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Research Article

Transformation of BL-general Fuzzy Automata

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Abstract

In this paper, we prove that any BL-general fuzzy automaton (BL-GFA) and its quotient have the same behavior. In addition, we obtain the minimal quotient BL-GFA and minimal quotient transformation of the BL-GFA, considering the notion of maximal admissible partition. Furthermore, we show that the number of input symbols and time complexity of the minimal quotient transformation of a BL-GFA are less than the minimal quotient BL-GFA.

Keywords : Homomorphism; Strong homomorphism; BL-general fuzzy automata; Quotient BL-general fuzzy automata; Transformation of BL-general fuzzy automata.

1 Introduction

Zadeh in 1965 [19] introduced the notion of fuzzy set as a method for representing uncertainty. Fuzzy set theory has become more and more mature in many fields such as fuzzy relation, fuzzy logic, fuzzy decision-making, fuzzy classification, fuzzy pattern recognition, fuzzy control, fuzzy optimization and fuzzy automata. The theory of fuzzy automata was introduced by Wee [17] in 1967 and Santos in 1968 [13]. E.T. Lee and L.A. Zadeh in 1969 [8] gave the concept of fuzzy finite state automata. Fuzzy finite automata have many important applications in the learning system, pattern recognition, neural networks, database theory and fuzzy discrete event systems [3, 5, 6, 9, 10, 11, 12, 18, 14]. M. Doostfatemeh and S.C. Kremer in 2005 [4] extended the notion of fuzzy automata and gave the notion of general fuzzy automata. Basic logic (BL) has been introduced by Hajek [7] in order to provide a general framework for formalizing statements of fuzzy nature. In 2012, Kh. Abolpour and M. M. Zahedi [2] extended the notion of general fuzzy automata and gave the notion of BL-general fuzzy automata.

In this paper, we define the concepts of homomorphism and strong homomorphism for a BL-general fuzzy automaton. A connection between strong homomorphism and admissible partition is presented. We present a quotient of the BL-GFA using the notion of strong homomorphism. Also, we show that this quotient BL-GFA and quotient BL-GFA defined in Definition 3.8 [15] have the same behavior. Then, we obtain the minimal quotient BL-general fuzzy automaton and minimal quotient transformation of BL-general fuzzy automaton considering the notions of maximal admissible partition. In addition, the authors show that the number of input symbols of the minimal

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quotient transformation of BL-GFA is not more than the minimal quotient BL-GFA. Therefore, the number of transitions and calculation of the minimal quotient transformation of a BL-GFA is not more than the minimal quotient BL-GFA.

2 Preliminaries

In this section, we give some definitions that is used in the rest of the paper.

Definition 2.1 *[7] A BL-algebra is an algebra* $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ *with four binary operations* $\wedge, \vee, *, \rightarrow$ *and two constants* $0, 1$ *such that: (i)* $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice, (ii)* $(L, *, 1)$ *is a commutative monoid, (iii)* $*$ *and* $\rightarrow$ *form an adjoint pair, i.e.,* $x \leq y \rightarrow z$ *if and only if* $x*y \leq z$ *for all* $x, y, z \in L$*, (iv)* $x \wedge y = x*(x \rightarrow y)$*, (v)* $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Definition 2.2 *[16] Let* $L = (L, \vee, \wedge, 0, 1)$ *be a bounded complete lattice. A BL-general fuzzy automaton (BL-GFA) as a ten-tuple machine is denoted by* $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$*, where*

(i) $\bar{Q} = P(Q)$, where Q is a finite set and $\bar{Q}$ is the power set of Q,

(ii) X is a finite set of input symbols,

(iii) $\tilde{R}$ is the set of fuzzy start states,

(iv) $\bar{Z}$ is a finite set of output symbols, where $\bar{Z}$ is the power set of Z,

(v) $\omega_l : \bar{Q} \rightarrow \bar{Z}$ is the output function defined by: $\omega_l(Q_i) = \{\omega(q) | q \in Q_i\}$,

(vi) $\delta_l : \bar{Q} \times X \times \bar{Q} \rightarrow L$ is the transition function defined by: $\delta_l(\{p\}, a, \{q\}) = \delta(p, a, q)$ and $\delta_l(Q_i, a, Q_j) = \vee_{q_i \in Q_i, q_j \in Q_j} \delta(q_i, a, q_j)$, for all $Q_i, Q_j \in P(Q)$ and $a \in X$,

(vii) $f_l : \bar{Q} \times X \rightarrow \bar{Q}$ is the next state map defined by: $f_l(Q_i, a) = \cup_{q_i \in Q_i} \{q_j | \delta(q_i, a, q_j) \in \Delta\}$,

(viii) $\tilde{\delta}_l : (\bar{Q} \times L) \times X \times \bar{Q} \rightarrow L$ is the augmented transition function defined $\tilde{\delta}_l((Q_i, \mu^t(Q_i)), a, Q_j) = F_1(\mu^t(Q_i), \delta_l(Q_i, a, Q_j))$,

(ix) $F_1 : L \times L \rightarrow L$ is called membership assignment function,

(x) $F_2 : L^* \rightarrow L$ is called multi-membership resolution function.

Suppose that the set of all transitions of $\tilde{F}$ be Δ and $Q_{act}(t_i)$ be the set of all active states at time t_i, for all $i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) | \exists q' \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta\}$, for all $i \geq 1$. Since $Q_{act}(t_i)$ is a fuzzy set, we write $q \in Domain(Q_{act}(t_i))$ to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$. Hereafter, we denote these notations by

$$q \in Q_{act}(t_i) \quad \text{and} \quad T \subseteq Q_{act}(t_i).$$

In the rest of this paper, L is a bounded complete lattice.

Definition 2.3 *[2] Let* $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ *be a BL-GFA. The run map of the BL-GFA* $\tilde{F}_l$ *is the map* $\rho : X^* \rightarrow \bar{Q}$ *defined by the following induction:*

$\rho(\Lambda) = \{q_0\}$ and $\rho(a_1a_2...a_n) = Q_{i_n}, \rho(a_1a_2...a_na_{n+1}) = f_l(Q_{i_n}, a_{n+1})$, where $(Q_{i_n}, \mu^{t_0+n}(Q_{i_n})) \in Q_{act}(a_1a_2...a_n)$ for every $a_1, ..., a_n \in X$.

Definition 2.4 *[15] Let* $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ *be a BL-GFA. The behavior of* $\tilde{F}_l$ *is the map*

$\beta = \omega_l \circ \rho : \mathcal{L}(\tilde{F}_l) \rightarrow \bar{Z}$, where

$$\mathcal{L}(\tilde{F}_l) = \{x \in X^* | \tilde{\delta}_l^*((\{q_0\}, \mu^{t_0}(\{q_0\})), x, P) > 0, \text{for some } P \in \bar{Q}\}.$$

Definition 2.5 *[15] Let* $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ *be a BL-GFA and* $\sim$ *be an equivalence relation on* $\bar{Q}$*. Then* $\sim$ *is an admissible relation on* $\bar{Q}$ *if and only if the followings hold:*

(i) If $Q', Q'' \in Q_{act}(t_i), x \in X^*, P' \in \bar{Q}, Q' \sim Q''$ and $\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, P') > 0$, then there exists $P'' \in \bar{Q}$ such that $\tilde{\delta}_l^*((Q'', \mu^{t_i}(Q'')), x, P'') \geq \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, P')$ and $P' \sim P''$.

(ii) If $Q' \sim Q''$, then $\omega_l(Q') = \omega_l(Q'')$.

Definition 2.6 *[15] Let* $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ *be a BL-GFA and* $H = \{Q_1, ..., Q_k\}$ *be a partition of* $\bar{Q}$. *Then* H *is called an admissible partition of* $\bar{Q}$ *if and only if the followings hold:*

(i) If $x \in X^*$, then for every l_1 there exists l_2, where $1 \leq l_1, l_2 \leq k$. For every $P_1, P_2 \in Q_{l_1}$ if $\tilde{\delta}_l^*((P_1, \mu^{t_i}(P_1)), x, R_1) > 0$ for some $R_1 \in \bar{Q}$, then there is $R_2 \in \bar{Q}$ such that $\tilde{\delta}_l^*((P_2, \mu^{t_i}(P_2)), x, R_2) \geq \tilde{\delta}_l^*((P_1, \mu^{t_i}(P_1)), x, R_1)$ and $R_1, R_2 \in Q_{l_2}$.

(ii) If $Q', Q'' \in Q_l$, where $1 \leq l \leq k$, then $\omega_l(Q') = \omega_l(Q'')$.

Definition 2.7 *[15] Let* $\tilde{F}_l$ *be a BL-GFA and* $\pi = \{H_l | l \in I\}$ *be an admissible partition of* $\bar{Q}$. *Let* π_1 *be a nontrivial partition. If for every admissible partition* π_2 *of* $\bar{Q}$ *where* $\pi_1 \leq \pi_2 \leq \{\bar{Q}\}$, *we have either* $\pi_2 = \pi_1$ *or* $\pi_2 = \{\bar{Q}\}$, *then* π_1 *is maximal.*

Definition 2.8 *[15] Let* $\tilde{F}_l$ *be a BL-GFA. Then* $\tilde{F}^*$ *is called minimal, if* $|\bar{Q}| > 1$ *and* 1_Q *and* $\{\bar{Q}\}$ *are the only admissible partitions of* $\bar{Q}$.

Theorem 2.1 *[15] Let* $\tilde{F}_l$ *be a BL-GFA and* $\pi = \{H_l | l \in I\}$ *be an admissible partition of* $\bar{Q}$. *Then* π *is maximal if and only if* $\frac{\tilde{F}_l}{\pi}$ *is minimal.*

Theorem 2.2 *[15] Let* $\tilde{F}_l$ *be a BL-GFA and* $\pi = \{H_l | l \in I\}$ *be an admissible partition of* $\bar{Q}$. *Then* $\beta_{\tilde{F}_l} = \beta_{\frac{\tilde{F}_l}{\pi}}$.

3 Quotient structures for BL-general fuzzy automata

This section attempts to introduce the concepts of homomorphism and strong homomorphism between BL-general fuzzy automata. Also, we present a quotient BL-general fuzzy automaton using the notion strong homomorphism. Finally, we obtain a minimal quotient BL-GFA.

Definition 3.1 *Let* $\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2$ *be two BL-GFAs. A pair* (ξ, φ) *of mappings* $\xi : \bar{Q}_1 \to \bar{Q}_2$ *and* $\varphi : X_1 \to X_2$ *is called a homomorphism, written as* $(\xi, \varphi) : \tilde{F}_{l1} \to \tilde{F}_{l2}$, *if*

$$\tilde{\delta}_{l1}((Q', \mu^t(Q')), a, Q'') \leq \tilde{\delta}_{l2}((\xi(Q'), \mu^t(\xi(Q'))), \varphi(a), \xi(Q'')),$$

and $\tilde{\omega}_{l1}(Q') \subseteq \tilde{\omega}_{l2}(\xi(Q'))$ for every $Q', Q'' \in \bar{Q}_{l1}$ and $a \in X_1 \cup \Lambda$.

The pair (ξ, φ) is called a strong homomorphism if

$$\tilde{\delta}_{l2}((\xi(Q'), \mu^t(\xi(Q'))), \varphi(a), \xi(Q'')) = \vee\{\tilde{\delta}_{l1}((Q', \mu^t(Q')), a, R) | \xi(R) = \xi(Q'')\},$$

and $\tilde{\omega}_{l1}(Q') = \tilde{\omega}_{l2}(\xi(Q'))$ for every $Q', Q'' \in \bar{Q}_{l1}$ and $a \in X_1 \cup \{\Lambda\}$.

A homomorphism (strong homomorphism) $(\xi, \varphi) : \tilde{F}_{l1} \to \tilde{F}_{l2}$ is called an isomorphism (strong isomorphism), if ξ and φ are both one-one and onto.

Theorem 3.1 *Let* $\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2$ *be two BL-GFAs. Let* $(\xi, \varphi) : \tilde{F}_{l1} \to \tilde{F}_{l2}$ *be a strong homomorphism. If* $\tilde{\delta}_{l2}((\xi(Q'), \mu^t(\xi(Q'))), \varphi(a), \xi(R)) > 0$, *then there exists* $R' \in \bar{Q}_{l1}$ *such that* $\tilde{\delta}_{l1}((Q', \mu^t(Q')), a, R') > 0$ *and* $\xi(R') = \xi(R)$ *for every* $Q', R \in \bar{Q}_{l1}$ *and* $a \in X_1 \cup \{\Lambda\}$. *Also, if* $\xi(Q') = \xi(Q'')$ *and* $\tilde{\delta}_{l2}((\xi(Q'), \mu^t(\xi(Q'))), \varphi(a), \xi(R)) > 0$, *then* $\tilde{\delta}_{l1}((Q', \mu^t(Q')), a, R') \geq \tilde{\delta}_{l1}((Q'', \mu^t(Q'')), a, R)$, *for some* $R' \in \bar{Q}_{l1}$.

Proof. By Definition 3.1, we have

$$\tilde{\delta}_{l2}((\xi(Q'), \mu^t(\xi(Q'))), \varphi(a), \xi(R)) = \bigvee\{\tilde{\delta}_{l1}((Q', \mu^t(Q')), a, R') | \xi(R) = \xi(R')\} > 0.$$

Therefore, there exists $R' \in \bar{Q}_{l1}$ such that $\tilde{\delta}_{l1}((Q', \mu^t(Q')), a, R') > 0$ and $\xi(R) = \xi(R')$. Now, let $\tilde{\delta}_{l2}((\xi(Q'), \mu^t(\xi(Q'))), a, \xi(R)) > 0$. Then there exists $R' \in \bar{Q}_{l1}$ such that $\xi(R) = \xi(R')$ and

$$\begin{aligned} \tilde{\delta}_{l1}((Q', \mu^t(Q')), a, R') &= \tilde{\delta}_{l2}((\xi(Q'), \mu^t(\xi(Q'))), a, \xi(R)) \\ &= \tilde{\delta}_{l2}((\xi(Q''), \mu^t(\xi(Q''))), a, \xi(R)) \\ &\geq \tilde{\delta}_{l1}((Q'', \mu^t(Q'')), a, R). \end{aligned}$$

Hence, the claim holds.

Definition 3.2 *Let* $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ *be a BL-GFA and* $\sim$ *be an admissible relation on* $\bar{Q}$. *We define* $[Q'] = \{P | P \sim Q'\}$ *for every* $Q' \in \bar{Q}$. *Now, consider the following notations:*

(i) $\frac{\bar{Q}}{\sim} = \{[Q'] | Q' \in \bar{Q}\}$ is a finite set of states,

(ii) X is a finite set of input symbols,

(iii) $\frac{\tilde{R}}{\sim} = [\{q_0\}]$ is the set of fuzzy start states,

(iv) $\bar{Z}$ is a finite set of output symbols, where $\bar{Z}$ is the power set of Z,

(v) $\frac{\omega_l}{\sim} : \frac{\bar{Q}}{\sim} \to \bar{Z}$ is the output function defined by: $\frac{\omega_l}{\sim}([Q_i]) = \omega_l(Q_i)$,

(vi) $\frac{\delta_l}{\sim} : \frac{\bar{Q}}{\sim} \times X \times \frac{\bar{Q}}{\sim} \to L$ is the transition function defined by: $\frac{\delta_l}{\sim}([Q'], a, [Q'']) = \vee\{\delta_l(Q', a, R') | R' \sim Q''\}$ for every $Q', Q'', R' \in \bar{Q}, a \in X$,

(vii) $f_l : \frac{\bar{Q}}{\sim} \times X \to P(\frac{\bar{Q}}{\sim})$ is the next state map defined by: $f_l([Q_i], a) = \cup_{R' \sim Q_i}\{R | \delta(R', a, R) \in \Delta\}$,

(viii) $\frac{\tilde{\delta}_l}{\sim} : (\frac{\bar{Q}}{\sim} \times L) \times X \times \frac{\bar{Q}}{\sim} \to L$ is the augmented transition function defined $\frac{\tilde{\delta}_l}{\sim}(([Q'], \mu^t([Q']), a, [Q'']) = \bigvee\{\tilde{\delta}_l((Q', \mu^t(Q')), a, R') | R' \sim Q''\}$,

(ix) $F_1 : L \times L \to L$ is the membership assignment function,

(x) $F_2 : L^* \to L$ is the multi-membership resolution function.

Now, we show that $\frac{\tilde{\delta}_l}{\sim}$ is well-defined. Let $[Q'] = [P'], a = b$ and $[Q''] = [P'']$, where $P', Q', Q'', P'' \in \bar{Q}$ and $a, b \in X$. Then $P' \sim Q'$ and $P'' \sim Q''$. So, $\frac{\tilde{\delta}_l}{\sim}(([Q'], \mu^t([Q'])), a, [Q'']) = \bigvee\{\tilde{\delta}_l((Q', \mu^t(Q')), a, R) | R \sim Q''\}$ and

$$\begin{aligned}
&\frac{\tilde{\delta}_l}{\sim}(([P'], \mu^t([P'])), b, [P'']) \\
&\quad = \frac{\tilde{\delta}_l}{\sim}(([P'], \mu^t([P'])), a, [P'']) \\
&\quad = \bigvee\{\tilde{\delta}_l((P', \mu^t(P')), a, R') | R' \sim P''\}.
\end{aligned}$$

Let $R \sim Q''$ such that $\tilde{\delta}_l((Q', \mu^t(Q')), a, R) > 0$. Then there is $R' \in \bar{Q}$ such that $\tilde{\delta}_l((P', \mu^t(P')), a, R') \geq \tilde{\delta}_l((Q', \mu^t(Q')), a, R)$ and $R \sim R'$. Also, if $\tilde{\delta}_l((P', \mu^t(P')), a, R') > 0$, where $P', R' \in \bar{Q}, a \in X$ and $R' \sim P''$, then there exists $R \in \bar{Q}$ such that $\tilde{\delta}_l((Q', \mu^t(Q')), a, R) \geq \tilde{\delta}_l((P', \mu^t(P')), a, R')$ and $R \sim R'$. Therefore,

$$\begin{aligned}
&\frac{\tilde{\delta}_l}{\sim}(([Q'], \mu^t([Q'])), a, [Q'']) \\
&\quad = \frac{\tilde{\delta}_l}{\sim}(([P'], \mu^t([P'])), a, [P'']).
\end{aligned}$$

Hence, $\frac{\tilde{\delta}_l}{\sim}$ is well-defined.

Clearly, $\frac{\omega_l}{\sim}$ is well-defined. Then $\frac{\tilde{F}_l}{\sim} = (\frac{\bar{Q}}{\sim}, X, \frac{\tilde{R}}{\sim} = ([\{q_0\}], \mu^{t_0}([\{q_0\}]) = \mu^{t_0}(\{q_0\})), \bar{Z}, \frac{\omega_l}{\sim}, \frac{\delta_l}{\sim}, \frac{f_l}{\sim}, \frac{\tilde{\delta}_l}{\sim}, F_1, F_2)$ is a BL-GFA.

Now, define $\xi : \bar{Q} \to \frac{\bar{Q}}{\sim}$ by $\xi(Q') = [Q']$ for every $Q' \in \bar{Q}$. It is clear that ξ is onto. Let $\varphi : X \to X$ be the identity map, $Q', Q'' \in \bar{Q}$ and $a \in X$. Then

$$\begin{aligned}
&\frac{\tilde{\delta}_l}{\sim}((\xi(Q'), \mu^t(\xi(Q'))), \varphi(a), \xi(Q'')) \\
&\quad = \frac{\tilde{\delta}_l}{\sim}(([Q'], \mu^t([Q'])), a, [Q'']) \\
&\quad = \vee\{\tilde{\delta}_l((Q', \mu^t(Q')), a, P'') | P'' \sim Q''\} \\
&\quad \geq \tilde{\delta}_l((Q', \mu^t(Q')), a, Q'').
\end{aligned}$$

Also, we have $\frac{\tilde{\omega}_l}{\sim}(\xi(Q')) = \omega_l(Q')$. Hence, (ξ, φ) is a homomorphism.

Example 3.1 *Let* $(L, \wedge, \vee, 0, 1)$ *be the given complete lattice in Figure 1.*

Let general fuzzy automaton $\tilde{F} = (Q, X, \tilde{\delta}, \tilde{R}, Z, \omega, F_1, F_2)$ as: $Q = \{q_0, q_1\}$

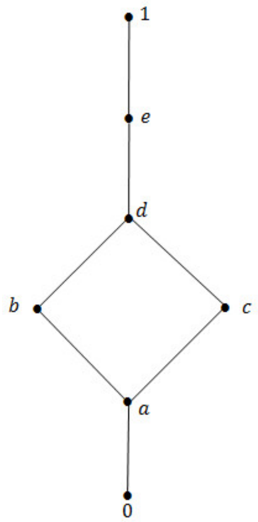

Figure 1: The complete lattice L of Example 3.1

$\tilde{R} = \{(q_0, 1)\}, X = \{\sigma_1, \sigma_1\}, Z = \{z\}, \omega(q_0) = \omega(q_1) = z$ and

$$\delta(q_0, \sigma_1, q_0) = a, \qquad \delta(q_0, \sigma_1, q_1) = b,$$
$$\delta(q_1, \sigma_1, q_0) = d, \qquad \delta(q_1, \sigma_1, q_1) = e,$$
$$\delta(q_1, \sigma_2, q_0) = d, \qquad \delta(q_1, \sigma_2, q_1) = e.$$

Then considering Definition 3.2, we have BL-general fuzzy automaton $\tilde{F}_l$ as follow:

$\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, where $\bar{Q} = \{\emptyset, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}$, $\bar{Z} = \{\emptyset, \{z\}\}$, $\omega_l(\{q_0\}) = \omega_l(\{q_1\}) = \omega_l(\{q_0, q_1\}) = \{z\}$ and

$$\delta_l(\{q_0\}, \sigma_1, \{q_0\}) = a,$$
$$\delta_l(\{q_0\}, \sigma_1, \{q_1\}) = b,$$
$$\delta_l(\{q_0\}, \sigma_1, \{q_0, q_1\}) = b,$$
$$\delta_l(\{q_1\}, \sigma_1, \{q_0\}) = d,$$
$$\delta_l(\{q_1\}, \sigma_1, \{q_1\}) = e,$$
$$\delta_l(\{q_1\}, \sigma_1, \{q_0, q_1\}) = e,$$
$$\delta_l(\{q_0, q_1\}, \sigma_1, \{q_0\}) = d,$$
$$\delta_l(\{q_0, q_1\}, \sigma_1, \{q_1\}) = e,$$
$$\delta_l(\{q_0, q_1\}, \sigma_1, \{q_0, q_1\}) = e,$$
$$\delta_l(\{q_1\}, \sigma_2, \{q_0\}) = d,$$
$$\delta_l(\{q_1\}, \sigma_2, \{q_1\}) = e,$$
$$\delta_l(\{q_1\}, \sigma_2, \{q_0, q_1\}) = e,$$
$$\delta_l(\{q_0, q_1\}, \sigma_2, \{q_0\}) = d,$$
$$\delta_l(\{q_0, q_1\}, \sigma_2, \{q_1\}) = e,$$
$$\delta_l(\{q_0, q_1\}, \sigma_2, \{q_0, q_1\}) = e.$$

Consider admissible relation $\sim$ as: $\{q_1\} \sim \{q_0, q_1\}$. Then we have $\frac{\tilde{F}_l}{\sim} = (\frac{\bar{Q}}{\sim}, X, \frac{\bar{R}}{\sim} = ([\{q_0\}], \mu^{t_0}([\{q_0\}]) = \mu^{t_0}(\{q_0\})), \bar{Z}, \frac{\omega_l}{\sim}, \frac{\delta_l}{\sim}, \frac{f_l}{\sim}, \frac{\tilde{\delta}_l}{\sim}, F_1, F_2)$, where $\frac{\bar{Q}}{\sim} = \{[\{q_0\}], [\{q_1\}]\}$, $\frac{\omega_l}{\sim}[\{q_0\}] = \frac{\omega_l}{\sim}[\{q_1\}] = \{z\}$ and

$$\frac{\delta_l}{\sim}([\{q_0\}], \sigma_1, [\{q_0\}]) = a,$$
$$\frac{\delta_l}{\sim}([\{q_0\}], \sigma_1, [\{q_1\}]) = b,$$
$$\frac{\delta_l}{\sim}([\{q_1\}], \sigma_1, [\{q_0\}]) = d,$$
$$\frac{\delta_l}{\sim}([\{q_1\}], \sigma_1, [\{q_1\}]) = e,$$
$$\frac{\delta_l}{\sim}([\{q_1\}], \sigma_2, [\{q_0\}]) = d,$$
$$\frac{\delta_l}{\sim}([\{q_1\}], \sigma_2, [\{q_1\}]) = e.$$

Now, let $\xi : \bar{Q} \to \frac{\bar{Q}}{\sim}$, where $\xi(\{q_0\}) = [\{q_0\}], \xi(\{q_1\}) = \xi(\{q_0, q_1\}) = [\{q_1\}]$ and $\varphi : X \to X$ be the identity map. It is clear that (ξ, φ) is an onto strong homomorphism.

Definition 3.3 *Let $\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2$ be two BL-GFAs. Let $\xi : \tilde{F}_{l1} \to \tilde{F}_{l2}$ be a strong homomorphism. Then the kernel of ξ, denoted by $Ker\xi$, is defined to be the set $Ker\xi = \{(Q', Q'') | \xi(Q') = \xi(Q'')\}$, where $Q', Q'' \in \bar{Q}_{l1}$.*

Theorem 3.2 *$Ker\xi$ is an admissible relation.*

Proof. It is clear that $Ker\xi$ is an equivalence relation. Let $Q', Q'' \in \bar{Q}_{l1act}(t_i), P' \in \bar{Q}_{l1}, (Q', Q'') \in Ker\xi$ and $\tilde{\delta}_{l1}((Q', \mu^{t_i}(Q')), a, P') > 0$. Then $\xi(Q') = \xi(Q'')$ and

$$\begin{aligned}&\tilde{\delta}_{l2}((\xi(Q''), \mu^{t_i}(\xi(Q''))), a, \xi(P')) \\ &\quad = \tilde{\delta}_{l2}((\xi(Q'), \mu^{t_i}(\xi(Q'))), a, \xi(P')) \\ &\quad \geq \tilde{\delta}_{l1}((Q', \mu^{t_i}(Q')), a, P') > 0.\end{aligned}$$

According Theorem 3.1, there exists $P'' \in \bar{Q}$ such that $\tilde{\delta}_{l1}((Q'', \mu^{t_i}(Q'')), a, P'') \geq \tilde{\delta}_{l1}((Q', \mu^{t_i}(Q')), a, P')$, where $\xi(P'') = \xi(P'), a \in$

$X \cup \{\Lambda\}$. Now, let $(Q', Q'') \in Ker\xi$. Then $\xi(Q') = \xi(Q'')$. Since ξ is a strong homomorphism, $\omega_{l1}(Q') = \omega_{l2}(\xi(Q')) = \omega_{l2}(\xi(Q'')) = \omega_{l1}(Q'')$. Hence, ξ is an admissible relation.

Theorem 3.3 *Let* $\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1,2$ *be two BL-GFAs and* $\xi' : \tilde{F}_{l1} \to \tilde{F}_{l2}$ *be an onto strong homomorphism. Then there exists a strong isomorphism*

$$\gamma : \frac{\tilde{F}_{l1}}{Ker\xi'} \to \tilde{F}_{l2},$$

such that $\xi' = \gamma \circ \xi$.

Proof. Define $\gamma : \frac{\bar{Q}_{l1}}{Ker\xi'} \to \bar{Q}_{l2}$ by $\gamma([Q']) = \xi'(Q')$, for some $Q' \in \bar{Q}_{l1}$. First, we show that γ is well defined. Let $[Q'], [Q''] \in \frac{\bar{Q}_{l1}}{Ker\xi'}$ and $[Q'] = [Q'']$. Then $(Q', Q'') \in Ker\xi'$. Thus, $\xi'(Q') = \xi'(Q'')$. Hence, the claim holds. Now, let $Q', Q'' \in \bar{Q}_{l1}, a \in X$ and

$$\begin{aligned}&\tilde{\delta}_{l2}((\gamma([Q']), \mu^t(\gamma([Q']))), a, \gamma([Q'']))\\ &\quad= \tilde{\delta}_{l2}((\xi'(Q'), \mu^t(\xi'(Q')), a, \xi'(Q''))\\ &\quad= \bigvee\{\tilde{\delta}_{l1}((Q', \mu^t(Q')), a, R') | \xi'(Q'') = \xi'(R')\}\\ &\quad= \frac{\tilde{\delta}_{l1}}{\xi'}((([Q'], \mu^t([Q']), a, [R']).\end{aligned}$$

Also, we have $\frac{\omega_{l1}}{Ker\xi'}([Q']) = \omega_{l1}(Q')$, where $Q' \in \bar{Q}_{l1}$. So, γ is a strong homomorphism. Clearly, γ is one-one and onto. Therefore, γ is a strong isomorphism.

Theorem 3.4 *Let* $\tilde{F}_{li}, i = 1,2$ *be two BL-GFAs and* $\xi' : \tilde{F}_{l1} \to \tilde{F}_{l2}$ *be an onto strong homomorphism. Then* $\beta_{\tilde{F}_{l1}} = \beta_{\tilde{F}_{l2}}$.

Proof. First, we show that $\mathcal{L}(\tilde{F}_{l1}) = \mathcal{L}(\tilde{F}_{l2})$. Let $x \in \mathcal{L}(\tilde{F}_{l1})$. Then, there exists $Q' \in \bar{Q}_1$ such that $\tilde{\delta}_{l1}((\{q_{01}\}, \mu^t(\{q_{01}\})), x, Q') > 0$. Since $\xi' : \tilde{F}_{l1} \to \tilde{F}_{l2}$ is a strong homomorphism, then $x \in \mathcal{L}(\tilde{F}_{l2})$. It is obvious that $\mathcal{L}(\tilde{F}_{l2}) \subseteq \mathcal{L}(\tilde{F}_{l1})$. Now, let ρ_1 and ρ_2 be the run relations of $\tilde{F}_{l1}$ and $\tilde{F}_{l2}$, respectively. Then we have $\beta_{\tilde{F}_{l1}} = \omega_{l1}(\rho_1(x)) = \omega_{l2}(\xi'(\rho_1(x))) \subseteq \omega_{l2}(\rho_2(x)) = \beta_{\tilde{F}_{l2}}$. Similarly, $\beta_{\tilde{F}_{l2}} = \omega_{l2}(\rho_2(x)) = \omega_{l2}(\xi'(Q')) = \omega_{l1}(Q') \subseteq \omega_{l1}(\rho_1(x)) = \beta_{\tilde{F}_{l1}}$. Hence, $\beta_{\tilde{F}_{l1}} = \beta_{\tilde{F}_{l2}}$.

Example 3.2 *Let* $\tilde{F}_l, \frac{\tilde{F}_l}{\sim}$ *be the BL-GFAs as in Example 3.1. We showed that* $\xi : \tilde{F}_l \to \frac{\tilde{F}_l}{\sim}$ *is an onto strong homomorphism. Then by Theorem 3.4,* $\beta_{\tilde{F}_l} = \beta_{\frac{\tilde{F}_l}{\sim}}$.

Corollary 3.1 *Let*

$$\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})),$$

$$\bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li},$$

$F_1, F_2), i = 1,2$ *be two BL-GFAs and* $\xi' : \tilde{F}_{l1} \to \tilde{F}_{l2}$ *be an onto strong homomorphism. Let* $\frac{\tilde{F}_{l1}}{Ker\xi'}$ *be as defined in Definition 3.2. Then* $\beta_{\frac{\tilde{F}_{l1}}{Ker\xi'}} = \beta_{\tilde{F}_{l2}}$.

Proof. Considering Theorems 3.3 and 3.4, the proof is clear.

Corollary 3.2 *Let* $\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1,2$ *be two BL-GFAs and* $\xi : \tilde{F}_{l1} \to \tilde{F}_{l2}$ *be a strong homomorphism. Then the set of all classes of* $Ker\xi$ *is an admissible partition of* $\bar{Q}_{l1}$.

Theorem 3.5 *Let* $\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1,2$ *be two BL-GFAs,* $\pi = \{H_l | l \in I\}$ *be a maximal admissible partition of* $\bar{Q}$ *and* $\xi' : \tilde{F}_{l1} \to \tilde{F}_{l2}$ *be an onto strong homomorphism. Let* $\frac{\tilde{F}_{l1}}{Ker\xi'}$ *be as defined in Theorem 3.3, and* $\frac{\tilde{F}_{l1}}{\pi}$ *be as defined in Definition 3.8 [15]. Then* $\beta_{\frac{\tilde{F}_{l1}}{\pi}} = \beta_{\frac{\tilde{F}_{l1}}{Ker\xi'}}$.

Proof. The proof is clear considering the proof of Theorem 3.4, Corollary 3.1, and Theorem 3.14. [15].

Theorem 3.6 *Let* $\tilde{F}_{li} = (\bar{Q}_{li}, X, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1,2$ *be two BL-GFAs and* $\xi' : \bar{Q}_{l1} \to \bar{Q}_{l2}$ *be a strong homomorphism. Then* $Ker\xi'$ *is a maximal admissible partition if and only if* $\frac{\tilde{F}_{l1}}{Ker\xi'}$ *is minimal.*

Proof. The proof is obvious considering Theorem 3.12 of [15] and Theorem 3.2.

Example 3.3 *Let $\tilde{F}_l, \dfrac{\tilde{F}_l}{\sim}$ be the BL-GFAs as in Example 3.1. We showed that $\xi : \tilde{F}_l \to \dfrac{\tilde{F}_l}{\sim}$ is an onto strong homomorphism.*

There exists a strong isomorphism $\gamma : \dfrac{\tilde{F}_l}{Ker\xi} \to \dfrac{\tilde{F}_l}{\sim}$, using Definition 3.3, and Theorem 3.3. Obviously, $Ker\xi$ is a maximal admissible partition. Therefore, by Theorem 3.6, $\dfrac{\tilde{F}_l}{Ker\xi}$ is minimal. Hence, $\dfrac{\tilde{F}_l}{\sim}$ is minimal.

4 Transformation for BL-general fuzzy automata

In this section, we define an equivalence relation on X^*. Using this equivalence relation, we present a transformation of BL-GFA. Also, we obtain a minimal quotient transformation of BL-GFA. Finally, we arrive in Corollary 4.2, that is one of the main results of this paper.

Definition 4.1 *Let $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GFA and $\equiv$ be a relation on X^*. Let $x, y \in X^*$. Then $x \equiv y$ if and only if $\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, Q'') = \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), y, Q'')$ for every $Q', Q'' \in \bar{Q}$.*

Theorem 4.1 *Let $\tilde{F}_l$ be a BL-GFA. Then $\equiv$ is a congruence relation on X^*.*

Proof. It is clear that $\equiv$ is an equivalence relation on X^*. Let $z \in X^*$ and $x \equiv y$. Then

$$\begin{aligned}
&\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), xz, Q'') \\
&\quad = \vee_{P\in\bar{Q}} \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, P) \\
&\quad \wedge \tilde{\delta}_l^*((P, \mu^{t_i}(P)), z, Q'') \\
&\quad = \vee_{P\in\bar{Q}} \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), y, P) \\
&\quad \wedge \tilde{\delta}_l^*((P, \mu^{t_i}(P)), z, Q'') \\
&\quad = \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), yz, Q'').
\end{aligned}$$

So, $xz \equiv yz$. Similarly, $zx \equiv zy$. Hence, $\equiv$ is a congruence relation on X^*. Let $x \in X^*$. Then we denote $[x] = \{y \in X^* | x \equiv y\}$ and $E(\tilde{F}_l) = \{[x] | x \in X^*\}$.

Definition 4.2 *Let $\tilde{F}_l$ be a BL-GFA. Define a binary operation $*$ on $E(\tilde{F}_l)$ by $[x] * [y] = [xy]$ for every $[x], [y] \in E(\tilde{F}_l)$.*

Theorem 4.2 *Let $\tilde{F}_l$ be a BL-GFA. Then $(E(\tilde{F}_l), *)$ is a finite monoid.*

Proof. First, we show that $*$ is well-defined and associative. Let $[x] = [u]$ and $[y] = [v]$, where $[x], [y], [u], [v] \in E(\tilde{F}_l)$. Then $[xy] = [x] * [y] = [u] * [v] = [uv]$. Also, $[x] * ([y] * [z]) = [x] * [yz] = [xyz] = [xy] * [z] = ([x] * [y]) * [z]$ for every $[x], [y], [z] \in E(\tilde{F}_l)$. Therefore, $(E(\tilde{F}_l), *)$ is well-defined and associative. Now, we have $[x] * [\Lambda] = [x\Lambda] = [x] = [\Lambda x] = [\Lambda] * [x]$ for every $[x] \in E(\tilde{F}_l)$. Since $Im(\delta_l)$ is finite, $Im(\delta_l^*)$ is finite. Hence, $(E(\tilde{F}_l), *)$ is a finite monoid.

Definition 4.3 *Let $\tilde{F}_l$ be a BL-GFA and $u, v \in X$. Then $\tilde{F}_l$ is called faithful if $\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), u, Q'') = \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), v, Q'')$ for every $Q', Q'' \in \bar{Q}$, implies that $u = v$.*

Example 4.1 *Let $\tilde{F}_l$ be the BL-general fuzzy automaton as in Example 3.1. Considering Definition 4.3, $\tilde{F}_l$ is a faithful BL-GFA.*

Theorem 4.3 *Let $\tilde{F}_l$ be a BL-GFA. Then $\tilde{F}_{lE(\tilde{F}_l)} = (\bar{Q}, E(\tilde{F}_l), (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_{lE}, f_{lE}, \tilde{\delta}_{lE}, F_1, F_2)$ is a faithful BL-GFA, where $\tilde{\delta}_{lE}^*((Q', \mu^{t_i}(Q')), [x], Q'') = \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, Q'')$, where $Q', Q'' \in \bar{Q}, x \in X$.*

Proof. Clearly, $\tilde{\delta}_{lE}$ is well-defined. Let $\tilde{\delta}_{lE}^*((Q', \mu^{t_i}(Q')), [x], Q'') = \tilde{\delta}_{lE}^*((Q', \mu^{t_i}(Q')), [y], Q'')$. Then $\tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), x, Q'') = \tilde{\delta}_l^*((Q', \mu^{t_i}(Q')), y, Q'')$. Therefore, $x \equiv y$ and so, $[x] = [y]$. Hence, $\tilde{F}_{lE(\tilde{F}_l)}$ is a faithful BL-GFA.

Let $\tilde{F}_l$ be a BL-GFA. Then $\tilde{F}_{lE(\tilde{F}_l)} = (\bar{Q}, E(\tilde{F}_l), (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_{lE}, f_{lE}, \tilde{\delta}_{lE}, F_1, F_2)$ is called the transformation of BL-GFA.

Example 4.2 *Let $(L, \wedge, \vee, 0, 1)$ be the given complete lattice in Figure 1. Consider BL-general fuzzy automaton $\tilde{F}_l = (\bar{Q}, X, (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, where*

$\bar{Q} = \{\emptyset, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}$, $\bar{Z} = \{\emptyset, \{z\}\}$, $\omega_l(\{q_0\}) = \omega_l(\{q_1\}) = \omega_l(\{q_0, q_1\}) = \{z\}$ and

$$\begin{aligned}
&\delta_l(\{q_0\}, \sigma_1, \{q_0\}) = a,\\
&\delta_l(\{q_0\}, \sigma_1, \{q_1\}) = b,\\
&\delta_l(\{q_0\}, \sigma_1, \{q_0, q_1\}) = b,\\
&\delta_l(\{q_1\}, \sigma_1, \{q_0\}) = d,\\
&\delta_l(\{q_1\}, \sigma_1, \{q_1\}) = e,\\
&\delta_l(\{q_1\}, \sigma_1, \{q_0, q_1\}) = e,\\
&\delta_l(\{q_0, q_1\}, \sigma_1, \{q_0\}) = d,\\
&\delta_l(\{q_0, q_1\}, \sigma_1, \{q_1\}) = e,\\
&\delta_l(\{q_0, q_1\}, \sigma_1, \{q_0, q_1\}) = e,\\
&\delta_l(\{q_0\}, \sigma_2, \{q_0\}) = a,\\
&\delta_l(\{q_0\}, \sigma_2, \{q_1\}) = b,\\
&\delta_l(\{q_0\}, \sigma_2, \{q_0, q_1\}) = b,\\
&\delta_l(\{q_1\}, \sigma_2, \{q_0\}) = d,\\
&\delta_l(\{q_1\}, \sigma_2, \{q_1\}) = e,\\
&\delta_l(\{q_1\}, \sigma_2, \{q_0, q_1\}) = e,\\
&\delta_l(\{q_0, q_1\}, \sigma_2, \{q_0\}) = d,\\
&\delta_l(\{q_0, q_1\}, \sigma_2, \{q_1\}) = e,\\
&\delta_l(\{q_0, q_1\}, \sigma_2, \{q_0, q_1\}) = e.
\end{aligned}$$

Then we have the transformation of BL-GFA $\tilde{F}_l$ as: $\tilde{F}_{lE(\tilde{F}_l)} = (\bar{Q}, E(\tilde{F}_l), (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_{lE}, f_{lE}, \tilde{\delta}_{lE}, F_1, F_2)$, where $E(\tilde{F}_l) = \{[\sigma_1]\}, [\sigma_1] = \{\sigma_1, \sigma_2\}$ and

$$\begin{aligned}
&\delta_l(\{q_0\}, [\sigma_1], \{q_0\}) = a,\\
&\delta_l(\{q_0\}, [\sigma_1], \{q_1\}) = b,\\
&\delta_l(\{q_0\}, [\sigma_1], \{q_0, q_1\}) = b,\\
&\delta_l(\{q_1\}, [\sigma_1], \{q_0\}) = d,\\
&\delta_l(\{q_1\}, [\sigma_1], \{q_1\}) = e,\\
&\delta_l(\{q_1\}, [\sigma_1], \{q_0, q_1\}) = e,\\
&\delta_l(\{q_0, q_1\}, [\sigma_1], \{q_0\}) = d,\\
&\delta_l(\{q_0, q_1\}, [\sigma_1], \{q_1\}) = e,\\
&\delta_l(\{q_0, q_1\}, [\sigma_1], \{q_0, q_1\}) = e.
\end{aligned}$$

Theorem 4.4 *Let $\tilde{F}_l$ be a BL-GFA and $\sim$ be an equivalence relation on $\bar{Q}$. Then $\sim$ is an admissible relation for $\tilde{F}_l$ if and only if $\sim$ is an admissible relation for $\tilde{F}_{lE(\tilde{F}_l)} = (\bar{Q}, E(\tilde{F}_l), (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_{lE}, f_{lE}, \tilde{\delta}_{lE}, F_1, F_2)$.*

Proof. Let $\sim$ be an admissible relation on $\tilde{F}_l$ and $Q', Q'' \in \bar{Q}, [x] \in E(\tilde{F}_l), P' \in \bar{Q}, Q' \sim Q''$ and $\tilde{\delta}^*_{lE}((Q', \mu^{t_i}(Q')), [x], P') = \tilde{\delta}^*_l((Q', \mu^{t_i}(Q')), x, P') > 0$. Then there exists $P'' \in \bar{Q}$ such that $\tilde{\delta}^*_l((Q'', \mu^{t_i}(Q'')), x, P'') \geq \tilde{\delta}^*_l((Q', \mu^{t_i}(Q')), x, P')$ and $P' \sim P''$. So, $\tilde{\delta}^*_{lE}((Q'', \mu^{t_i}(Q'')), [x], P'') \geq \tilde{\delta}^*_{lE}((Q', \mu^{t_i}(Q')), [x], P')$. Hence, $\sim$ is an admissible relation for $\tilde{F}_{lE(\tilde{F}_l)}$.

Theorem 4.5 *Let $\tilde{F}_l$ be a BL-GFA and $\tilde{F}_{lE(\tilde{F}_l)}$ be a transformation of the BL-GFA. Then $\beta_{\tilde{F}_{l1}} = \beta_{\tilde{F}_{lE(\tilde{F}_l)}}$.*

Proof. Considering Definition 2.4, and Theorem 4.3, the proof is obvious.

Theorem 4.6 *Let $\tilde{F}_l$ be a faithful BL-GFA. Then $\tilde{F}_{lE(\tilde{F}_l)}$ is isomorphism to $\tilde{F}_l$.*

Proof. Let $f : \bar{Q} \to \bar{Q}$ be an identity map. Define $g : X \to E(\tilde{F}_l)$ by $g(x) = [x]$ for every $Q' \in \bar{Q}$ and $x \in X$. Let $x, y \in X^*$ and $g(x) = g(y)$. Then $[x] = [y]$. Thus, $\tilde{\delta}^*_{lE}((Q', \mu^{t_i}(Q')), [x], Q'') = \tilde{\delta}^*_{lE}((Q', \mu^{t_i}(Q')), [y], Q'')$ for every $Q', Q'' \in \bar{Q}$. So, $\tilde{\delta}^*_l((Q', \mu^{t_i}(Q')), x, Q'') = \tilde{\delta}^*_l((Q', \mu^{t_i}(Q')), y, Q'')$ for every $Q', Q'' \in \bar{Q}$. Since $\tilde{F}_l$ is faithful, then $x = y$. Therefore, g is injective. Clearly, g is surjective. Also, we have

$$\begin{aligned}
&\tilde{\delta}^*_{lE}((f(Q'), \mu^{t_i}(f(Q'))), g(x), f(Q''))\\
&\quad = \tilde{\delta}^*_{lE}((Q', \mu^{t_i}(Q')), [x], Q'')\\
&\quad = \tilde{\delta}^*_l((Q', \mu^{t_i}(Q')), x, Q'').
\end{aligned}$$

Hence, $(f, g) : \tilde{F}_l \to \tilde{F}_{lE}$ is a strong isomorphism.

Theorem 4.7 *Let $\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2$ be two BL-GFAs. Let $(\alpha, \beta) : \tilde{F}_{l1} \to \tilde{F}_{l2}$ be a strong homomorphism with α one-one and onto. Then there exists a strong homomorphism $(f_\alpha, g_\beta) : \tilde{F}_{l1E} \to \tilde{F}_{l2E}$.*

Proof. Define $f_\alpha : \bar{Q}_{l1} \to \bar{Q}_{l2}$ by $f_\alpha(Q'_1) = \alpha(Q'_1)$ for every $Q'_1 \in \bar{Q}_{l1}$ and $g_\beta : E(\tilde{F}_{l1}) \to E(\tilde{F}_{l2})$ by $g_\beta([x]) = [\beta^*(x)]$ for every $[x] \in E(\tilde{F}_{l1})$. Let $[x], [y] \in E(\tilde{F}_{l1})$ and $[x] = [y]$. Then $\tilde{\delta}^*_{l1}((Q', \mu^{t_i}(Q')), x, Q'') = \tilde{\delta}^*_{l1}((Q', \mu^{t_i}(Q')), y, Q'')$

for every $Q', Q'' \in \bar{Q}_{l1}$. So,

$$\begin{aligned}&\tilde{\delta}^*_{l2}((\alpha(Q'), \mu^{t_i}(\alpha(Q'))), \beta^*(x), \alpha(Q''))\\ &\quad = \tilde{\delta}^*_{l1}((Q', \mu^{t_i}(Q')), x, Q'')\\ &\quad = \tilde{\delta}^*_{l1}((Q', \mu^{t_i}(Q')), y, Q'')\\ &\quad = \tilde{\delta}^*_{l2}((\alpha(Q'), \mu^{t_i}(\alpha(Q'))), \beta^*(y), \alpha(Q'')),\end{aligned}$$

for every $Q', Q'' \in \bar{Q}_{l1}$. Since α is onto, then $[\beta^*(x)] = [\beta^*(y)]$. Therefore, g_β is well-defined. Also,

$$\begin{aligned}&\tilde{\delta}^*_{l2E}((f_\alpha(Q'), \mu^{t_i}(f_\alpha(Q'))), g_\beta([x]), f_\alpha(Q''))\\ &\quad = \tilde{\delta}^*_{l2}((\alpha(Q'), \mu^{t_i}(\alpha(Q'))), \beta^*(x), \alpha(Q''))\\ &\quad = \tilde{\delta}^*_{l1}((Q', \mu^{t_i}(Q')), x, Q'')\\ &\quad = \tilde{\delta}^*_{l1E}((Q', \mu^{t_i}(Q')), [x], Q'').\end{aligned}$$

Hence, (f_α, g_β) is a strong homomorphism.

Corollary 4.1 *Let* $\tilde{F}_{li} = (\bar{Q}_{li}, X_i, (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2$ *be two BL-GFAs and* (α, β) *be a strong homomorphism with* α *one-to-one and onto. Then Ker* α *is a maximal admissible partition of* $\bar{Q}_{l1}$ *if and only if* $\frac{\tilde{F}_{lE1}}{Ker\alpha}$ *is a minimal BL-GFA.*

Proof. The proof is clear considering Theorems 3.6 and 4.7.

Corollary 4.2 *Let* $\tilde{F}_{li}, i = 1, 2$ *be two BL-GFAs and* (α, β) *be a strong homomorphism with* α *one-to-one and onto. If Ker* α *is a maximal admissible partition of* $\bar{Q}_{l1}$*, then* $\frac{\tilde{F}_{l1}}{Ker\alpha}$ *and* $\frac{\tilde{F}_{lE1}}{Ker\alpha}$ *are minimal BL-GFA. But the number of input symbols of* $\frac{\tilde{F}_{lE1}}{Ker\alpha}$ *is not more than* $\frac{\tilde{F}_{l1}}{Ker\alpha}$.

Example 4.3 *Consider BL-general fuzzy automaton* $\tilde{F}_l$ *in Example 4.2. By Theorem 4.5, it is obvious that* $\beta_{\tilde{F}_{l1}} = \beta_{\tilde{F}_{lE(\tilde{F}_l)}}$. *Consider the admissible relation* $\sim$ *as* $\{q_1\} \sim \{q_0, q_1\}$. *Clearly,* $\sim$ *is an admissible relation for* $\tilde{F}_l$. *By Theorem 4.4,* $\sim$ *is an admissible relation for* $\tilde{F}_{lE(\tilde{F}_l)}$. *Also, we have*

$\frac{\tilde{F}_{lE(\tilde{F}_l)}}{\sim} = (\frac{\bar{Q}}{\sim}, E(\tilde{F}_l), \frac{\tilde{R}}{\sim} = ([\{q_0\}], \mu^{t_0}([\{q_0\}]) = \mu^{t_0}(\{q_0\})), \bar{Z}, \frac{\omega_l}{\sim}, \frac{\delta_{lE}}{\sim}, \frac{f_l}{\sim}, \frac{\tilde{\delta}_l}{\sim}, F_1, F_2)$, where $\frac{\bar{Q}}{\sim} = \{[\{q_0\}], [\{q_1\}]\}, E(\tilde{F}_l) = [\sigma_1]$, $\frac{\omega_l}{\sim}[\{q_0\}] = \frac{\omega_l}{\sim}[\{q_1\}] = \{z\}$ and

$$\begin{aligned}\frac{\delta_{lE}}{\sim}([\{q_0\}], \sigma_1, [\{q_0\}]) &= a,\\ \frac{\delta_{lE}}{\sim}([\{q_0\}], \sigma_1, [\{q_1\}]) &= b,\\ \frac{\delta_{lE}}{\sim}([\{q_1\}], \sigma_1, [\{q_0\}]) &= d,\\ \frac{\delta_{lE}}{\sim}([\{q_1\}], \sigma_1, [\{q_1\}]) &= e.\end{aligned}$$

Define $\xi : \bar{Q} \to \frac{\bar{Q}}{\sim}$ by $\xi(\{q_0\}) = [\{q_0\}], \xi(\{q_1\}) = \xi(\{q_0, q_1\}) = [\{q_1\}]$ and $\varphi : X \to E(\tilde{F}_l)$ by $\varphi(\sigma_1) = [\sigma_1]$ the identity map. Obviously, (ξ, φ) is an onto strong homomorphism. By Definition 3.3, and Theorem 3.3, there exists a strong isomorphism $\gamma : \frac{\tilde{F}_{lE(\tilde{F}_l)}}{Ker\xi} \to \frac{\tilde{F}_{lE(\tilde{F}_l)}}{\sim}$. Clearly, $Ker\xi$ is a maximal admissible partition. Therefore, considering Theorem 3.6, $\frac{\tilde{F}_{lE(\tilde{F}_l)}}{Ker\xi}$ is minimal. Hence, $\frac{\tilde{F}_{lE(\tilde{F}_l)}}{\sim}$ is the minimal quotient transformation of the BL-general fuzzy automaton. Also, we have $\frac{\tilde{F}_l}{\sim} = (\frac{\bar{Q}}{\sim}, X, \frac{\tilde{R}}{\sim} = ([\{q_0\}], \mu^{t_0}([\{q_0\}]) = \mu^{t_0}(\{q_0\})), \bar{Z}, \frac{\omega_l}{\sim}, \frac{\delta_l}{\sim}, \frac{f_l}{\sim}, \frac{\tilde{\delta}_l}{\sim}, F_1, F_2)$, where $\frac{\bar{Q}}{\sim} = \{[\{q_0\}], [\{q_1\}]\}$, $\frac{\omega_l}{\sim}[\{q_0\}] = \frac{\omega_l}{\sim}[\{q_1\}] = \{z\}$ and

$$\begin{aligned}\frac{\delta_l}{\sim}([\{q_0\}], \sigma_1, [\{q_0\}]) &= a,\\ \frac{\delta_l}{\sim}([\{q_0\}], \sigma_1, [\{q_1\}]) &= b,\\ \frac{\delta_l}{\sim}([\{q_1\}], \sigma_1, [\{q_0\}]) &= d,\\ \frac{\delta_l}{\sim}([\{q_1\}], \sigma_1, [\{q_1\}]) &= e,\\ \frac{\delta_l}{\sim}([\{q_0\}], \sigma_2, [\{q_0\}]) &= a,\\ \frac{\delta_l}{\sim}([\{q_0\}], \sigma_2, [\{q_1\}]) &= b,\\ \frac{\delta_l}{\sim}([\{q_1\}], \sigma_2, [\{q_0\}]) &= d,\\ \frac{\delta_l}{\sim}([\{q_1\}], \sigma_2, [\{q_1\}]) &= e.\end{aligned}$$

Now, let $\xi : \bar{Q} \to \frac{\bar{Q}}{\sim}$, where $\xi(\{q_0\}) = [\{q_0\}], \xi'(\{q_1\}) = \xi(\{q_0, q_1\}) = [\{q_1\}]$ and φ :

$X \to X$ be the identity map. Similarly, $\frac{\tilde{F}_l}{\sim}$ is minimal quotient BL-general fuzzy automaton.

This example showed that the number of input symbols of the minimal quotient transformation of a BL-general fuzzy automaton is less than the minimal quotient BL-general fuzzy automaton. Hence, the number of transitions and calculation of the minimal quotient transformation of a BL-general fuzzy automaton is less than the minimal quotient BL-general fuzzy automaton.

5 Conclusion

In this paper, a connection between strong homomorphism and admissible partition is presented. Also, we showed that any quotient of a given BL-GFA and the BL-GFA itself have the same behavior. The researchers obtained the minimal quotient BL-GFA and minimal quotient transformation of BL-GFA using the notions of maximal admissible partition. It is shown that the number of input symbols of the minimal quotient transformation of a BL-general fuzzy automaton is not more than the minimal quotient BL-general fuzzy automaton. Hence, the number of transitions and the number of computations of the minimal quotient transformation of a BL-GFA are not more than the minimal quotient BL-GFA.

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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