



Ranking Efficient DMUs

Using the Ideal point and Norms

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Abstract

In this paper, presenting two simple methods for ranking of efficient DMUs in DEA models that included to add one virtual DMU as ideal DMU and is using the additive model. Note that, we use an ideal point just for comparing efficient DMUs with. Although these methods are simple, they have ability for ranking all efficient DMUs, extreme points and the others, also they are capable of ranking the whole DMUs at special cases that previous methods could not ranked them or they can be ranked with hard computing.

Keywords : Data envelopment analysis; Ranking; Ideal point; Efficiency; Additive model; Norm.

1 Introduction

Data envelopment analysis (DEA) by Charnes et al. (1978), is a mathematical programming technique for identifying efficient frontiers for peer decision making units (DMUs). In most models of DEA (such as CCR and BCC), the best performers have efficiency score unity and from experience, we know that there are usually plural DMUs which have this “efficient status”. To discriminate between these efficient DMUs is an interesting research subject. Several authors have proposed methods for ranking the best performers. See Sexton et al. (1986), Andersen and Petersen (AP) (Super-efficiency) (1993), Torgersen et al. (benchmarking methods) (1996), Seiford and Zhu (1999), Mehrabian et al. (MAJ) (1999) and Zhu (2001) among others. In some cases, the AP and MAJ models are infeasible (see Thrall (1996)), In addition to this difficulty, the AP model may be unstable because of extreme sensitivity to small variations in the data when some DMUs have relatively small values for some of their inputs (Mehrabian et al. (1999)). Therefore, some methods cannot rank non-extreme points and some have hard conclusions.

In this paper, we introduce a new simple ranking method based on the additive model that including to add a virtual DMU as ideal DMU to the model calculated distance between evaluated DMU and ideal point by L_1 -norm. Therefore, we use L_∞ -norm on this model and make another similar ranking

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model. We can rank all efficient DMUs in DEA by these two models. The proposed method removes the difficulties arising from other ranking models. Note that, we use an ideal point just for comparing efficient DMUs with. Although these methods are simple, they have ability for ranking of all efficient DMUs, extreme points and the others, and ranking all DMUs at special cases that previous methods cannot be ranked them or they can be ranked them with hard conclusions.

Some similar ranking models such as Topsis methods and Wang and Luo (2006) used two ideal points, positive and negative ideal points, in their models. In this paper, we use just positive point in different way.

This paper consist the following sections: In Section 2, some necessary definition and models such Additive model and ideal point are expressed. New ranking method presented in section 3. In section 4, some numerical examples are given. We used L_∞ -norm on our approach and introduced another similar model for ranking in section 5; and conclusion is put forward.

2 Background DEA

DEA is a mathematical model that measures the relative efficiency of decision making units (DMUs) with multiple inputs and outputs but with no obvious production function to aggregate the data in its entirety. By comparing n units with s outputs denoted by y_{rj} , $r=1, \dots, s$, and m inputs denoted by x_{ij} , $i=1, \dots, m$, that all of them are non-negative and each DMU has at least one strictly positive input and output.

2.1 The additive model

An alternative formulation proposed by Charnes et al. (1985) utilizes slacks alone in the objective function which is called the additive model and we express that under both constant returns to scale (model (1)) and variable returns to scale (model (2)) assumptions.

$$\begin{aligned}
 & \text{Max} \quad \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+, \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1, \dots, m, \\
 & \quad \quad \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \dots, s, \\
 & \quad \quad \lambda_j \geq 0, \quad s_i^- \geq 0, \quad s_r^+ \geq 0, \quad j = 1, \dots, n, \quad i = 1, \dots, m, \quad r = 1, \dots, s.
 \end{aligned} \tag{1}$$

And

$$\begin{aligned}
 &Max \quad \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+, \\
 &s.t. \quad \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1, \dots, m, \\
 &\quad \quad \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \dots, s, \\
 &\quad \quad \sum_{j=1}^n \lambda_j = 1, \\
 &\quad \quad \lambda_j \geq 0, \quad s_i^- \geq 0, \quad s_r^+ \geq 0, \quad j = 1, \dots, n, \quad i = 1, \dots, m, \quad r = 1, \dots, s.
 \end{aligned} \tag{2}$$

DMU_o might be evaluated, s_i^- and s_r^+ are input and output slacks, respectively.

Definition.1. (ADD-efficient DMU)

DMU_o is ADD-efficient iff $\forall i, s_i^- = 0$ and $\forall r, s_r^+ = 0$.

2.2 Ideal point (IDMU)

The ideal DMU (IDMU) is a virtual DMU which can use the least inputs to generate the most outputs. In other word we can express that as follows:

$$IDMU = (x_i^I, y_r^I) = (Min_j \{x_{ij}\}, Max_j \{y_{rj}\}) \quad , \quad i = 1, 2, \dots, m ; \quad r = 1, 2, \dots, s. \tag{3}$$

Note that a virtual IDMU may not exist in practical production activity at least at current technical level.

3 The proposed method

First, we generate the efficient DMUs from one of the basis DEA's models such as BCC or Additive model, then use the Additive model just for these efficient DMUs and IDMU which generate from (3), so we have:

$$\begin{aligned}
 &Max \quad \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+, \\
 &s.t. \quad s_i^- = x_{io} - x_i^I, \quad i = 1, \dots, m, \\
 &\quad \quad s_r^+ = y_r^I - y_{ro}, \quad r = 1, \dots, s,
 \end{aligned} \tag{4}$$

Now, we normalize data; x_{ij} divide by x_i^{\max} and y_{rj} by y_r^{\max} to render the model unit-invariant. We set $Eff = \{j; DMU_j \text{ is efficient}\}$ and define:

$$x_i^{\max} = (\text{Max}_{j \in Eff} \{x_{ij}\}) \quad , \quad i = 1, 2, \dots, m, \quad (5)$$

$$y_r^{\max} = (\text{Max}_{j \in Eff} \{y_{rj}\}) \quad , \quad r = 1, 2, \dots, s, \quad (6)$$

And we will have:

$$\begin{aligned} \text{Max } S &= \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+, \\ \text{s.t. } s_i^- &= \frac{x_{io} - x_i^I}{x_i^{\max}}, \quad i = 1, \dots, m, \\ s_r^+ &= \frac{y_r^I - y_{ro}}{y_r^{\max}}, \quad r = 1, \dots, s, \end{aligned} \quad (7)$$

At the first, we solve model (7) for ranking efficient DMUs, in comparing two DMUs, if a DMU has minimum value of objective function in model (7) (S^*), then it has the best rank.

Theorem.1. The model (7) is feasible and bounded.

Proof: by section 2.2 we have:

$\forall i ; s_i^- \geq 0$, and $s_r^+ \geq 0$, for all r , so clearly, exist a feasible solution for the model (7).

In other hand, it can see for all i and r , $0 \leq \frac{x_{io} - x_i^I}{x_i^{\max}} \leq 1$, $0 \leq \frac{y_r^I - y_{ro}}{y_r^{\max}} \leq 1$,

And because the efficient DMU's are infinite, the most value of s_i^- and s_r^+ , are unique for all i and r .

Furthermore, the values of \underline{m} (number of inputs) and \underline{s} (number of outputs) are infinite too, so the maximum value of objective function is equal to $\underline{m} + \underline{s}$, shown that by S^* , Therefore, the model (7) is bounded. \square

Note.1. It can be seen that the model (7) is unit invariant.

Note.2. from relations (5) and (6), the model above is more stable than the model (4), because the median of the inputs more stable than their maximum when the inputs of inefficient DMUs are change.

Note.3. the model above calculated the distance between an efficient DMU and the ideal point by L_1 -norm.in comparing two DMUs, if a DMU has minimum value of objective function in model (7) then that DMU has the best rank.

4 Examples

Example.1. Consider eight DMUs with one input and one output (Table 1). As you can see, DMUs A–E are BCC efficient and DMU B and D are non-extreme while the others are extreme. The ideal point (IDMU) obtain by (5) and (6) is shown in table 1.

Table 1

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>IDMU</i>
Input	1	2	4	8	12	8	6	11	1
Output	0.5	1	2	3	4	2	1	3	4

Table 2

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
S^*	0.875	0.833333333333332	0.75	0.833333333333333	0.9167
Rank	4	2	1	3	5

A production possibility set (PPS) before and after adding the ideal point is shown in Fig .1. The bold line show PPS before adding the ideal point and the dot line show the PPS after that. The results of the proposed approach are summarized in Table 2.

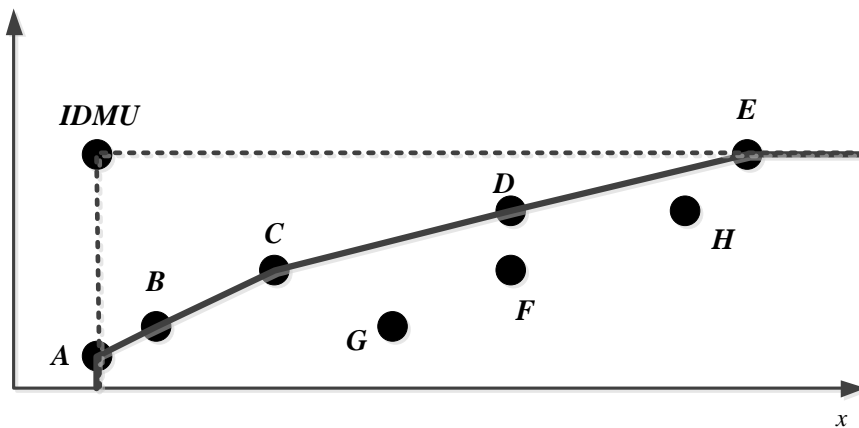


Fig.1. A production possibility set (PPS) before and after adding the ideal point.

Example.2. (An empirical example)

Consider 19 DMUs with two inputs and two outputs (Table 3). Here we apply our new ranking method to the case of that studied in jahanshahloo et al. (2005).

Table 3

<i>DMU</i>	<i>Input 1</i>	<i>Input 2</i>	<i>Output 1</i>	<i>Output 2</i>
1	81	87.6	5191	205
2	85	12.8	3629	0
3	56.7	55.2	3302	0
4	91	78.8	3379	8
5	216	72	5368	639
6	58	25.6	1674	0
7	112.2	8.8	2350	0
8	293.2	52	6315	414
9	186.6	0	2865	0
10	143.4	105.2	7689	66
11	108.7	127	2165	266
12	105.7	134.4	3963	315
13	235	236.8	6643	236
14	146.3	124	4611	128
15	57	203	4869	540
16	118.7	48.2	3313	16
17	58	47.4	1853	230
18	146	50.8	4578	217
19	0	91.3	0	508

DMUs 1, 2, 5, 9, 15 and 19 are CCR Efficient. The results of the proposed method are compared with AP, MAJ and Monte Carlo methods in Table 4.

Table 4

DMU		1	2	5	9	15	19
AP	Value	115	174	130	Inf.	133	Inf.
	Rank	4	1	3	–	2	–
MAJ	Value	105	109	110	104	106	128
	Rank	5	3	2	6	4	1
Monte Carlo	Value	10155	3	74964	2	42252	16
	Rank	3	5	1	6	2	4
Proposed Method	Value	1.8106	1.9846	1.6565	2.4913	1.7856	1.6548
	Rank	4	5	2	6	3	1

5 Ranking by the L_∞ - norm and the ideal point

Consider model (4) again, now we use the L_∞ -norm to calculate the distance between an efficient DMU and the ideal point. We will have:

$$\begin{aligned}
 & \text{Min } \text{Max}_{i,r} \{s_i^-, s_r^+; i = 1, \dots, m, r = 1, \dots, s\} \\
 \text{s.t. } & s_i^- = x_{i_0} - x_i^I, \quad i = 1, \dots, m, \\
 & s_r^+ = y_r^I - y_{r_0}, \quad r = 1, \dots, s,
 \end{aligned} \tag{8}$$

The linear form is:

$$\begin{aligned}
 & \text{Min } \varphi \\
 \text{s.t. } & \varphi \geq x_{i_0} - x_i^I, \quad i = 1, \dots, m, \\
 & \varphi \geq y_r^I - y_{r_0}, \quad r = 1, \dots, s,
 \end{aligned} \tag{9}$$

Where $\varphi = \text{Max}_{i,r} \{s_i^-, s_r^+; i = 1, \dots, m, r = 1, \dots, s\}$. Similar to part 3, in comparing two DMUs, if a DMU has minimum value of objective function in model (9) then that DMU has the best rank.

Theorem.2. The model (9) is feasible and bounded.

The proof is similar to the proof of Theorem 1.

Note.4. for normalized data, because of the special form in this model and for higher benchmarking power we use definitions (5) and (6) again, but we set $j = 1, 2, \dots, n$, to calculated x_i^{\max} and y_r^{\max} .

Note.5. we use this model when there is not exists a zero in input or output data. Because if we have some zeros in data then this model could not rank some DMUs. On this time we suggest to use the model (7) for ranking DMUs.

6 Example

Now the example 1 is revisited using model (9) for ranking. The results of ranking using the proposed method (model (9)) are in Table 4.

Table 5

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
φ^*	0.875	0.75	0.5	0.5833	0.9167
Rank	4	3	1	2	5

7 Conclusion

In this paper, we introduce two ranking methods for extreme and non-extreme efficient decision making units based on the additive model that including to add a virtual DMU as ideal DMU to the model and calculated distance between evaluated DMU's and ideal point L_1 and L_∞ -norms in data envelopment analysis that corresponding problem of each efficient DMU is feasible and bounded. And we can use it for ranking all DMUs at special cases that previous methods could not rank them or they ranked but with hard computing.

Because of removing the inefficient DMUs on first proposed ranking method, we have smaller model and so we have a simple calculating.

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