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A Class of High Pass Filters Derived From Elliptic Differential Operators

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Abstract

In this study, a new type of filters based on elliptic differential operators is introduced. First, we review the elliptic operators and filters, and due to a wide range of elliptic operators, focus on a batch of elliptic operators with constant coefficients second order and then generalizing them to a higher order. Finally by discretization of the elliptic operators, we express and prove two theorems and show that the obtained filters are the high pass.

Keywords : Filters; High pass filter; Differential operator; Principal symbol; Elliptic operators.

1 Introduction

There are many ways to analyze code and rebuild signals which needs special operators on them. One of the most important operators that can analyze them is filter which are used to extract needed frequency components from signals [1, 2]. Signal processing is discussed as one of the most important fields that still attracts many researchers. One of the main parts of signal processing is filter designing [3, 4, 5, 6]. So, filtering is extensive topic in signal processing and has many applications, for example, the Gabor filters are amongst the most important filters in the field of defect detection [7]. Gabor filters is obtained by scaling and orienting the mother Gabor wavelet [8].

The Wiener filter is another important approach which researchers are using this extensively and in technical applications for noise reduction in time domain. This filter is always able to reduce the noise embedded in a signal. Though, signal degradation accompanies the noise reduction quantity [9]. Kalman-Bucy filter in the stochastic differential equation is used for the modeling of RL circuit [10]. There are two known groups of old types of filters which are weighted mean and median filters. Though, fuzzy versions of these filters like Weighted Fuzzy Mean (WFM) and also Fuzzy Median Filter (FMF) as fuzzy forms of these filters have improved significantly [11, 12, 13]. Adaptive filters make a distinct class of filters. Many types of adaptive filters can also be found in the literature [14, 15]. Nowadays one

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of the most important methods of state estimation, is Kalman Filter (KF) which has a variety application, like target tracking [16, 17, 18], climate forecast [19, 20] and Neural Network training [21, 22].
Differential operators as elliptic ones are considered as simplify the Laplace operator in the theory of partial differential equations. Elliptic operators are characteristic of potential concept, and they seem often in electrostatics, in hydrodynamics and the theory of elasticity and continuum mechanics [23]. In 1936, the Dirichlet realization P of a additional command elliptic operator was considered by Carleman in a bounded domain $U \subseteq R^n$ [24]. Feature researchers, like Agmon [25, 26], Agranovich [27], Markus [28] and Matseev [29] have achieved serious and important results in the field of Keldysh. In the broader level can be noted in the application and new concepts such as spectral properties and negative spectra of elliptic operators [30, 31, 32]. In this paper, in order to prove that elliptic filters are high-pass, first we express two theorems of Askari Hemaat [33] and Inspired by them, we show that the partial derivatives result in a high-pass filter.

The organization of this paper is as follows: Section 2 has devoted to a review of elliptic differential operators. Section 3 is devoted to reviewing filters, properties, and theorems on filters. Discrete and high-pass filters resulted from Partial derivatives operations have introduced in section 4. Finally, in section 5, the main results of this paper have concluded.

2 Partial differential operators

Let U be an open set in the Euclidean $n-$space E^n, for all $n \in N$ the real Euclidean space E^n is the finite dimensional Hilbert space

Eˆn:=$\{X, Y \in R^n$ whit $i) \|X\|= (\sum_{j=1}^{n} |x_j|^2)^{1/2}$ $ii) \langle X, Y\rangle = \sum_{j=1}^{n} x_j y_j\}$
We consider the linear partial operator $P : C^\infty(U) \longrightarrow C^\infty(U)$ defined by

P(X, D)u := $\sum_{|\alpha|\leq r} a_\alpha(X) D^\alpha u =$

$$\sum_{|\alpha|\leq r} a_{\alpha_1,\ldots,\alpha_n}(x_1, \ldots, x_n) \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots . \partial x_n^{\alpha_n}} \quad (2.1)$$

The number r is called the order of the differential operator (2.1), if there exists some α whit $|\alpha|= r$ and $a_\alpha(X) \neq 0$ on U. With complex-valued coefficients $a_\alpha \in C^\infty(\overline{U})$, in the compact closure $\overline{U}$of a bounded region $U \in E^n$.Where multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots., \alpha_n)$ with $\alpha_j \in N = \{0, 1, 2, \ldots\}$ for $j = 1, 2, \ldots, n$ and $|\alpha|= \sum_{j=1}^{n} \alpha_j$, real coordinate vectors $X = (x_1, x_2, \ldots, x_n)$ in Euclidean space E^n, classical and weak partial derivatives in $E^n$are denoted by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots . \partial x_n^{\alpha_n}}$$

for all $\alpha = (\alpha_1, \alpha_2, \ldots., \alpha_n)$ with total order $|\alpha|> 0$[34, 35].

Definition 2.1 We assume that the partial differential expression $P(., D)$ of (2.1) has order $r \geq 1$. That is, the (highest order) principal polynomial

p(X, ξ) := $\sum_{|\alpha|=r} i^r a_\alpha(X) \xi^\alpha =$ $i^r \sum_{|\alpha|=r} a_{\alpha_1,\alpha_2,\ldots,\alpha_n}(x_1, x_2, \ldots, x_n) \xi_1^{\alpha_1} \xi_2^{\alpha_2} \ldots \xi_n^{\alpha_n}$
$\xi \in R^n, X \in \overline{U}$.

The principal symbol of P is roughly speaking its "r^{th} order part". More explicitly it is the function on $U \times R^n$.

Definition 2.2 The operator P is called elliptic at the point $X \in \overline{U}$, if $p(X, \xi) \neq 0$ for all $\xi \in R^n$, except, of course, for $\xi = 0$. P is called elliptic on $\overline{U}$, if P is elliptic at all points $X \in \overline{U}$.
Note that, ellipticity is defined in terms of the principal symbol of P, The lower-order terms that appear in (2.1) don't play role.

Definition 2.3 The operator P is called strongly elliptic at the point $X \in \overline{U}$,if there exists a complex constant γ with $Re(\gamma.p(X, \xi)) \neq 0$ for all $\xi \in R^n$. P is called strongly elliptic on $\overline{U}$, if P is strongly elliptic at all points $X \in \overline{U}$ with the constant γ independent of X.

Theorem 2.1 If P is strongly elliptic at $X \in \overline{U}$, then P is elliptic at $X \in \overline{U}$ [36].

For example, the operator defined by

$$P(X, D)u = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + (ix - y)\frac{\partial u}{\partial t}$$

is not elliptic in the $(x, y, t)-$space, if $\xi = (y, -x, 1) \neq 0$, then $p(X, \xi) = i(\xi_1 + i\xi_2 + (ix - y)\xi_3) = 0$.Therefore by theorem 2.1 P is not strongly elliptic.

On the other hand, in the case $ordP = r = 2, n \geq 2$, let

P(X, D)u $:= \sum_{j,k=1}^{n} a_{jk}(X) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} a_j(X) \frac{\partial u}{\partial x_j} + a_0(X)u$

Then, the principal symbol

p(X, ξ) $= \sum_{j,k=1}^{n} i^2 a_{jk}(X)\xi_j\xi_k = -\sum_{j,k=1}^{n} a_{jk}(X)\xi_j\xi_k$

is a quadratic form. If the coefficients $a_{jk}(X)$ are real, then P is strongly elliptic at X, if the quadratic form $p(X, \xi)$ is positive or negative definite. In particular the principal symbol of Laplacian operator Δ,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$$

is $\Delta(X, \xi) = -(\xi_1^2 + ... + \xi_n^2)$, so that Δ and all its powers Δ^n are strongly elliptic. Therefore by theorem 2.1 they are elliptic [36, 37].

3 Filters

special operators on signals are necessary to evaluate, code, rebuild signals and a few more. One of the most important operators that can analyze them is filter which are used to extract needed frequency components from signals. Such as high frequency mechanisms of a signal usually have not only the noise but also the actuations, that often have to be detached from the signal. To decay signals by their frequency bands, low and high pass filters should be used. A low pass filter reduces high frequency components of a signal while a high pass filter behaves oppositely (figure 1). In this section, linear filters are described, which are convolution operators on l^2[1].

Definition 3.1 An operator S on l^2 is called a shift operator (also called a time-delay operator) if

$$(Sx)[n] = x[n-1] \qquad , \qquad x \in l^2$$

an operator H on l^2 is called time-invariant if $SH = HS$ and an operator Hon l^2 is called a

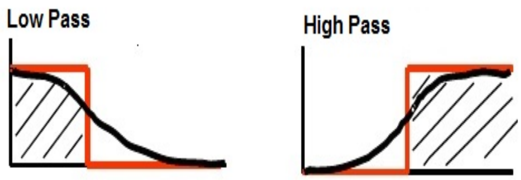

Figure 1: A RLC circuit with fuzzy current and fuzzy source.

linear operator if for any $x \in l^2$,

$$Hx = H(\sum_{k \in Z} x[k]\delta_k) = \sum_{k \in Z} x[k]H\delta_k.$$

A linear and time-invariant operator is called a filter. If H is a filter, then Hx is called the response of x.

Definition 3.2 *The (discrete) convolution of two sequences h and x is a sequence $h * x$ given by*

$$(h * x)[n] = \sum_{k} h[k]x[n-k] \qquad , \qquad n \in Z \tag{3.2}$$

Provided the series in (3.2) is convergent for each $n \in Z$

The following theorem identifies the filter with a sequence.

Theorem 3.1 (In discrete mode) H is a filter if and only if there is a sequence h such that $Hx = h * x$.

Theorem 3.2 (In continuous mode) Let L be a linear, time-invariant transformation on the space of signals that are piecewise continuous functions. Then there exists an integrable function, h, such that $L(f) = f * h$ for all signals f.

Since $h(t)$ can be obtained the impact of L on an impulse input signal so $h(t)$ is called the impulse response function. $\widehat{h}(\lambda)$ Fourier transform of the impulse response is called the frequency response $(L(\lambda) = \widehat{h}(\lambda))$ [1, 38].

3.1 Design Filters

Designing a time-invariant filter is equivalent to constructing the impulse function, h, since any such filter can be written as $L(f) = f * h$ (In continuous mode) or $Hx = h * x$ (In discrete mode) by Theorem 3.1 and Theorem 3.2 The construction of h depends on what the filter is designed to do. In this paper, we consider filters that reduce low frequencies, but leave the high frequencies virtually unchanged. Such filters are called high -pass filters [1, 33, 38].

3.2 Low Pass and High Pass Filters theorems and properties

As noted above, filters are really significant type of linear time-invariant systems. the term frequency-selective filter proposes a system which passes certain frequency components and completely discards all others, but in other words filter is a system which adjusts sure frequencies relative to others [2].
According to Theorem 3.1, in the discrete case, H is a filter if and only if there is a sequence h such that $Hx = h * x$. Where $x = x[n]$ is discrete signal and

$$(h * x)[n] = \sum_k h[k]x[n-k] \qquad , \qquad n \in Z \tag{3.3}$$

is the (discrete) convolution of two sequences h and x.
Note that, filter coefficients, $h = \{h[k]\}_{k \in Z}$, can complex and its called impulse response .If claim that

$$x[n] = e^{in\omega} \qquad , \qquad -\pi \leq \omega \leq \pi$$

Then by (3.2)
Hx[n] $= \sum_k h[k]e^{i(n-k)\omega} = (\sum_k h[k]e^{-ik\omega})e^{in\omega} = H(\omega)x[n]$
Where $\sum_k h[k]e^{-ik\omega}$is called frequency response function [33].

Theorem 3.3 let $\sum_k h[k]e^{-ik\omega}$ be a frequency response function, which satisfies the following conditions

1. $\sum_k (-1)^k h[k] = 1$
2. $\sum_k h[k] = 0$

Then it is a high pass filter, i.e. the output signal at π as the input signal, and at zero is zero [33].

Remark 3.1 According to the definition of high-pass filter, it is clear that the condition 1 in Therorem (3.3) with the condition $\sum_k (-1)^k h[k] \neq 0$ can be replaced.

Theorem 3.4 suppose that $\sum_k h[k]e^{-ik\omega}$ is a frequency response function, which satisfies in conditions

1. $\sum_k (-1)^k h[k] = 0$

2. $\sum_k h[k] = 1$

Then is a low pass filter, i.e. the output signal at zero as the input signal, and at π is zero [33].

Remark 3.2 According to the definition of low-pass filter, it is clear that the condition 2 in Therorem (3.3) with the condition $\sum_k h[k] \neq 0$ can be replaced.

4 Discretization and high-pass filters resulted from partial derivatives operations

In this section, according to Theorem 3.3 show that the partial derivatives operators result in the high-pass filter. For this purpose the conditions of Theorem 3.3 for the high-pass mode to examine the derivative operators. Note, in the process provided below, initially intended to be a one-dimensional model, and then will be extended to two-dimensional and above.

Lemma 4.1 The partial derivatives operators (in R and R^2) result in the high-pass filter.

Proof. We know if $u \in C^\infty(U)$, $U \subseteq R$ then in the case of one-dimensional and based on the central difference derivative

$$\frac{du}{dx} = \frac{u(x+h) - u(x-h)}{2h}$$

Without lose of the generality, by choosing $h = 1$ we have

$$\frac{du}{dx} = \frac{1}{2}[u(x+1) - u(x-1)]$$

By following this process in generally for order $n-th$ derivative u with respect to x we have

$$\frac{d^n u}{dx^n}=\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}u(x+n-2k)$$

Now if we put $h[k]=\frac{(-1)^k}{2^n}\binom{n}{k}$, it is clear that

1. $\sum_k (-1)^k h[k]=1$

2. $\sum_k h[k]=0$

Therefore, by Theorem 3.3 the corresponding filter is high-pass filter.

Hence, the above process is extended for $u\in C^\infty(U)$, $U\subseteq R^n$. Actually the discretization of the partial derivatives order $|\alpha|-th$ u that $|\alpha|=\alpha_1+\alpha_2+...+\alpha_n$ and $\alpha_i\in\{0,1,2,....\}$ as follows

$$\frac{\partial^{|\alpha|}u}{\partial {x_1}^{\alpha_1}\partial {x_2}^{\alpha_2}...\partial {x_n}^{\alpha_n}}=$$

$$\frac{1}{2^{\alpha_1}}\sum_{i_1=0}^{\alpha_1}(-1)^{i_1}\binom{\alpha_1}{i_1}\frac{1}{2^{\alpha_2}}\sum_{i_2=0}^{\alpha_2}(-1)^{i_2}\binom{\alpha_2}{i_2}\cdots$$
$$\frac{1}{2^{\alpha_n}}\sum_{i_n=0}^{\alpha_n}(-1)^{i_n}\binom{\alpha_n}{i_n}u(x_1+\alpha_1-2i_1,$$
$$x_2+\alpha_2-2i_2,...,x_n+\alpha_n-2i_n)$$

Equivalently

$$\frac{\partial^{|\alpha|}u}{\partial {x_1}^{\alpha_1}\partial {x_2}^{\alpha_2}...\partial {x_n}^{\alpha_n}}=\frac{1}{2^{|\alpha|}}$$
$$\sum_{i_1,i_2,\cdots,i_n}(-1)^{i_1+i_2+\cdots+i_n}\binom{\alpha_1}{i_1}\binom{\alpha_2}{i_2}\cdots\binom{\alpha_n}{i_n}$$
$$u(x_1+\alpha_1-2i_1,\ x_2+\alpha_2-2i_2,...,x_n+\alpha_n-2i_n)$$

As before, if we put

$$h[i_1,i_2,\cdots,i_n]=\frac{(-1)^{i_1+i_2+\cdots+i_n}}{2^{\alpha_1+\alpha_2+...+\alpha_n}}$$
$$\binom{\alpha_1}{i_1}\binom{\alpha_2}{i_2}\cdots\binom{\alpha_n}{i_n}$$

, it is clear that

1. $\sum_{i_1,\cdots,i_n}(-1)^{i_1+\cdots+i_n}h[i_1,\cdots,i_n]=1$

2. $\sum_{i_1,i_2,\cdots,i_n}h[i_1,i_2,\cdots,i_n]=0$

Therefore, by Theorem 3.4 and Remark 3.2 the corresponding filter is high-pass filter.

5 Conclusion

According to what was said in Lemma 4.1, partial derivatives operators alone are high-pass filters. At first, we express the paper's main claim for filters resulted from elliptic differential operators with rank $r=2, n=2$ with constant coefficients according to equation:

$$P(X,D)u=A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}$$
$$+E\frac{\partial u}{\partial y}+Fu$$

And prove that and then extend it to the arbitrary n,r.

Theorem 5.1 *For any elliptic differential operator of the form*

$$P(X,D)u=A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2},$$

the obtained filter is high pass.

Proof. Let $H(\omega)$ be frequency response and $h[i_1,i_2]$ be its equivalent impulse response. Since derivatives are alone high-pass, conditions (1) and (2) of Theorem 3.3 for them are established. Consequently

$$H(\omega)=\sum_{i_1,i_2}(-1)^{i_1,i_2}h[i_1,i_2]=$$
$$A\times 1+B\times 1+C\times 1,$$

since P is elliptic operator then $A+B+C\neq 0$ (if $A+B+C=0$, then the corresponding principal symbol i.e.

$$p(X,\xi)=-(A\xi_1^2+B\xi_1\xi_2+C\xi_2^2),$$

becomes zero at $\xi=(1,1,1)\neq 0$, and this contradicts the ellipticity of the P) and this shows that the condition (1) of Theorem 3.3 is established. On the other hand

$$H(0)=\sum_{i_1,i_2}h[i_1,i_2]=A\times 0+B\times 0+C\times 0$$

This shows that the condition (2) of Theorem 3.3 is established. So the obtained filter is high-pass.

Theorem 5.2 *For any elliptic differential operator of the form*

$$P(X,D)u = \sum_{|\alpha|=r} a_\alpha(X)D^\alpha u =$$

$$\sum_{|\alpha|=r} a_{\alpha_1,\alpha_2,\cdots\alpha_n}(x_1,x_2,\cdots,x_n)$$

$$\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}},$$

the obtained filter is high-pass.

Proof. Let $H(\omega)$ be frequency response and $h[i_1,i_2,\cdots,i_n]$ be its equivalent impulse response. Since derivatives are alone high-pass, conditions (1) and (2) of Theorem 3.3 for them is established. Consequently

$$H(\omega) = \sum_{i_1,i_2,\cdots,i_n} (-1)^{i_1,i_2,\cdots,i_n} h[i_1,i_2,\cdots,i_n]$$

$$= \sum_{|\alpha|=r} a_{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_1,x_2,\cdots,x_n)\times 1,$$

since P is elliptic operator then $\sum_{|\alpha|=r} a_{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_1,x_2,\cdots,x_n) \neq 0$ (if it is equal to zero, then the corresponding principal symbol i.e.

$$p(X,\xi) = (i)^{|\alpha|}$$

$$\sum_{|\alpha|=r} a_{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_1,x_2,\cdots,x_n)\xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n},$$

becomes zero at $\xi = (1,1,\cdots,1) \neq 0$, and this contradicts the ellipticity of the P)and this shows that the condition (1) of Theorem 3.3 is established. On the other hand

$$H(0) = \sum_{i_1,i_2,\cdots,i_n} h[i_1,i_2,\cdots,i_n] =$$

$$\sum_{|\alpha|=r} a_{\alpha_1,\alpha_2}(x_1,x_2,\cdots,x_n)\times 0 = 0$$

This shows that the condition (2) of Theorem 3.3 is established. So the obtained filter is high-pass.

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