Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 14, No. 4, 2022 Article ID IJIM-1556, 12 pages
DOI: http://dx.doi.org/10.30495/ijim.2022.64041.1556
Research Article

Order Reduction, μ-Symmetry and μ-Conservation Law of The Generalized mKdV Equation with Constant-coefficients and Variable-coefficients

Kh. Goodarzi *†

Received Date: 2021-10-29 Revised Date: 2022-06-06 Accepted Date: 2022-06-25

Abstract

The goal of this paper is to calculate the order reduction of the generalized mKdV equation with constant-coefficients ($gmKdV_{cc}$) and the generalized mKdV equation with variable-coefficients ($gmKdV_{vc}$) using the μ-symmetry method. Moreover we obtain Lagrangian and μ-conservation law of the $gmKdV_{cc}$ equation and the $gmKdV_{vc}$ equation using the variational problem method.

Keywords : Symmetry; μ-symmetry; μ-conservation law; Variational problem; Order reduction.

1 Introduction

Partial differential equations (PDEs) have been a most important subject of study in all areas of mathematical physics, engineering sciences and other technical arena. At present time, different methods are being established to order reduction and conservation law of nonlinear PDEs such as the symmetries method [21], the direct method [22], the general theorem [22] and the Noether theorem [9].

The Korteweg-de Vries (KdV) equation is as:

$$u_t + a_1 u u_x + b_1 u_{xxx} = 0,$$

where a_1 and b_1 are real constants. The modified KdV (mKdV) equation

$$u_t + a_2 u^2 u_x + b_2 u_{xxx} = 0,$$

where a_2 and b_2 are real constants, is one of the most popular partial differential equations by Korteweg and de Vries in the 19^{th} century as water waves equations.

The KdV type equation can be shown as follows:

$$u_t + a_1 u u_x + b_1 u_{xxx} + c_1 u_x = 0,$$

where a_1, b_1 and c_1 are real constants. The mKdV type equation is as:

$$u_t + a_2 u^2 u_x + b_2 u_{xxx} + c_2 u_x = 0,$$

where a_2, b_2 and c_2 are real constants. Both of them used to model water waves, plasma physics, harmonic lattices, elastic rods and nonlinear long dynamo waves observed in the Sun [10, 15].

*Corresponding author. kh.goodarzi@iaub.ac.ir, Tel:+98(66)42518049.
†Department of Mathematics, Borujerd Branch, Islamic Azad University, Borujerd, Iran.

The generalized mKdV equation with constant-coefficients ($gmKdV_{cc}$) is as follows:

$$u_t + au^2u_x + bu_{xxx} + cu_x + d(xu_x + u) = 0, \quad (1.1)$$

where a, b, c and d are real constants. The most of mathematical methods are related to the partial differential equations with constant-coefficient models [6, 7].

Recently, the study of the variable-coefficient nonlinear equations has attracted much attention [24] because most of real nonlinear physical equations possess variable-coefficients. The variable-coefficient models can describe the true physical systems in various fields, they bring more difficulties to be solved analytically [26, 27]. It is also important to study the nonlinear wave equations with variable-coefficients.

We consider the generalized mKdV equation with variable-coefficient ($gmKdV_{vc}$) in the form:

$$u_t + \alpha(t)u^2u_x + \beta(t)u_{xxx} + \gamma(t)u_x + \delta(t)(xu_x + u) = 0, \quad (1.2)$$

where $\alpha(t)$, $\beta(t)$,$\gamma(t)$ and $\delta(t)$ are arbitrary functions of time t.

This equation is a generalization of the variable-coefficient mKdV equation which is well known as a model equation describing the propagation of weakly nonlinear and weakly dispersive waves in inhomogeneous media[11, 28]. Eq. (1.2) is used as a mathematical model to study physical phenomena arising in several areas of interest. For example, in the study of coastal waves in ocean and liquid drops and bubbles, in the issues of atmospheric blocking phenomenon and dipole blocking [2, 25].

Many researchers studied the $gmKdV_{cc}$ equation and the $gmKdV_{vc}$ equation for obtaining solutions, etc. But these equations are not investigated via the μ-symmetry, Lagrangian and μ-conservation law.

In this article we calculate an order reduction of the $gmKdV_{cc}$ and the $gmKdV_{vc}$ equations using the μ-symmetry method. Moreover we calculate Lagrangian and μ-conservation law of these equations using the variational problem method.

The outline of this paper is as follows. Firstly, μ-symmetry and reduced equations for the $gmKdV_{cc}$ and the $gmKdV_{vc}$ equations are provided. Secondly, lagrangian for the $gmKdV_{cc}$ and the $gmKdV_{vc}$ equations are shown in potential form. Finally, μ-conservation law for the $gmKdV_{cc}$ and the $gmKdV_{vc}$ equations are described.

2 Background

In 2001, Muriel and Romero introduced a new method to order reduction of ordinary differential equations (ODEs), and they called it as λ-symmetries method to order reduction of ODEs. In 2004, Gaeta and Morando expanded λ-symmetries method of ODEs to μ-symmetries method of the partial differential equations (PDEs) frame with p independent variables $x = (x^1, ..., x^p)$ and q dependent variables $u = (u^1, ..., u^q)$, where $\mu = \lambda_i dx^i$ is a horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and also μ is a compatible, i.e. $D_i\lambda_j - D_j\lambda_i = 0$.

In 2006, Muriel, Romero and Olver have expanded the concept of variational problem and conservation law in the case of symmetries to the case of λ-symmetries of ODEs. They have presented an adapted formulation of the Nother's theorem for λ-symmetry of ODEs. In 2007, Cicogna and Gaeta have generalized the results obtained by Muriel, Romero and Olver in the case of λ-symmetries for ODEs to the case of μ-symmetries for PDEs, and in the case of μ-symmetry of the Lagrangian, the conservation law is referred as μ-conservation law.

3 μ-symmetry and μ-conservation law

In this section, the foundational results of μ-prolongation, μ-symmetry and μ-conservation law are briefly introduced.

Let $\mu = \lambda_i dx^i$ be horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and compatible [12], i.e. $D_i\lambda_j - D_j\lambda_i = 0$, where D_i is total derivative x^i and $\lambda_i : J^{(1)}M \longrightarrow \mathbb{R}$.

μ-symmetry:

Suppose $\Delta(x, u^{(k)}) = 0$ is a scalar PDEs of order

k for $u = u(x^1, ..., x^p)$, i.e. involving p independent variables $x = (x^1, ..., x^p)$ and one dependent variable. Let $X = \xi^i \partial_{x^i} + \varphi \partial_u$ be a vector field on M. We define

$$Y = X + \sum_{J=1}^{k} \Psi_J \, \partial_{u_J}$$

on k-th order jet space $J^k M$ as μ-prolongation of X if its coefficient satisfies the μ-prolongation formula

$$\Psi_{J,i} = (D_i + \lambda_i)\Psi_J - u_{J,m}(D_i + \lambda_i)\xi^m \,, \quad (3.3)$$

where $\Psi_0 = \varphi$. Let $\mathcal{S} \subset J^{(k)}M$ be the solution manifold for Δ. If $Y : \mathcal{S} \longrightarrow T\mathcal{S}$, we say that X is a μ-symmetry for Δ.

if $\mu = 0$ in (3.3), then it can be assumed that ordinary prolongation is as 0-prolongation in μ-prolongation and ordinary symmetry is as 0-symmetry in μ-symmetry framework.

symmetry of exponential type:

We consider an equation Δ such that $\mu = \lambda_i dx^i$ is a horizontal 1-form and compatible on $\mathcal{S}_\Delta$. Suppose $V = \exp\left(\int \mu\right)X$ is an exponential vector field, where X is a vector field on M. Then V is a general symmetry for Δ if and only if X is a μ-symmetry for Δ.

order reduction of PDEs:

In paper [12], an order reduction of PDEs under μ-symmetries is shown as the following theorem.

Theorem 3.1. *Let Δ be a scalar PDE of order k for $u = u(x^1, ..., x^p)$. Let $X = \xi^i(\frac{\partial}{\partial x^i}) + \varphi(\frac{\partial}{\partial u})$ be a vector field on M, with characteristic $Q = \varphi - u_i \xi^i$, and let Y be the μ-prolong of order k of X. If X is a μ-symmetry for Δ, then $Y : \mathcal{S}_X \longrightarrow T\mathcal{S}_X$, where $\mathcal{S}_X \subset J^{(k)}M$ is the solution manifold for the system Δ_X made of Δ and of $E_J := D_J Q = 0$ for all J with $| J |= 0, 1, ..., k-1$.*

μ-symmetry of given equations (PDE):

In order to determine μ-symmetry of a given equation Δ of order n, the same way as for ordinary symmetries is considered that a generic vector field X acting in M, and its μ-prolongation Y of order n for a generic $\mu = \lambda_i dx^i$, acting in $J^{(n)}M$. Then applies Y to Δ, and restricts the obtained expression to the solution manifold $\mathcal{S}_\Delta \subset J^{(n)}M$. The equation Δ_* resulting by requiring this is zero is the determining equation for μ-symmetries of Δ; this is an equation for ξ, τ, φ and λ_i. If we require λ_i are functions on $J^{(k)}M$, all the dependences on u_J will be explicit, and one obtains a system of determining equation. This system should be complemented with the compatibility conditions between the λ_i. If we determine a priori the form μ, we are left with a system of linear equation for ξ, τ, φ; similarly, if we fix a vector field X and try to find the μ for which it is a μ-symmetry of the given equation Δ, we have a system of quasilinear equation for the λ_i [12].

μ-conservation law:

A conservation law is a relation $\mathrm{Div}\,\mathbf{P} := \sum_{i=1}^{p} D_i P^i = 0$, where $\mathbf{P} = (P^1, \cdots, P^p)$ is a $p-$dimensional vector. Let $\mu = \lambda_i dx^i$ be a horizontal one-form and compatibility condition, i.e, $D_i \lambda_j = D_j \lambda_i$. A *μ-conservation law* is a relation as

$$(D_i + \lambda_i)P^i = 0,$$

where P^i is a vector and the $M-$vector P^i is called a μ-conserved vector.

The following theorem about the existence of $M-$vector P^i and μ-conservation law can be seen in [9]:

Theorem 3.2. *Consider the $n-th$ order Lagrangian $\mathcal{L} = \mathcal{L}(x, u^{(n)})$, and vector field X, then X is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}] = 0$ if and only if there exists $M-$vector P^i satisfying the μ-conservation law $(D_i + \lambda_i)P^i = 0$.*

Using the other theorems in [9] and Theorem 3.2, the $M-$vector P^i is obtained for first and second order Lagrangian, as the following:

- For first order Lagrangian $\mathcal{L}(x, t, u, u_x, u_t)$ and the vector field $X = \varphi\,(\partial/\partial u)$ is a μ-symmetry for $\mathcal{L}$, then the $M-$vector $P^i := \varphi\,(\partial\mathcal{L}/\partial u_i)$, is a μ-conserved vector.

- For second order Lagrangian $\mathcal{L}$ and the vector field $X = \varphi\,(\partial/\partial u)$ is a μ-symmetry for

$\mathcal{L}$, then the $M-$vector

$$P^i := \varphi \frac{\partial \mathcal{L}}{\partial u_i} + ((D_j + \lambda_j)\varphi) \frac{\partial \mathcal{L}}{\partial u_{ij}} - \varphi D_j \frac{\partial \mathcal{L}}{\partial u_{ij}}, \quad (3.4)$$

is a μ-conserved vector.

4 order reduction of the $gmKdV_{cc}$ equation and the $gmKdV_{vc}$ equation using the μ-symmetry method

In this section, we want to compute order reduction of the $gmKdV_{cc}$ equation in subsection (*4.1*) and order reduction of the $gmKdV_{vc}$ equation in subsection (*4.2*) using the μ-symmetry method.

4.1 order reduction of the $gmKdV_{cc}$ equation using the μ-symmetry method

The generalized mKdV equation with constant-coefficients ($gmKdV_{cc}$) can be shown as follows:

$$u_t + au^2 u_x + bu_{xxx} + cu_x + d(xu_x + u) = 0,$$

where a, b, c and d are real constants and this equation a scalar PDE of order 3 for $u = u(x,t)$. Let $\mu = \lambda_1 dx + \lambda_2 dt$ be a horizontal one-form and with the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $u_t + au^2 u_x + bu_{xxx} + cu_x + d(xu_x + u) = 0$. Suppose $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ is a vector field on M. In order to compute μ-prolongation Y of order 3 of X, we can use of (3.3); therefore, μ-prolongation Y of X is as

$$Y = X + \Psi^x \partial_{u_x} + \Psi^t \partial_{u_t} + \Psi^{xx} \partial_{u_{xx}} + \dots + \Psi^{ttt} \partial_{u_{ttt}},$$

where coefficients Y are as the following

$$\begin{aligned}
\Psi^x &= (D_x + \lambda_1)\varphi - u_x(D_x + \lambda_1)\xi \\
&\quad -u_t(D_x + \lambda_1)\tau\,, \\
\Psi^t &= (D_t + \lambda_2)\varphi - u_x(D_t + \lambda_2)\xi \\
&\quad -u_t(D_t + \lambda_2)\tau\,, \\
\Psi^{xx} &= (D_x + \lambda_1)\Psi^x - u_{xx}(D_x + \lambda_1)\xi \\
&\quad -u_{xt}(D_x + \lambda_1)\tau\,, \\
\Psi^{xt} &= (D_t + \lambda_2)\Psi^x - u_{xx}(D_t + \lambda_2)\xi \\
&\quad -u_{xt}(D_t + \lambda_2)\tau\,, \\
\Psi^{tt} &= (D_t + \lambda_2)\Psi^t - u_{tx}(D_t + \lambda_2)\xi \\
&\quad -u_{tt}(D_t + \lambda_2)\tau\,, \qquad (4.5) \\
\Psi^{xxx} &= (D_x + \lambda_1)\Psi^{xx} - u_{xxx}(D_x + \lambda_1)\xi \\
&\quad -u_{xxt}(D_x + \lambda_1)\tau, \\
\Psi^{xxt} &= (D_t + \lambda_2)\Psi^{xx} - u_{xxx}(D_t + \lambda_2)\xi \\
&\quad -u_{xxt}(D_t + \lambda_2)\tau, \\
\Psi^{xtt} &= (D_t + \lambda_2)\Psi^{xt} - u_{xtx}(D_t + \lambda_2)\xi \\
&\quad -u_{xtt}(D_t + \lambda_2)\tau\,, \\
\Psi^{ttt} &= (D_t + \lambda_2)\Psi^{tt} - u_{ttx}(D_t + \lambda_2)\xi \\
&\quad -u_{ttt}(D_t + \lambda_2)\tau\,.
\end{aligned}$$

By applying Y to Eq. (1.1) and substituting

$$\frac{-1}{b}\Big(u_t + au^2 u_x + cu_x + d(xu_x + u)\Big),$$

for u_{xxx} , we obtain the following system [1] [1]:

$$\begin{gathered}
-3b\tau_u = 0,\ -3b\tau_{uu} = 0,\ -b\tau_{uuu} = 0, \\
-3b\xi_u = 0,\ -6b\xi_{uu} = 0,\ -b\xi_{uuu} = 0, \\
-3b(\lambda_1\tau + \tau_x) = 0, \\
\vdots \qquad (4.6) \\
-3b(\tau_{xx} + \tau\lambda_{1x} + 2\lambda_1\tau_x + \lambda_1^2\tau) = 0.
\end{gathered}$$

For any choice of the type

$$\begin{aligned}
\lambda_1 &= D_x[f(x,t)] + g(x), \\
\lambda_2 &= D_t[f(x,t)] + h(t), \qquad (4.7)
\end{aligned}$$

where $f(x,t)$, $g(x)$ and $h(t)$ are arbitrary functions and λ_1 and λ_2 satisfy to the compatibility condition, i.e. $D_t\lambda_1 = D_x\lambda_2$ on solutions to Eq. (1.1). For instance, two cases are studied to obtain in μ-symmetry of Eq. (1.1) as follows:

[1]1 using Maple

- When $g(x) = 0$ and $h(t) = d$ in the functions of (4.7), then by substituting the functions

$$\lambda_1 = D_x f(x,t) \ , \ \lambda_2 = D_t f(x,t) + d$$

into the system of (4.6) and solving them, we obtain

$$\xi = F(x,t), \qquad \tau = 0, \qquad \varphi = 0,$$

where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then

$$\begin{aligned} X &= \xi\partial_x + \tau\partial_t + \varphi\partial_u \\ &= F(x,t)\partial_x \end{aligned}$$

is μ-symmetry of Eq. (1.1) and corresponds to an ordinary symmetry

$$V = \exp\Big(\int(D_x f(x,t)dx + (D_t f(x,t) + d)dt)\Big)X\,,$$

of exponential type. In this case, using Theorem 3.1, reduction of Eq. (1.1) is

$$\begin{aligned} Q &= \varphi - \xi u_x - \tau u_t \\ &= -F(x,t)u_x. \end{aligned} \tag{4.8}$$

- When $g(x) = 0$ and $h(t) = 0$ in the functions of (4.7), then by substituting the functions

$$\lambda_1 = D_x f(x,t) \ , \ \lambda_2 = D_t f(x,t),$$

into the system of (4.6) and solving them, we obtain

$$\xi = 0\,, \qquad \tau = F(x,t)\,, \qquad \varphi = 0\,,$$

where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then

$$\begin{aligned} X &= \xi\partial_x + \tau\partial_t + \varphi\partial_u \\ &= F(x,t)\partial_t, \end{aligned}$$

is μ-symmetry of Eq. (1.1) and corresponds to an ordinary symmetry

$$V = \exp\Big(\int D_x f(x,t)dx + D_t f(x,t)dt\Big)X,$$

of exponential type. In this case, using Theorem 3.1, reduction of Eq. (1.1) is

$$\begin{aligned} Q &= \varphi - \xi u_x - \tau u_t \\ &= -F(x,t)u_t, \end{aligned} \tag{4.9}$$

4.2 order reduction of the $gmKdV_{vc}$ equation using the μ-symmetry method

The generalized mKdV equation with variable-coefficients ($gmKdV_{vc}$) can be shown as follows:

$$u_t + \alpha(t)u^2u_x + \beta(t)u_{xxx} + \gamma(t)u_x + \delta(t)(xu_x + u) = 0,$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are arbitrary functions of time t and its a scalar PDE of order 3 for $u = u(x,t)$. Let $\mu = \lambda_1 dx + \lambda_2 dt$ be a horizontal one-form and with the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $u_t + \alpha(t)u^2u_x + \beta(t)u_{xxx} + \gamma(t)u_x + \delta(t)(xu_x + u) = 0$. Suppose $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ is a vector field on M. In order to compute μ-prolongation Y of order 3 of X, we can use of (3.3); therefore, μ-prolongation Y of X is as

$$Y = X + \Psi^x\partial_{u_x} + \Psi^t\partial_{u_t} + \Psi^{xx}\partial_{u_{xx}} + \ldots + \Psi^{ttt}\partial_{u_{ttt}},$$

where coefficients Y are as of (4.5). By applying Y to Eq. (1.1) and substituting

$$\frac{-1}{\beta(t)}\Big(u_t + \alpha(t)u^2u_x + \gamma(t)u_x + \delta(t)(xu_x + u)\Big),$$

for u_{xxx} , we obtain the following system [12]:

$$\begin{gathered} -3\beta(t)\tau_u = 0, -3\beta(t)\tau_{uu} = 0, \\ -\beta(t)\tau_{uuu} = 0, -3\beta(t)\xi_u = 0, \\ -6\beta(t)\xi_{uu} = 0, -\beta(t)\xi_{uuu} = 0, \\ -3\beta(t)(\lambda_1\tau + \tau_x) = 0, \\ \vdots \\ -3\beta(t)(\tau_{xx} + \tau\lambda_{1x} + 2\lambda_1\tau_x + \lambda_1^2\tau) = 0. \end{gathered} \tag{4.10}$$

Let λ_1 and λ_2 are functions of (4.7) and satisfy to the compatibility condition, i.e. $D_t\lambda_1 = D_x\lambda_2$ on solutions to Eq. (1.2). For instance, two cases are studied to obtain in μ-symmetry of Eq. (1.2) as follows:

- When $g(x) = 0$ and $h(t) = \delta(t)$ in the functions of (4.7), also $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are arbitrary functions, then by substituting the functions

$$\lambda_1 = D_x f(x,t) \ , \ \lambda_2 = D_t f(x,t) + \delta(t)$$

[21] using Maple

into the system of (4.10) and solving them, we obtain

$$\xi = F(x,t), \qquad \tau = 0, \qquad \varphi = 0\,,$$

where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then

$$X = \xi\partial_x + \tau\partial_t + \varphi\partial_u = F(x,t)\partial_x\,,$$

is μ-symmetry of Eq. (1.2) and corresponds to an ordinary symmetry

$$V = \exp\Big(\int D_x f(x,t)dx + (D_t f(x,t) +\delta(t))dt\Big)X\,,$$

of exponential type. In this case, using Theorem 3.1, reduction of Eq. (1.2) is

$$\begin{aligned} Q &= \varphi - \xi u_x - \tau u_t \\ &= -F(x,t)u_x. \end{aligned} \tag{4.11}$$

- When $g(x) = 0$ and $h(t) = 1/t$ in the functions of (4.7), also $\alpha(t) = c_1/t^2$, $(t) = c_2 t^2$, $\gamma(t) = c_3$ and $\delta(t) = c_4/t$ where c_1, c_2, c_3 and c_4 are arbitrary constants, then by substituting the functions

$$\lambda_1 = D_x f(x,t)\ ,\ \lambda_2 = D_t f(x,t) + 1/t$$

into the system of (4.10) and solving them, we obtain

$$\xi = \frac{x}{t}F(x,t), \tau = F(x,t), \varphi = \frac{u}{t}F(x,t),$$

where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then

$$X = F(x,t)\Big(\frac{x}{t}\partial_x + \partial_t + \frac{u}{t}\partial_u\Big),$$

is μ-symmetry of Eq. (1.2) and corresponds to an ordinary symmetry

$$V = \exp\Big(\int D_x f(x,t)dx + (D_t f(x,t) +1/t)dt\Big)X\,,$$

of exponential type. In this case, using Theorem 3.1, reduction of Eq. (1.2) is

$$\begin{aligned} Q &= \varphi - \xi u_x - \tau u_t \\ &= F(x,t)\Big(\frac{u}{t} - \frac{x}{t}u_x - u_t\Big). \end{aligned} \tag{4.12}$$

5 Lagrangian of the $gmKdV_{cc}$ and the $gmKdV_{vc}$ equations in potential form using the variational problem method

In this section, we show that the $gmKdV_{cc}$ equation and the $gmKdV_{vc}$ equation do not admit a variational problem since they are of odd order, but the $gmKdV_{cc}$ equation and the $gmKdV_{vc}$ equation in potential form admitting a variational problem. We obtain Lagrangian of the $gmKdV_{cc}$ equation in subsection (*5.1*) and Lagrangian of the $gmKdV_{vc}$ equation in subsection (*5.2*).

In the book [22], a system admits a variational formulation if and only if its Frechet derivative is self-adjoint. In fact, we have the following theorem.

Theorem 5.1. *Let $\Delta = 0$ be a system of differential equation. Then Δ is the Euler-Lagrange expression for some variational problem $\mathfrak{L} = \int L dx$, i.e. $\Delta = E(L)$, if and only if the Frechet derivative D_Δ is self-adjoint: $D^*_\Delta = D_\Delta$. In this case, a Lagrangian for Δ can be explicitly constructed using the homotopy formula $L[u] = \int_0^1 u.\Delta[\lambda u]d\lambda$.*

5.1 Lagrangian of the $gmKdV_{cc}$ equation in potential form

We consider the $gmKdV_{cc}$ as

$$\begin{aligned} \Delta_{Ku_{cc}} : u_t + au^2u_x + bu_{xxx} \\ +cu_x + d(xu_x + u) = 0. \end{aligned} \tag{5.13}$$

The Frechet derivative of $\Delta_{Ku_{cc}}$ is

$$\begin{aligned} D_{\Delta_{Ku_{cc}}} = D_t + 2auu_x + bD_x^3 \\ +\Big(au^2 + c + xd\Big)D_x + d. \end{aligned}$$

Obviously, it does not admit a variational problem since $D^*_{\Delta_{Ku_{cc}}} \neq D_{\Delta_{Ku_{cc}}}$. But the well-known differential substitution $u = v_x$ yields the related transformed the $gmKdV_{cc}$ as the following

$$\begin{aligned} \Delta_{Kv_{cc}} : v_{xt} + av_x^2v_{xx} + bv_{xxxx} + cv_{xx} \\ +d(xv_{xx} + v_x) = 0. \end{aligned} \tag{5.14}$$

This equation is called "the $gmKdV_{cc}$ in potential form" and its Frechet derivative is

$$D_{\Delta_{Kv_{cc}}} = D_x D_t + \Big(2av_x v_{xx} + d\Big)D_x + \Big(av_x^2 + c + xd\Big)D_x^2 + bD_x^4 .$$

which is self-adjoint: $D^*_{\Delta_{Kv_{cc}}} = D_{\Delta_{Kv_{cc}}}$. By Theorem 5.1, the $gmKdV_{cc}$ in potential form $\Delta_{Kv_{cc}}$ has a Lagrangian of the form

$$\begin{aligned} L[v] &= \int_0^1 v.\Delta_{Kv_{cc}}[\lambda v]d\lambda \\ &= -\frac{1}{2}\Big(v_x v_t - bv_{xx}^2 + cv_x^2 + (1/6)av_x^4 \\ &\quad + xdv_x^2\Big) + \mathrm{Div}P. \end{aligned}$$

Hence, Lagrangian of the $gmKdV_{cc}$ in potential form $\Delta_{Kv_{cc}}$, up to Div-equivalence is

$$\mathcal{L}_{\Delta_{Kv_{cc}}}[v] = -\frac{1}{2}\Big(v_x v_t - bv_{xx}^2 + cv_x^2 + (1/6)av_x^4 + xdv_x^2\Big). \tag{5.15}$$

5.2 Lagrangian of the $gmKdV_{vc}$ equation in potential form

We consider the $gmKdV_{vc}$ as

$$\Delta_{Ku_{vc}} : u_t + \alpha(t)u^2 u_x + \beta(t)u_{xxx} + \gamma(t)u_x + \delta(t)(xu_x + u) = 0. \tag{5.16}$$

The Frechet derivative of $\Delta_{Ku_{vc}}$ is

$$D_{\Delta_{Ku_{vc}}} = D_t + 2\alpha(t)uu_x + \beta(t)D_x^3 + \Big(\alpha(t)u^2 + \gamma(t) + x\delta(t)\Big)D_x + \delta(t).$$

Obviously, it does not admit a variational problem since $D^*_{\Delta_{Ku_{vc}}} \neq D_{\Delta_{Ku_{vc}}}$. But the well-known differential substitution $u = v_x$ yields the related transformed the $gmKdV_{vc}$ as the following

$$\Delta_{Kv_{vc}} : v_{xt} + \alpha(t)v_x^2 v_{xx} + \beta(t)v_{xxxx} + \gamma(t)v_{xx} + \delta(t)(xv_{xx} + v_x) = 0. \tag{5.17}$$

This equation is called "the $gmKdV_{vc}$ in potential form" and its Frechet derivative is

$$D_{\Delta_{Kv_{vc}}} = D_x D_t + \Big(2\alpha(t)v_x v_{xx} + \delta(t)\Big)D_x + \Big(\alpha(t)v_x^2 + \gamma(t) + x\delta(t)\Big)D_x^2 + \beta(t)D_x^4 .$$

which is self-adjoint: $D^*_{\Delta_{Kv_{vc}}} = D_{\Delta_{Kv_{vc}}}$. By Theorem 5.1, the $gmKdV_{vc}$ in potential form $\Delta_{Kv_{vc}}$ has a Lagrangian of the form

$$\begin{aligned} L[v] &= \int_0^1 v.\Delta_{Kv_{vc}}[\lambda v]d\lambda \\ &= -\frac{1}{2}\Big(v_x v_t - \beta(t)v_{xx}^2 + \gamma(t)v_x^2 \\ &+ (1/6)\alpha(t)v_x^4 + +x\delta(t)v_x^2\Big) + \mathrm{Div}P. \end{aligned}$$

Hence, Lagrangian of the $gmKdV_{vc}$ in potential form $\Delta_{Kv_{vc}}$, up to Div-equivalence is

$$\mathcal{L}_{\Delta_{Kv}}[v] = -\tfrac{1}{2}\Big(v_x v_t - \beta(t)v_{xx}^2 + \gamma(t)v_x^2 + (1/6)\alpha(t)v_x^4 + x\delta(t)v_x^2\Big). \tag{5.18}$$

6 μ-conservation laws of the $gmKdV_{cc}$ equation in potential form

In this section, we want to compute μ-conservation law for the $gmKdV_{cc}$ equation in potential form $\Delta_{Kv_{cc}}$ in subsection (*6.1*) and using it, we compute μ-conservation law for the $gmKdV_{cc}$ equation $\Delta_{Ku_{cc}}$ in subsection (*6.2*).

6.1 μ-conservation laws of the $gmKdV_{cc}$ equation in potential forms

We consider the second order Lagrangian (5.15), i.e.

$$\mathcal{L}_{\Delta_{Kv_{cc}}}[v] = -\frac{1}{2}\Big(v_x v_t - bv_{xx}^2 + cv_x^2 + (1/6)av_x^4 + xdv_x^2\Big),$$

for the $gmKdV_{cc}$ equation in potential form

$$\begin{aligned} \Delta_{Kv_{cc}} &= v_{xt} + av_x^2 v_{xx} + bv_{xxxx} + cv_{xx} \\ &+ d(xv_{xx} + v_x) \\ &= E(\mathcal{L}_{\Delta_{Kv}}). \end{aligned} \tag{6.19}$$

Suppose $X = \varphi\partial_v$ is a vector field for $\mathcal{L}_{\Delta_{Kv_{cc}}}[v]$. Let $\mu = \lambda_1 dx + \lambda_2 dt$ be a horizontal one-form and with the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $\Delta_{Kv_{cc}} = 0$. In order to compute μ-prolongation of order 2 of X, we can use of (3.3), we have,

$$Y = \varphi\partial_v + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} + \Psi^{xx}\partial_{v_{xx}} + \Psi^{xt}\partial_{v_{xt}} + \Psi^{tt}\partial_{v_{tt}},$$

where coefficients Y are as the following:

$$\begin{aligned} \Psi^x &= (D_x+\lambda_1)\varphi, \quad \Psi^t = (D_t+\lambda_2)\varphi, \\ \Psi^{xx} &= (D_x+\lambda_1)\Psi^x, \Psi^{xt} = (D_t+\lambda_2)\Psi^x, \\ \Psi^{tt} &= (D_t+\lambda_2)\Psi^t. \end{aligned} \tag{6.20}$$

Thus, the μ-prolongation Y acts on the $\mathcal{L}_{\Delta_{Kv_{cc}}}[v]$, and substituting $\Big(-bv_{xx}^2 + cv_x^2 + (1/6)av_x^4 + xdv_x^2\Big)/-v_x$ for v_t, we obtain the system as the following:

$$\begin{gathered} b\varphi_{vv} = 0, (-1/6)a\varphi_v = 0, \\ -(1/2)b(\varphi_x+\lambda_1\varphi) = 0, \\ (-1/4)a(\varphi_x+\lambda_1\varphi) = 0\,, \\ b(\lambda_{1v}\varphi + 2\lambda_1\varphi_v + 2\varphi_{xv}) = 0\,, \\ b(2\lambda_1\varphi_x + \lambda_{1x}\varphi + \varphi_{xx} + \lambda_1^2\varphi) = 0\,, \\ (-1/2)(xd\varphi_x + \varphi_t + c\lambda_1\varphi + \lambda_2\varphi + c\varphi_x \\ +xd\lambda_1\varphi) = 0\,. \end{gathered} \tag{6.21}$$

Suppose $\varphi = F(x,t)$, where $F(x,t)$ is an arbitrary positive function satisfying $\mathcal{L}_{\Delta_{Kv_{cc}}}[v]=0$, then a special solution of the system (6.21) is given by

$$\lambda_1 = -\frac{F_x(x,t)}{F(x,t)},\ \lambda_2 = -\frac{F_t(x,t)}{F(x,t)}, \tag{6.22}$$

where λ_1 and λ_2 are satisfying to $D_t\lambda_1 = D_x\lambda_2$. Hence,

$$X = F(x,t)\partial_v$$

is a μ-symmetry for $\mathcal{L}_{\Delta_{Kv_{cc}}}[v]$, then, using Theorem 3.2, there exists $M-$vector P^i satisfying the μ-conservation law $(D_i+\lambda_i)P^i = 0$. Then, by of (3.4), the $M-$vector P^i is as

$$\begin{aligned} P^1 &= \frac{-1}{6}\Big(3v_t + 2av_x^3 + 6bv_{xxx} + 6cv_x \\ &\quad +6xdv_x\Big)F(x,t), \\ P^2 &= -\frac{v_x}{2}F(x,t), \end{aligned} \tag{6.23}$$

and $(D_x+\lambda_1)P^1 + (D_t+\lambda_2)P^2 = 0$, or corresponds to $D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0$, is a μ-conservation law for second order Lagrangian $\mathcal{L}_{\Delta_{Kv_{cc}}}[v]$. Therefore we have obtained the following corollary:

Corollary 6.1. *μ-conservation law for the $gmKdV_{cc}$ equation in potential form $\Delta_{Kv_{cc}} = E(\mathcal{L}_{\Delta_{Kv_{cc}}})$ is as*

$$D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0, \tag{6.24}$$

where P^1 and P^2 are the $M-$vector P^i of (6.23).

Remark 6.1. *μ-conservation law for the $gmKdV_{cc}$ equation in potential form $\Delta_{Kv_{cc}}$, satisfying to the Noether's Theorem for μ-symmetry, i.e.*

$$\begin{aligned} (D_i+\lambda_i)P^i &= (D_x+\lambda_1)P^1 + (D_t+\lambda_2)P^2 \\ &= F(x,t)\Big(v_{xt} + av_x^2v_{xx} + bv_{xxxx} \\ &\quad +cv_{xx} + d(xv_{xx}+v_x)\Big) \\ &= QE(\mathcal{L}_{\Delta_{Kv_{cc}}}). \end{aligned}$$

6.2 μ-conservation laws of the $gmKdV_{cc}$ equation

We consider the $gmKdV_{cc}$ equation in potential form

$$\Delta_{Kv_{cc}} = v_{xt} + av_x^2v_{xx} + bv_{xxxx} + cv_{xx} + d(xv_{xx}+v_x) = 0,$$

or equivalently

$$D_x\Big(v_t + (1/3)av_x^3 + bv_{xxx} + cv_x + xdv_x\Big) = 0,$$
$$v_t + (1/3)av_x^3 + bv_{xxx} + cv_x + xdv_x = F_1(t),$$

where $F_1(t)$ is an arbitrary function. If we substitute

$$F_1(t) - (1/3)av_x^3 - bv_{xxx} - cv_x - xdv_x$$

for v_t and substitute u for v_x in the $M-$vector P^i of (6.23), then, we obtain $M-$vectors P^1 and P^2 as the following

$$\begin{aligned} P^1 &= -\frac{1}{6}\Big(3F_1(t) + au^3 + 3bu_{xx} + 3cu \\ &\quad +3xdu\Big)F(x,t), \\ P^2 &= -\frac{u}{2}F(x,t)\,. \end{aligned} \tag{6.25}$$

Therefore we have obtained the following corollary:

Corollary 6.2. *μ-conservation law for the $gmKdV_{cc}$ equation is as*

$$D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0, \qquad (6.26)$$

where P^1 and P^2 are the $M-$vector P^i of (6.25).

Remark 6.2. *The $gmKdV_{cc}$ equation satisfying to the characteristic form, i.e.*

$$\begin{aligned}(D_i+\lambda_i)P^i &= (D_x+\lambda_1)P^1 + (D_t+\lambda_2)P^2\\ &= F(x,t)(u_t + au^2u_x + bu_{xxx} + cu_x\\ &\quad + d(xu_x+u))\\ &= Q\Delta_{Ku_{cc}}.\end{aligned}$$

7 μ-conservation laws of the $gmKdV_{vc}$ equation in potential form

In this section, we want to compute μ-conservation law for the $gmKdV_{vc}$ equation in potential form $\Delta_{Kv_{vc}}$ in subsection (*7.1*) and using it, we compute μ-conservation law for the $gmKdV_{vc}$ equation $\Delta_{Ku_{vc}}$ in subsection (*7.2*).

7.1 μ-conservation laws of the $gmKdV_{vc}$ equation in potential forms

We consider the second order Lagrangian (5.18), i.e.

$$\begin{aligned}\mathcal{L}_{\Delta_{Kv_{vc}}}[v] = -\frac{1}{2}\Big(&v_xv_t - \beta(t)v_{xx}^2 + \gamma(t)v_x^2\\ &+(1/6)\alpha(t)v_x^4 + x\delta(t)v_x^2\Big),\end{aligned}$$

for the $gmKdV_{vc}$ equation in potential form

$$\begin{aligned}\Delta_{Kv} &= v_{xt} + \alpha(t)v_x^2v_{xx} + \beta(t)v_{xxxx}\\ &\quad + \gamma(t)v_{xx} + \delta(t)(xv_{xx}+v_x)\\ &= E(\mathcal{L}_{\Delta_{Kv}}). \qquad (7.27)\end{aligned}$$

Suppose $X = \varphi\partial_v$ is a vector field for $\mathcal{L}_{\Delta_{Kv_{vc}}}[v]$. Let $\mu = \lambda_1dx + \lambda_2dt$ be a horizontal one-form and with the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $\Delta_{Kv_{vc}} = 0$. In order to compute μ-prolongation of order 2 of X, we can use of (3.3), we have,

$$\begin{aligned}Y = \varphi\partial_v + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} + \Psi^{xx}\partial_{v_{xx}}\\ + \Psi^{xt}\partial_{v_{xt}} + \Psi^{tt}\partial_{v_{tt}},\end{aligned}$$

where coefficients Y are as of (6.20). Thus, the μ-prolongation Y acts on the $\mathcal{L}_{\Delta_{Kv_{vc}}}[v]$, and substituting $\Big(-\beta(t)v_{xx}^2 + \gamma(t)v_x^2 + (1/6)\alpha(t)v_x^4 + x\delta(t)v_x^2\Big)/-v_x$ for v_t, we obtain the system as the following:

$$\begin{aligned}&\beta(t)\varphi_{vv} = 0,\ (-1/6)\alpha(t)\varphi_v = 0,\\ &-(1/2)\beta(t)(\varphi_x + \lambda_1\varphi) = 0,\\ &(-1/4)\alpha(t)(\varphi_x+\lambda_1\varphi) = 0,\\ &\beta(t)(\lambda_{1v}\varphi + 2\lambda_1\varphi_v + 2\varphi_{xv}) = 0, \qquad (7.28)\\ &\beta(t)(2\lambda_1\varphi_x + \lambda_{1x}\varphi + \varphi_{xx} + \lambda_1^2\varphi) = 0,\\ &(-1/2)(x\delta(t)\varphi_x + \varphi_t + \gamma(t)\lambda_1\varphi + \lambda_2\varphi\\ &\quad + \gamma(t)\varphi_x + x\delta(t)\lambda_1\varphi) = 0.\end{aligned}$$

Suppose $\varphi = F(x,t)$, where $F(x,t)$ is an arbitrary positive function satisfying $\mathcal{L}_{\Delta_{Kv_{vc}}}[v] = 0$, then a special solution of the system (7.28) is given by

$$\lambda_1 = -\frac{F_x(x,t)}{F(x,t)}, \lambda_2 = -\frac{F_t(x,t)}{F(x,t)}, \qquad (7.29)$$

where λ_1 and λ_2 are satisfying to $D_t\lambda_1 = D_x\lambda_2$. Hence,

$$X = F(x,t)\partial_v$$

is a μ-symmetry for $\mathcal{L}_{\Delta_{Kv_{vc}}}[v]$, then, using Theorem 3.2, there exists $M-$vector P^i satisfying the μ-conservation law $(D_i+\lambda_i)P^i = 0$. Then, by of (3.4), the $M-$vector P^i is as

$$\begin{aligned}P^1 &= \frac{-1}{6}\Big(3v_t + 2\alpha(t)v_x^2 + 6\beta(t)v_{xxx}\\ &\quad + 6\gamma(t)v_x + 6x\delta(t)v_x\Big)F(x,t), \qquad (7.30)\\ P^2 &= -\frac{v_x}{2}F(x,t),\end{aligned}$$

and $(D_x+\lambda_1)P^1 + (D_t+\lambda_2)P^2 = 0$, or corresponds to $D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0$, is a μ-conservation law for second order Lagrangian $\mathcal{L}_{\Delta_{Kv_{vc}}}[v]$. Therefore we have obtained the following corollary:

Corollary 7.1. *μ-conservation law for the $gmKdV_{vc}$ equation in potential form $\Delta_{Kv_{vc}} = E(\mathcal{L}_{\Delta_{Kv_{vc}}})$ is as*

$$D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0, \qquad (7.31)$$

where P^1 and P^2 are the $M-$vector P^i of (7.30).

Remark 7.1. *μ-conservation law for the $gmKdV_{vc}$ equation in potential form $\Delta_{Kv_{vc}}$, satisfying to the Noether's Theorem for μ-symmetry, i.e.*

$$\begin{aligned}(D_i+\lambda_i)P^i &= (D_x+\lambda_1)P^1+(D_t+\lambda_2)P^2\\ &= F(x,t)\Big(v_{xt}+\alpha(t)v_x^2v_{xx}+\beta(t)v_{xxxx}\\ &\quad +\gamma(t)v_{xx}+\delta(t)(xv_{xx}+v_x)\Big)\\ &= QE(\mathcal{L}_{\Delta_{Kv_{vc}}}).\end{aligned}$$

7.2 μ-conservation laws of the $gmKdV_{vc}$ equation

We consider the $gmKdV_{vc}$ equation in potential form

$$\begin{aligned}\Delta_{Kv} &= v_{xt}+\alpha(t)v_x^2v_{xx}+\beta(t)v_{xxxx}\\ &\quad +\gamma(t)v_{xx}+\delta(t)(xv_{xx}+v_x)=0,\end{aligned}$$

or equivalently

$$\begin{aligned}&D_x\Big(v_t+\frac{1}{3}\alpha(t)v_x^3+\beta(t)v_{xxx}+\gamma(t)v_x\\ &\qquad +\delta(t)xv_x\Big)=0,\\ &v_t+\frac{1}{3}\alpha(t)v_x^3+\beta(t)v_{xxx}+\gamma(t)v_x\\ &\qquad +\delta(t)xv_x=F_1(t),\end{aligned}$$

where $F_1(t)$ is an arbitrary function. If we substitute

$$F_1(t)-\frac{1}{3}\alpha(t)v_x^3-\beta(t)v_{xxx}-\gamma(t)v_x-\delta(t)xv_x$$

for v_t and substitute u for v_x in the $M-$vector P^i of (7.30), then, we obtain $M-$vectors P^1 and P^2 as the following

$$\begin{aligned}P^1 &= -\frac{1}{6}\Big(3F_1(t)+\alpha(t)u^3+3\beta(t)u_{xx}\\ &\quad +3\gamma(t)u+3x\delta(t)u\Big)F(x,t),\\ P^2 &= -\frac{u}{2}F(x,t).\end{aligned} \tag{7.32}$$

Therefore we have obtained the following corollary:

Corollary 7.2. *μ-conservation law for the $gmKdV_{vc}$ equation is as*

$$D_xP^1+D_tP^2+\lambda_1P^1+\lambda_2P^2=0, \tag{7.33}$$

where P^1 and P^2 are the $M-$vector P^i of (7.32).

Remark 7.2. *The $gmKdV_{vc}$ equation satisfying to the characteristic form, i.e.*

$$\begin{aligned}(D_i+\lambda_i)P^i &= (D_x+\lambda_1)P^1+(D_t+\lambda_2)P^2\\ &= F(x,t)(u_t+\alpha(t)u^2u_x+\beta(t)u_{xxx}\\ &\quad +\gamma(t)u_x+\delta(t)(xu_x+u))\\ &= Q\Delta_{Ku_{vc}}.\end{aligned}$$

8 Conclusion

In this paper, we provided μ-symmetry and reduced equations for the $gmKdV_{cc}$ and the $gmKdV_{vc}$ equations and lagrangian for the $gmKdV_{cc}$ and the $gmKdV_{vc}$ equations are shown in potential form. Finally, we described μ-conservation law for the $gmKdV_{cc}$ and the $gmKdV_{vc}$ equations.

References

[1] H. Airault, Rational solutions of Painlevé equation, *Studies in Applied Mathematics* 61 (1979) 31-53.

[2] A. Biswas, Solitary wave solution for the generalized KdV equation with time- dependent damping and dispersion, *Commun. Nonlinear Sci. Numer. Simulat* 14 (2009) 3503-3506.

[3] T. B. Benjamin, J. L. Bona, J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Trans. R. Soc. (Lond) ser.A* 272 (1992).

[4] W. Bluman, F. Cheviakov, C. Anco, Construction of conservation law: how the direct method generalizes Nother's theorem, *Group analysis of differential equations and integrability* (2009) 1-23.

[5] J. L. Bona, P. J. Bryant, A mathematical model for long waves gener- ated by wave makers in nonlinear dispersive systems, *Proc. Cambridge Phil. Soc* (1973).

[6] H. H. Chen, C. S. Liu, Solitons in nonuniform media, *Phys. Rev. Lett* 37 (1976).

[7] W. L. Chan, L. Kam-Shun, Nonpropagating solitons of the variable coefficient and nonisospectral Kortewegde Vries equation, *J. Math. Phys* 30 (1989) 2521-2526.

[8] G. Cicogna, G. Gaeta, P. Morando, On the relation between standard and μ-symmetries for PDEs, *J. Phys. A* 37 (2004) 9467-9486.

[9] G. Cicogna, G. Gaeta, Norther theorem for μ-symmetries, *J. Phys.A* 40 (2007) 11899-11921.

[10] P. G. Drazin, R. S. Johnson, Solitons: An Introduction, *Cambridge University press. London* (1983).

[11] Z. T. Fu, S. D. Liu, S. K. Liu, New exact solutions to KdV equations with variable coefficients or forcing, *Appl. Math. Mech* 25 (2004) 73-79.

[12] G. Gaeta, P. Morando, On the geometry of lambda-symmetries and PDEs reduction, *J. Phys. A* 37 (2004) 6955-6975.

[13] G. Gaeta, Lambda and mu-symmetries, *SPT2004.World Scientific. Singapore* (2005).

[14] KH. Goodarzi, M.Nadjafikhah, μ-symmetry and μ-conservation law for the extended mKdV equation, *JNMP* 3 (2014) 371-381.

[15] D. J.Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves, *Phil. Mag* 39 (1895) 422-443.

[16] A. Kudryashov, I. Sinelshchikov, A note on the Lie symmetry analysis and exact solutions for the extended mKdV equation, *Acta. Appl.Math* 113 (2011) 41-44.

[17] E. V. Krishnan, Y. Peng, Exact solutions to the combined KdV-mKdV equation by the extended mapping method, *Phys. Scr* 73 (2006) 405-409.

[18] C. Muriel, J. L. Romero, New methods of reduction for ordinary differential equation, *IMA J. Appl. Math* 66 (2001) 111-125.

[19] C. Muriel, J. L. Romero, C^{∞}-symmetries and reduction of equation without Lie point symmetries, *J. Lie Theory* 13 (2003) 167-188.

[20] C. Muriel, J. L. Romero, P.J. Olver, Variationl C^{∞}-symmetries and Euler-Lagrange equations, *J. Diff. Eqs* 222 (2006) 164-184.

[21] C. Muriel, J. L. Romero, Prolongations of vector fields and the invariants-by-derrivation property, *Theor. Math. Phys* 133 (2002) 1565-1575.

[22] P. J. Olver, Applications of Lie Groups to Differential Equations, *New York* (1986).

[23] C. Vasconcellos, P. N. da Silva, Stabilization of the linear Kawahara equation, *Applied and Computational Mathematics* 3 (2015).

[24] ML.Wang, YM. Wang, *Phys. Lett* (2001) 287- 291.

[25] T. Xiao-Yan, H. Fei, L. Sen-Yue, Variable coecient KdV equation and the analytical diagnosis of a dipole blocking life cycle, *Chinese Phys. Lett* 23 (2006) 887-890.

[26] X. Yu, Y. T. Gao, Z. Y. Sun, Y. Liu. Solitonic propagation and interaction for a generalized variable-coefficient forced Korteweg-de Vries equation in fluids, *J. Phys. Rev. E* 83 (2011).

[27] X. Yu, Y. T. Gao, Z. Y. Sun, Y. Liu. N-soliton solutions, Bcklund transformation and Lax pair for a generalized variable-coefficient fifth-order Korteweg-de Vries equation, *J.Phys. Scr* 81 (2010).

[28] S. Zhang, Application of Exp-function method to a KdV equation with variable coefficients, *Phys. Lett.A* 365 (2007) 448-453.

Khodayar Goodarzi is a PhD in Department of Mathematics, Faculty of Mathematics, Borujerd Branch of Azad University, Borujerd, Iran. His research interests include Order Reduction, -Symmetry and -Conservation Law method of the partial differential equations (PDEs).