Available online at http://ijdea.srbiau.ac.ir

Int. J. Data Envelopment Analysis (ISSN 2345-458X)

Vol.3, No.4, Year 2015 Article ID IJDEA-00344, 8 pages

Research Article

International Journal of Data Envelopment Analysis   Science and Research Branch (IAU)

Differential Characteristics of Efficient Frontiers in DEA with Weight Restrictions

Sevan Sohraiee* ,

Department of Mathematics, Faculty of Sciences, Tehran North Branch, Islamic Azad University, Tehran, Iran

Received 15 January 2016, Revised 2 April 2016, Accepted 20 April 2016

Abstract

The non-differentiability and implicit definition of boundary of production possibility set (PPS) in data envelopment analysis (DEA) are two important difficulties for obtaining directional characteristics, including different elasticity measures and marginal rates of substitution. Also, imposing weight restrictions in DEA models have some shortcomings and misunderstandings. In this paper we utilize the core concept of directional derivative theorem to calculate different elasticity measures in DEA models with weight restrictions. Some theorems have been proved in order to overcome the problem.

***Keywords**: Data Envelopment Analysis, Elasticity Measure, Directional derivative, Efficient Frontier*

* Corresponding Author: S_sohraiee@iau-tnb.ac.ir

1. Introduction

DEA is a mathematical programming approach to the assessment of efficiency of decision making units (DMUs). The DEA methodology makes use of a pair of dual linear programs referred to as the multiplier and envelopment models (Cooper, Seifrod and Tone, 2007, Charnes, Cooper and Rhodes, 1978).

Different weight restrictions are often imposed in multiplier DEA models, to incorporate additional information and better discrimination on the resulting efficiency scores (Podinovski, 2004 and 2005). Also, different problems can be occurred using weight restrictions in DEA models such as infeasibility of models and zero or negative value of efficiency (Podinovski, 2013).

Directional characteristics of efficient frontiers are important for the analysis production technologies. If the frontier is smooth, the partial derivatives of the production transformation function that defines the efficient frontier can be used for the calculation of various efficient frontiers. In practical applications of DEA, production technologies are modeled by polyhedral sets that envelop the observed production units (DMUs). Because the efficient frontiers of DEA are generally not smooth, the classical concepts of elasticity measures cannot be directly applied to their analysis. The concept of directional derivatives has been utilized to overcome the abovementioned problem (Podinovski, 2013).

In this paper we utilize the concept of directional derivatives to obtain different elasticity measures in DEA models with weight restrictions. The approach developed in this paper leads to a complete analytical solution for the problem of calculating elasticity measures. Different class of elasticity measures has been considered that represent the response of any output or input bundle to marginal changes of any mixed input and output bundle, at any point of the efficient frontier.

The rest of paper, is as follows: Section 2 considers some preliminaries. Scale elasticity for output sets has been investigated in section 3. Section 4 provides scale elasticity for input sets. A numerical example has been demonstrated in section 5.

2. Preliminaries

Consider the variable returns to scale (VRS) technology with m inputs and s outputs. Let I be set of all inputs and O the set of all outputs. Observed units are denoted $(X_j, Y_j), j = 1,..., n$, where $X_j \in R_+^m \setminus \{0\}$ and $Y_j \in R_+^s \setminus \{0\}$ are its input and output vectors, respectively. Let $\bar{X}$ and $\bar{Y}$ be the $m \times n$ and $s \times n$ matrices whose columns are, respectively, the input and output vectors X_j and Y_j, j=1,…,n. Also, throughout this paper we assume that all inputs and outputs can be divided into three disjoint sets: A, B and C, and we investigate the response of the factors included in the set B to marginal changes of the factors in set A under the assumption that the factors in set C remain constant.

DEA multiplier models are stated in terms of vectors of output and input weights $u \in R_+^s$ and $v \in R_+^m$, respectively. Weight restrictions are additional constraints on vectors u and v that may be incorporated in such models. Below we consider weight restrictions that can be considered as follows:

$$u^t Q_t - v^t P_t \le 0, t = 1,\ldots K. \quad (1)$$

where $Q_t \in R^s$ and $P_t \in R^m$ are constant vectors whose components can be positive, negative, or zero. Weight restrictions are referred to as homogeneous because the constant on the right- hand side is zero. Also, the VRS technology with weight restrictions can be considered as follows (Podinovski, 2004):

$$T_{VRS-To} = \{(X,Y) \big| X \ge \bar{X}\lambda + \sum_{t=1}^{k} \pi_t P_t, Y \le \bar{Y}\lambda + \sum_{t=1}^{k} \pi_t P_t, e\lambda = 1, \lambda \ge 0, \pi_t \ge 0, t = 1,\ldots, K\} \quad (2)$$

Now any unit $(X_o, Y_o) \in T_{VRS-TO}$ can be represented as follows:

$$(X_o, Y_o) = (X_o^A, X_o^C, Y_o^A, Y_o^B, Y_o^C) \quad (3)$$

where the superscripts indicate the sub-vectors of X_o and Y_o corresponding to the sets A, B and C. Assuming that the sub-vector of outputs Y_o^B has at least one strictly positive component, let us consider the largest amount of β of the output bundle Y_o^B that can be produced in technology T_{VRS-TO}, given the amount α of the mixed bundle (X_o^A, Y_o^A), under the condition that the remaining inputs X_o^C and outputs Y_o^C do not change. This leads to the following output response function:

$$\bar{\beta}(\alpha) = \max \left\{ \beta \left| (\alpha X_o^A, X_o^C, \alpha Y_o^A, \beta Y_o^B, Y_o^C) \in T_{VRS-TO} \right. \right\} \quad (4)$$

It is trivial that the domain of $\bar{\beta}(\alpha)$ includes $\alpha = 1$ and it can be bounded or unbounded. Without loss of generality we can assume that the unit (X_o, Y_o) is efficient and therefore $\bar{\beta}(1) = 1$.

Definition 1. (Podinovski, 2010) Assume that the function $\bar{\beta}(\alpha)$ is differentiable at $\alpha = 1$. Then the elasticity of response of the output bundle Y_o^B with respect to the mixed bundle (X_o^A, Y_o^A) is

$$\varepsilon_{A,B}(X_o, Y_o) = \bar{\beta}'(1) \quad (5)$$

The Definition 1 is an extension of definition made by Hanoch, (1970). Utilizing the implicit function theorem, the efficient frontier can be defined by function $\Phi(X,Y) = 0$.

3. Scale elasticity for output sets

The extension of elasticity $\varepsilon_{A,B}(X_o, Y_o)$ to T_{VRS-TO} leads us to achieve a new model for obtaining the elasticity measure in the case which weight restrictions are considered in technology. By restating (4), the output response function $\bar{\beta}(\alpha)$ can be achieved by the following linear programming:

$$\bar{\beta}(\alpha) = \max \ \beta \quad (6)$$

$$s.t. \quad \bar{X}^A \lambda + \sum_{t=1}^{k} \pi_t P_t^A \le \alpha X_o^A,$$

$$\bar{X}^C\lambda + \sum_{t=1}^{k}\pi_t P_t^C \le X_o^C ,$$

$$-\bar{Y}^A\lambda - \sum_{t=1}^{k}\pi_t Q_t^A \le -\alpha Y_o^A ,$$

$$-\bar{Y}^B\lambda - \sum_{t=1}^{k}\pi_t Q_t^B + \beta Y_o^B \le 0,$$

$$-\bar{Y}^C\lambda - \sum_{t=1}^{k}\pi_t Q_t^C \le -Y_o^C ,$$

$$e\lambda = 1, \lambda, \pi_t \ge 0, t = 1,..., k,$$

$$\beta \quad \text{free in sign.}$$

where $\bar{X}^A$ and $\bar{X}^C$ are the rows of matrix $\bar{X}$ corresponding to bundle sets of input A and C, $\bar{Y}^A, \bar{Y}^B$ and $\bar{Y}^C$ are the rows of matrix $\bar{Y}$ corresponding to bundle sets of output A, B and C, P_t^A and P_t^C are the rows of matrix P_t corresponding to bundle sets of input A and C, and Q_t^A, Q_t^B and Q_t^C are the rows of matrix Q_t corresponding to bundle sets of output A, B and C. Also, the dual problem of model (6) for $\alpha = 1$ is as follows:

$$\bar{\beta}(1) = \min \quad v^A X_o^A + v^C X_o^C - \mu^A Y_o^C - \mu^C Y_o^C + \mu_o , \tag{7}$$

$$s.t. \quad v\bar{X} - \mu\bar{Y} + \mu_o e \ge 0 ,$$

$$\mu^B Y_o^B = 1 ,$$

$$vP_t - \mu Q_t \ge 0 , \quad t = 1,...,k,$$

$$v = (v^A, v^C) \ge 0,$$

$$\mu = (\mu^A, \mu^B, \mu^C) \ge 0,$$

$$\mu_0 \quad \text{free in sign.}$$

Theorem 1.

a) If the function $\bar{\beta}(\alpha)$ is defined in some right (left) neighborhood of $\alpha = 1$, then it has a finite right- hand (respectively, left-hand) derivative, which can be calculate as follows:

$$\bar{\beta}'_+(1) = \min \quad (v^A X_o^A - \mu^A Y_o^A), \tag{8}$$

$$s.t. \quad (v, \mu, \mu_0) \in \Omega ,$$

And,

$$\bar{\beta}'_-(1) = \max \quad (v^A X_o^A - \mu^A Y_o^A), \tag{9}$$

$$s.t. \quad (v, \mu, \mu_0) \in \Omega ,$$

b) If the function $\bar{\beta}(\alpha)$ is undefined in some right (left) neighborhood of $\alpha = 1$ (that is, the required right (left) neighborhood of $\alpha = 1$ does not exist) and $Q_t^A \le 0, t = 1,..., K,$ then the objective function in (8) and (9) is unbounded (The set Ω is the set of all optimal solutions of model (7)).

Proof. To prove (a), the function $\bar{\beta}(\alpha)$ can be considered as a function $\Phi(Z)$ of the vector $Z = (\alpha X_o^A, X_o^C, -\alpha Y_o^A, 0, -Y_o^C, 1) \in R^{m+s+1}$ on the right-hand side of (6). Then $\bar{\beta}'_+(1)$ is equal to the directional derivative of the function $\Phi(Z)$ taken at $\hat{Z} = (X_o^A, X_o^C, -Y_o^A, 0, -Y_o^C, 1) \in R^{m+s+1}$ in the direction $\hat{d} = (X_o^A, 0, -Y_o^A, 0, 0, 0) \in R^{m+s+1}$, provided one of the two derivatives exists:

$$\bar{\beta}'_+(1) = \lim_{\alpha \downarrow 1} \frac{\bar{\beta}(\alpha) - \bar{\beta}(1)}{\alpha - 1} =$$

$$\lim_{\tau \downarrow 0} \frac{\Phi(\hat{Z} + \tau\hat{d}) - \Phi(\hat{Z})}{\tau} = \Phi'(\hat{Z}; \hat{d}).$$

Similarly, $\bar{\beta}'_-(1) = -\Phi'(\hat{Z}; -\hat{d})$. By reformulation of Theorem 2.2 in Shapiro (1979), the directional derivatives of $\Phi(Z)$ at

$\hat{Z}$ exist and $\Phi'(\hat{Z};\hat{d}) = \min\{w\hat{d} | w \in \Omega\}$, $\Phi'(\hat{Z};-\hat{d}) = \min\{w(-\hat{d}) | w \in \Omega\}$

This completes the proof of part (a) (Podinovski, 2010) .

To prove (b), assume that $\overline{\beta}(\alpha)$ is undefined to the right of $\alpha = 1$. We know that set A has at least one non-zero output. Also, the problem (8) has the same constraints as (7) and additional condition: $\nu^A X_o^A + \nu^C X_o^C - \mu^A Y_o^C - \mu^C Y_o^C + \mu_o = 1$.

Suppose that, contrary to Theorem 1, program (8) has a finite optimal solution. Then its dual is feasible:

$$\max \ \beta + \delta \tag{10}$$

$$\begin{aligned}
s.t.\quad & \overline{X}^A \lambda + \delta X_o^A + \sum_{t=1}^{k} \pi_t P_t^A \le X_o^A, \\
& \overline{X}^C \lambda + \delta X_o^C + \sum_{t=1}^{k} \pi_t P_t^C \le 0, \\
& -\overline{Y}^A \lambda - \delta Y_o^A - \sum_{t=1}^{k} \pi_t Q_t^A \le -Y_o^A, \\
& -\overline{Y}^B \lambda - \sum_{t=1}^{k} \pi_t Q_t^B + \beta Y_o^B \le 0, \\
& -\overline{Y}^C \lambda - \delta Y_o^C - \sum_{t=1}^{k} \pi_t Q_t^C \le 0, \\
& e\lambda + \delta = 1,\ \lambda, \pi_t \ge 0, t = 1,\dots,k, \\
& \beta, \delta \quad free\ in\ sign.
\end{aligned}$$

Since $\lambda \ge 0$ we have $\delta \le 0$. If $\delta = 0$ then equality $e\lambda + \delta = 1$ implies that $\lambda = 0$ and since $Q_t^A \le 0, t = 1,\dots,K,$ from the third constraint of model (10), we have $Y_o^A \le 0$, which is impossible. Therefore, $\delta < 0$. By dividing the constraints of (10) to $-\delta > 0$, and rearranging the variables as follows:

$$\tilde{\lambda} = -\frac{\lambda}{\delta}, \alpha = 1 - \frac{1}{\delta} \quad \text{and} \quad \tilde{\beta} = -\frac{\beta}{\delta}.$$

This means that the unit $(\alpha X_o^A, X_o^C, \alpha Y_o^A, \beta Y_o^B, Y_o^C) \in T_{VRS-TO}$. Thus, the assumption that $\overline{\beta}(\alpha)$ is undefined to the right of $\alpha = 1$ is incorrect. Therefore, (8) cannot have a finite optimal solution. This completes the proof.

Utilizing Theorem 1the right-hand and left-hand elasticities can be defined by the following statements:

$$\varepsilon^+_{A,B}(X_o, Y_o) = \overline{\beta}'_+(1) \tag{11}$$

$$\varepsilon^-_{A,B}(X_o, Y_o) = \overline{\beta}'_-(1) \tag{12}$$

The equations (8) and (9) imply $\varepsilon^+_{A,B}(X_o, Y_o) \le \varepsilon^-_{A,B}(X_o, Y_o)$.

4. Scale elasticity for input sets

By considering $(X_o, Y_o) \in T_{VRS-TO}$ and discovering the elasticity of response of its input bundle X_o^B to its mixed bundle (X_0^A, Y_o^A) while the remaining inputs and outputs are unchanged the following input response function can be achieved:

$$\hat{\beta}(\alpha) = \min\{\beta \ge 0 | (\alpha X_o^A, \beta X_o^B, X_o^C, \alpha Y_o^A, Y_o^C) \in T_{VRS-TO}\} \tag{13}$$

As before, again consider that the unit $(X_o, Y_o) \in T_{VRS-TO}$ is efficient. It means that $(X_o, Y_o) \in T_{VRS-TO}$ is on the boundary of efficient frontier and thus, $\hat{\beta}(1) = 1$.

Definition 2. (Podinovski, 2010) Assume that the function $\hat{\beta}(\alpha)$ is differentiable at $\alpha=1$. Then the elasticity of response of the input bundle X_o^B with respect to the mixed bundle (X_o^A, Y_o^A) is

$$\rho_{A,B}(X_o, Y_o) = \hat{\beta}'(1) \quad (14)$$

By restating (14), the input response function $\hat{\beta}(\alpha)$ can be obtained by the following linear programming:

$$\begin{aligned} \hat{\beta}(\alpha) = \min\ & \beta \qquad (15)\\ s.t.\ & -\bar{X}^A\lambda - \sum_{t=1}^{k}\pi_t P_t^A \ge -\alpha X_o^A,\\ & -\bar{X}^B\lambda - \sum_{t=1}^{k}\pi_t P_t^B + \beta X_o^B \ge 0,\\ & -\bar{X}^C\lambda - \sum_{t=1}^{k}\pi_t P_t^C \ge -X_o^C,\\ & \bar{Y}^A\lambda + \sum_{t=1}^{k}\pi_t Q_t^A \ge \alpha Y_o^A,\\ & \bar{Y}^C\lambda + \sum_{t=1}^{k}\pi_t Q_t^C \ge Y_o^C,\\ & e\lambda = 1, \lambda, \pi_t \ge 0, t = 1,\dots,k,\\ & \beta \quad \textit{free in sign.} \end{aligned}$$

The dual of model (15) for $\alpha = 1$ is as follows:

$$\begin{aligned} \hat{\beta}(1) = \max\ & -v^A X_o^A - v^C X_o^C + \mu^A Y_o^C + \mu^C Y_o^C + \mu_o, \qquad (16)\\ s.t.\ & -v\bar{X} + \mu\bar{Y} + \mu_o e \le 0,\\ & v^B X_o^B = 1,\\ & -vP_t + \mu Q_t \le 0, \quad t = 1,\dots,k,\\ & v = (v^A, v^B, v^C) \ge 0,\\ & \mu = (\mu^A, \mu^C) \ge 0,\\ & \mu_0 \quad \textit{free in sign.} \end{aligned}$$

Theorem 2.

a) If the function $\hat{\beta}(\alpha)$ is defined in some right (left) neighborhood of $\alpha=1$, then it has a finite right- hand (respectively, left-hand) derivative, which can be calculate as follows:

$$\begin{aligned} \hat{\beta}'_+(1) = \max\ & (-v^A X_o^A + \mu^A Y_o^A), \qquad (17)\\ s.t.\ & (v, \mu, \mu_0) \in \Delta, \end{aligned}$$

And,

$$\begin{aligned} \hat{\beta}'_-(1) = \min\ & (-v^A X_o^A + \mu^A Y_o^A), \qquad (18)\\ s.t.\ & (v, \mu, \mu_0) \in \Delta, \end{aligned}$$

b) If the function $\hat{\beta}(\alpha)$ is undefined in some right (left) neighborhood of $\alpha=1$ (that is, the required right (left) neighborhood of $\alpha=1$ does not exist) and $P_t^A \ge 0, t = 1,\dots,K,$ then the objective function in (17) and (18) is unbounded.

Proof. The proof is similar to the proof of Theorem 1 and omitted.

This leads to the following right-hand and left-hand elasticities:

$$\rho^+_{A,B}(X_o, Y_o) = \hat{\beta}'_+(1) \quad (19)$$

$$\rho^-_{A,B}(X_o, Y_o) = \hat{\beta}'_-(1) \quad (20)$$

5.Numerical example

Consider two DMUs, with one input and one output that data are listed in Table 1. The frontier of VRS technology before and after imposing weight restriction $u - 2v \ge 0$ can be seen in Fig. 1.

Table1. Data of Example 1.

	Input	Output
M	7	3
N	5	1

The second and third columns of Table 2 show the one-sided elasticity measures without weight restrictions and the third and forth columns of Table 2 indicates the one-sided elasticity measures with weight restrictions (models (8) and (9)). Note that in both cases $A = I$ and $B = O$. For example the model (8) for unit M is as follows:

$$\begin{aligned} \bar{\beta}'_{+}(1) = \min \quad & 7v \\ s.t. \quad & 5v - \mu + \mu_0 \geq 0, \\ & 7v - 3\mu + \mu_0 \geq 0, \\ & 3\mu = 1, \\ & 7v + \mu_0 = 1, \\ & -2v + \mu \geq 0, \\ & v, \mu \geq 0, \mu_0 \ \ free\, in\, sign. \end{aligned}$$

As can be seen in Table 2, the left-hand elasticity of unit M has been changed after imposing weight restrictions, although the right-hand elasticity has not been changed. But for unit N the problem is completely different because the extreme efficient unit N is not an extreme efficient unit after imposing weight restrictions. Therefore, models (8) and (9) are infeasible.

Table 2. Result of Example 1.

	$\varepsilon^{-}_{I,O}$	$\varepsilon^{+}_{I,O}$	$\varepsilon^{-}_{I,O}$	$\varepsilon^{+}_{I,O}$
M	Undefined	5	infeasibility	Infeasibility
N	1.4	0	1.166	0

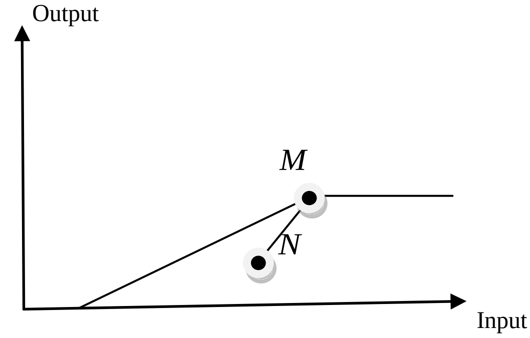

Figure 1. PPS of Example 1.

6. Conclusion

In this paper, differential characteristics of efficient frontier of PPS with weight restriction have been investigated. We utilize the core concept of directional derivative to obtain a linear programming problem for achieving the right-hand and left-hand derivative of boundary units. In actual fact, it becomes easier to understand and to prove the basic concepts of scale elasticity. We hope that the concept of this paper will be a great value and significance to the readers.

References

[1] Chanes A, Cooper WW, Rhodes E, (1978) Measuring the efficiency of decision making unit. *Eur. J. Oper. Res.* 2(6):429-444.

[2] Cooper WW, Seiford LM, Tone K, (2007) *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-solver software* (Springer, New York).

[3] Hanoch G, (1970) Homotheticity in joint production. *J. Econ. Theory* 2:423-426.

[4] Podinovski VV (2004) Production trade-offs and weight restrictions in data

envelopment analysis. *J. Oper. Res Soc.*. 55(12):1311-1322.

[5] Podinovski VV (2005) The explicit role of weight bounds in models of data envelopment analysis. *J Oper. Res. Soc.* 56(12):14081418.

[6] Podinovski VV, Forsund FR, (2010) Differential characteristics of efficient frontiers in data envelopment analysis. *Oper. Res.* 58(6):1743-1754.

[7] Shapiro JF, (1979) *Mathematical programming: structures and algorithms.* John Wiley & Sons, New York