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Malmquist Productivity Index with Dynamic Network Structure

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Abstract

Data envelopment analysis (DEA) measures the relative efficiency of decision making units (DMUs) with multiple inputs and multiple outputs. DEA-based Malmquist productivity index measures the productivity change over time. We propose a dynamic DEA model involving network structure in each period within the framework a DEA. We have previously published the network DEA (NDEA) and the dynamic DEA (DDEA) models separately. Hence, this article is a composite of these two models. Vertically, we deal with multiple divisions connected by links of network structure within each period and, horizontally, we combine the network structure by means of carry-over activities between succeeding periods. We also introduce dynamic Malmquist index by which we can compare divisional performances over time.

Keywords: Malmquist productivity index- DEA- network- overall efficiency.

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1. Introduction

Efficiency and productivity measurement in organizations has enjoyed a great deal of interest among researchers studying performance analysis. Data envelopment analysis (DEA) is a popular method for comparing the inputs and outputs of a set of homogenous decision-making units (DMUs) by evaluating their relative efficiency.

Network data envelopment analysis (NDEA) concerns using the DEA technique to measure the relative efficiency of a system, taking into account its internal structure.

However, DMUs might include internal or network structures, which consist of several interactive processes; for instance, see Färe and Grosskopf [6], Castelli et al, [3] and Tone and Tsutsui[10]. During the past two decades, the possibility of assessing the efficiency related to sub-processes or decision making sub-units (DMUs) of DMUs within the DEA framework has been investigated by several authors. (To obtain a suitable reference for these approaches, see Castelli et al, [2].

In addition, companies' activity generally continues across multiple periods. The dynamic DEA model was developed to evaluate DMUs performance from a long-term perspective using carry-over Variables [1, 4, 8, 9].

We proposed a model for the developed dynamic and network DEA (with parallel network structure). [7]This combined model enables us not only to obtain the overall efficiency of DMUs over the entire observed period, but also to conduct further analysis, that is, observing dynamic change of the period efficiency and dynamic change of the divisional efficiency of DMUs.

In this paper, we propose a Malmquist index corresponding to the dynamic and network framework. Using this model, we can measure the efficiency score DMUs more realistic that their divisions have parallel network structure.

The rest of this paper unfolds as follows. In section2, we describe mathematical formulations of dynamic and network DEA model. Malmquist index with dynamic network structure is introduced in section3. Section 4 concludes the paper.

2. Dynamic Network DEA

In this section, we define the multi-stage models with serial and parallel structure for network DEA model [7] and after that, we define the dynamic network DEA model and formulate it as a programing problem.

In dynamic DEA with network structure we deal with decision making units n (DMU_j, j=1,…, n). Each DMU is divided to k divisions (k=1,…, K) that their efficiencies and the desired DMU efficiencies in T time period (t=1,…,T) is examined. Let $I = \{1,...,m\}$, $O = \{1,...,s\}$, $M = \{1,...,D\}$ and $K = \{1,...,K\}$ the index sets of the inputs, outputs, intermediate products and processes, respectively. Also, let $I^K \subseteq I$, $0^K \subseteq 0$ and $M^K \subseteq M$, the corresponding index sets of inputs, outputs and intermediate products in process k, respectively.

The Total amount of input i used by DMU_j is the sum of those used by all of its DMU_s, i.e. $x_{ij} = \sum_{k \in K} x_{ij}^k$. Also, the Total amount of output r produced by DMU_j is the sum of those. Produced by all of its DMUs, i.e. $y_{ij} = \sum_{k \in K} y_{rj}^k$.

Let DMU_j be the series system under evaluation. To measure the efficiency of the system is to find the multipliers u_{rk}, v_{ik}, w_{dk} and w'_{fk} which produce the maximum efficiency under the constraint that aggregation of the outputs is less than or equal to that of the inputs for all processes. In symbols, this is:

$$
\begin{aligned}
&{}_{series}E_o = \max \frac{1}{K}\sum_{k=1}^{K}\Big(\sum_{r\in O^k} u_{rk}y_{rok} + \sum_{f\in N^k} w'_{fk}z_{fok} + u_{0k}\Big)\\
&s.t.\\
&\frac{1}{K}\sum_{k=1}^{K}\Big(\sum_{i\in I^k} v_{ik}x_{iok} + \sum_{d\in M^k} w_{dk}z_{dok}\Big) = 1 \qquad (1)\\
&\sum_{k=1}^{K}\Big(\sum_{r\in O^k} u_{rk}y_{rjk} + \sum_{f\in N^k} w'_{fk}z_{fjk} + u_0 - \sum_{i\in I^k} v_{ik}x_{ijk} + \sum_{d\in M^k} w_{dk}z_{djk}\Big) \le 0 \quad j = 1,\dots,n\\
&u_r \ge 0, r = 1,\dots,s_k \qquad v_i \ge 0, i = 1,\dots,m_k\\
&w_d \ge 0, d = 1,\dots,D \qquad w'_f \ge 0, f = 1,\dots,D-1\\
&u_{0k} : free
\end{aligned}
$$

If let DMU_j be the parallel system under evaluation than the model for measuring the system overall efficiency can be formulated as:

$$
\begin{aligned}
&{}_{parallel}E_o = \max \frac{1}{K}\sum_{k=1}^{K}\Big(\sum_{r\in O^k} u_{rk}y_{rok} + u_{0k}\Big)\\
&s.t.\\
&\frac{1}{K}\sum_{k=1}^{K}\Big(\sum_{i\in I^k} v_{ik}x_{iok}\Big) = 1 \qquad (2)\\
&\sum_{k=1}^{K}\Big(\sum_{r\in O^k} u_{rk}y_{rjk} + u_{0k} - \sum_{i\in I^k} v_{ik}x_{ijk}\Big) \le 0 \qquad j = 1,\dots,n\ ,\\
&u_r \ge 0, r = 1,\dots,s_k \qquad v_i \ge 0, i = 1,\dots,m_k\\
&u_{0k} : free
\end{aligned}
$$

x_{ijk} is input resource i to DMU_j for division k.

y_{rjk} is output product r from DMU_j for division k.

z_{fjk} is intermediate products f from DMU_j at division k with treated as output.

z_{djk} is intermediate products d from DMU_j at division k with treated as input.

These models will be able to calculate the efficiency of the desired DMU according to sub-unit.

The dynamic structure model consists of internal connections that transport intermediate products of t period to t+1 period. In the first period, we don't have any connection from previous period besides, in the last period of T, we didn't consider any connection for the next period.

Dynamic network DEA with the series system can be expressed as follows:

$$
\begin{aligned}
& {}_{series}E_o = \max \frac{1}{TK}\sum_{t=1}^{T}\Big(\sum_{k=1}^{K}\Big(\sum_{r\in O^k} u_{rk}^{t} y_{rok}^{t} + \sum_{f\in N^k} w_{fk}^{\prime t} z_{fok}^{t} + u_{0k}\Big)\Big) \\
& s.t. \\
& \frac{1}{TK}\sum_{t=1}^{T}\Big(\sum_{k=1}^{K}\Big(\sum_{i\in I^k} v_{ik}^{t} x_{iok}^{t} + \sum_{d\in M^k} w_{dk}^{t} z_{dok}^{t}\Big)\Big) = 1 \qquad (3) \\
& \sum_{t=1}^{T}\Big(\sum_{k=1}^{K}\Big(\sum_{r\in O^k} u_{rk}^{t} y_{rjk}^{t} + \sum_{f\in N^k} w_{fk}^{\prime t} z_{fjk}^{t} + u_0 - \sum_{i\in I^k} v_{ik}^{t} x_{ijk}^{t} + \sum_{d\in M^k} w_{dk}^{t} z_{djk}^{t}\Big)\Big) \le 0 \quad j = 1,\ldots,n \\
& u_r \ge 0, r = 1,\ldots,s_k \qquad v_i \ge 0, i = 1,\ldots,m_k \\
& w_d \ge 0, d = 1,\ldots,D \qquad w_f' \ge 0, f = 1,\ldots,D-1 \\
& u_{0k} : free
\end{aligned}
$$

As well as dynamic DEA with parallel network can be defined like following:

$$
{}_{parallel}E_o = \max \frac{1}{TK}\sum_{t=1}^{T}(\sum_{k=1}^{K}(\sum_{r\in O^k} u_{rk}^t y_{rok}^t + u_{0k}))
$$

$$
\begin{aligned}
& s.t. \\
& \frac{1}{TK}\sum_{t=1}^{T}(\sum_{k=1}^{K}(\sum_{i\in I^k} v_{ik}^t x_{iok}^t)) = 1 \qquad (4) \\
& \sum_{t=1}^{T}(\sum_{k=1}^{K}(\sum_{r\in O^k} u_{rk}^t y_{rjk}^t + u_{0k} - \sum_{i\in I^k} v_{ik}^t x_{ijk}^t)) \le 0 \qquad j = 1,\dots,n \ , \\
& u_r \ge 0, r = 1,\dots,s_k \qquad v_i \ge 0, i = 1,\dots,m_k \\
& u_{0k} : free
\end{aligned}
$$

Obtaining overall efficiencies, period efficiencies, divisional efficiencies and period-divisional efficiencies in each period of time and in each part of DMUs' decision making sub-units could be assumed as one of the merits of this method considering the volatile links & connections.

3 Malmquist index with dynamic network structure

Cooper et al [5] express Malmquist index as anevaluate of productivity change of decision making unit (DMU) between two time periods and is an example in comparative statistic. Therefore, Malmquist index is defined into product of Catch-up and Frontier-shift terms. The terms explain catch-up (recovery) relates to the degree to which the DMU improves or worsens its efficiency. In addition, frontier-shift (innovation) reflects the change in the efficient frontiers between the two time periods.

The proposed dynamic network DEA models with parallel network structure in the current study generate relative period efficiency scores based on efficiency frontiers of each period, while they do not capture the absolute position of each frontier. In this case, the absolute progress or regress of efficiency performance of each DMU cannot be measured.

The Malmquist index will be an effective measure to incorporate frontier-shift effect into evaluation, and thus result in capturing the absolute productivity change of each DMU in the dynamic DEA model. In this section, we define overall and divisional Malmquist indices as follows.

3.1 Period-divisional catch-up index

As the ratio of the period-divisional efficiencies (ρ_{ok}^{t}) between *t* and *t*+1, that can evaluate the efficiency for a division in the period. Thus period-divisional catch-up index, k division efficiency in t and t+1 period for the decision making units is defined as follows and will be represented by $\gamma_{ok}^{t \to t+1}$.

We define the as

$$\gamma_{ok}^{t \to t+1} = \frac{\rho_{ok}^{t+1^*}}{\rho_{ok}^{t^*}} \qquad (o = 1, ..., n; k = 1, ..., \mathrm{K}; \mathrm{t} = 1, ... \mathrm{T} - 1) \tag{5}$$

$\gamma_{ok}^{t \to t+1}$ >1, = 1, and <1 indicate respectively progress, status quo and regress in catch-up effect, respectively.

3.2 Period-divisional frontier-shift effect

The frontier technology determined by the efficient frontier is estimated using DEA for a set of DMUs. However, the frontier technology for a particular DMU under evaluation is only represented by a section of the DEA frontier or a facet and indicates the distance covered by the efficient frontier from one period to another and is thus a measure of technological improvements between the periods and indicates the distance covered by the efficient frontier from one period to another and is thus a measure of technological improvements between the periods. We define period-divisional frontier-shift effect from *t* to *t*+1 as

$$\sigma_{ok}^{t\to t+1} = \left[\frac{\rho_{ok}^{t^*}}{\pi_{ok}^{t\,(t+1)}} \times \frac{\pi_{ok}^{t+1(t)}}{\rho_{ok}^{t+1^*}} \right]^{1/2} \qquad (o = 1,...,n;\, k = 1,...,\mathrm{K};\, \mathrm{t} = 1,...\mathrm{T}-1) \quad (6)$$

Where $\pi_{ok}^{t\,(t+1)}$ (or $\pi_{ok}^{t\,(t+1)}$) is DNDEA (Keikha-Javan et al., [7]) score of DMU_{ok} at period t (or t+1) evaluated by the division k frontier at t+1 (or t). If the division k has no inputs or no outputs, we define $\sigma_{ok}^{t\to t+1} = 1$.

3.3 Period-divisional Malmquist index

According to what has been said in the past about, catch-up index and frontier-shift effect, we define the period-divisional Malmquist index as $\mu_{ok}^{t\to t+1}$ at $t \to t+1$ in division k.

$$\mu_{ok}^{t\to t+1} = \gamma_{ok}^{t\to t+1}\,\sigma_{ok}^{t\to t+1} \qquad (o = 1,...,n;\, k = 1,...,\mathrm{K};\, \mathrm{t} = 1,...\mathrm{T}-1) \quad (7)$$

3.4 Period Malmquist index

Divisional Malmquist index can be obtained as the weighted geometric mean of the period-divisional Malmquist indices as

$$\mu_{ok}^{t\to t+1} = \prod_{k=1}^{K} (\mu_{ok}^{t\to t+1})^{w_k} \qquad (o = 1,...,n;\, k = 1,...,\mathrm{K};\, \mathrm{t} = 1,...\mathrm{T}-1) \quad (8)$$

Where $w_k \geq 0$ is the weight to division *k* with $\sum_{k=1}^{K} w_k = 1$.

3.5 Divisional Malmquist index

However the above Malmquist index is defined between two-period ($t \to t+1$) for each DMU and all their divisions, we can find the divisional Malmquist indices (d-MI) in a long-time period (based on the Period 1 to *t*), which can be divided into divisional catch-up index (d-CU) and divisional frontier-shift effect (d-FS) as follows:

$$\begin{aligned} d-MI &= \mu_{ok}^{1\to t} = \prod_{t'=1}^{t} \mu_{ok}^{t'\to t'+1} \\ &= d-CU \times d-FS \\ &= \prod_{t'=1}^{t} (\gamma_{ok}^{t'\to t'+1} . \sigma_{ok}^{t'\to t'+1}) \qquad (o=1,\dots,n; k=1,\dots,\mathrm{K}; \mathrm{t}=1,\dots \mathrm{T}-1) \end{aligned} \tag{9}$$

Overall Malmquist index with network structure (O-MI) is defined as follows:

$$\begin{aligned} &O-MI = \mu_{ok}^{1\to t} = \prod_{k=1}^{K} (\mu_{ok}^{1\to t})^{w_k} \\ &(o=1,\dots,n; \mathrm{t}=1,\dots \mathrm{T}-1) \end{aligned} \tag{10}$$

This index enables us to capture continuous productivity change of each DMU from the first period.

5. Conclusion

For real formulate models that specific details are included, it is necessary to study the many different relationships. Subunits of a single decision-maker can be used in series or parallel, both of these cases, the real issues are many practical applications.

In this paper, at first our calculated the catch-up index and frontier-shift effect of each part of the desired DMU in a time period and then according to the weighted geometric mean of all parts, we counted the Malmquist index in different time period and ultimately in the overall period..

The proposed model has parallel network structure that is proposed to researcher to do Serial network evaluation with mentioned conditions and extensions to the radial DEA model.

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