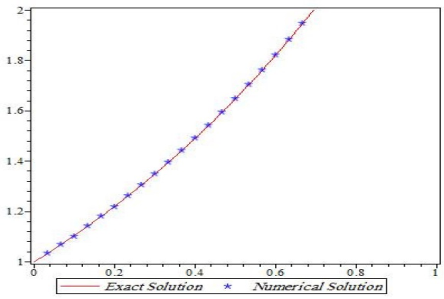

Figure 2: Exact and approximate solutions for Example 5.2 ($\gamma = 0.01$).

Example 5.2 *Consider the Fk1 integral equation*

$$\int_0^1 e^{xs}\phi(s)\,ds = \frac{e^{x+1}-1}{x+1}.$$

with exact solution $\phi(x) = \exp(x)$. The absolute error is tabulated in Table 3, that shows, a comparison with method in [10].

Example 5.3 *Consider the Fk1 integral equation with exact solution $\phi(x) = x$.*

$$\int_0^1 \sqrt{x^2+s^2}\phi(s)\,ds = 1/3\big(1+x^2\big)^{3/2} - 1/3x^3$$

These results are compared by the method which has been prepared in [13] and tabulated in Table 4.

6 Conclusion

Although the Fredholm integral equations of the first kind are an ill posed problem, there is a systematic expression method that converts it to the second kind integral equation and its use was very useful based on the numerical solution. The cases mentioned in the error and convergence analysis section showed that the approximate method

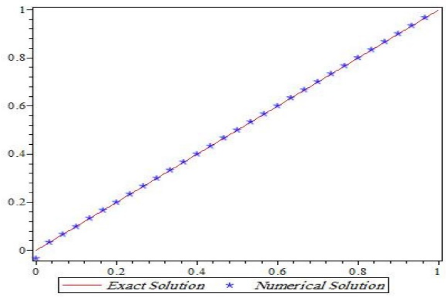

Figure 3: Exact and approximate solutions for Example 5.3 ($\gamma = 0.01$).

using the B-splines cube requires a fourth derivative. Also, numerical results in the examples show that the absolute error of the solution obtained with the B-spline method is less than the previous methods. The results of the tables show that the method error decreases with decreasing gamma value. In other words, the accuracy increases, which is one of the advantages of this method.

References

[1] H. Adibi, P. Assari, Chebyshev wavelet method for numerical solution of Fredholm integral equation of the first kind, *Math. Prob. Eng*, (2010).

[2] R. S. Anderssen, F. R. De hogg, M. A. Lukas, The Application and Numerical Solution of Integral Equations, *Sijhoff and Noordhoff International Publishers*, (1980).

[3] B. Davies, Integral Transforms and Their Applications, *Springer-Verlag, New York*, (1978).

[4] C. DE boor, Good approximation by spline with variable knots, in Spline Functions and Approximation Theory, *Birkhauser, Basel*, (1973).

[5] J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, *Yale University Press, New Haven*, (1923).

[6] P. K. Kythe, P. Puri, Computational Methods for Linear Integral Equations, *Springer Science*, (2002).

[7] K. Maleknejad, T. Lotfi, K. Mahdiani, Numerical solution of first kind Fredholm integral equation with wavelets-Galerkin method (WGM) and wavelets precondition, *Appl. Math. Comput* 186 (2007) 794-800.

[8] K. Maleknejad, H. Almasieh, M. Roodaki , Preconditioned technique for solving Fredholm integral equations of the first kind with orthogonal triangular functions, *World Applied Sciences Journal* 7 (2009) 162-167.

[9] K. Maleknejad, R. Mollapourasl, K. Nouri , Convergence of Numerical Solution of the Fredholm Integral Equation of the First Kind with Degenerate Kernel, *Applied Mathematics and Computation* 181 (2006) 1000-1007.

[10] K. Maleknejad, S. Sohrabi, Numerical solution of Fredholm integral equation of the first kind by using Legendre wavelets, *Appl. Math. Comput* 186 (2007) 836-843.

[11] M. S. Pilant, W. Rundell, An Inverse problem for a nonlinear parabolic equation, *Communications in Partial Differential Equations* (1986) 445-457.

[12] P. M. Prenter, Spline and Variational Methods, *Wiley, New york* (1975).

[13] M. T. Rashed, Numerical solutions of the integral equations of the first kind, *Applied Mathematics and Computation* 145 (2003) 413-420.

[14] M. Razzaghi, F. Ahmad , A collocation-type method for the solution of inverse problems in dispersive scattering theory, *Microwave and Optical Technology Letters* 9 (1995) 14-17.

[15] X. Shang, D. Han, Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets, *Applied Mathematics and Computation* 191 (2007) 440-444.

[16] A. N. Tikhonov, On the solution of ill-posed problems and the method of regularization, *Soviet Math. Dokl* 4 (1963) 1035-1038.

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