Available online at http://ijdea.srbiau.ac.ir

Int. J. Data Envelopment Analysis (ISSN 2345-458X)

Vol. 11, No. 3, Year 2023 Article ID IJDEA-00422, Pages 24-32
Research Article

International Journal of Data Envelopment Analysis

Science and Research Branch (IAU)

Fair distribution of weights for ranking decision-making units using cross-efficiency method in DEA

S.S.Kassaei [1], A. Amirteimoori [*] [2], B. Rahmani Parchkolaee [3]

[1] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran,

[2] Department of Applied Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran,

[3] Department of Applied Mathematics, Noor Branch, Islamic Azad University, Noor, Iran

Received 3 April 2023, Accepted 24 June 2023

Abstract

Cross-efficiency is a frequently used method for ranking decision-making units in Data Envelopment Analysis (DEA). A fundamental weakness of this method which has been quite problematic is the presence of multiple optimal weights along with selection of zero values by many of these multiple weights in calculating cross-efficiency. In the current paper it is tried to provide a method which through utilizing fair distribution of weights resolve the mentioned problems and, in this way, give more validity to the cross-efficiency method in raking decision-making units.

Keywords: Data Envelopment Analysis; Cross-efficiency; Zero weights; Fair weights.

[*] Corresponding author: aamirteimoori@gmail.com

1. Introduction

Data Envelopment Analysis (DEA) is a linear programming technique which was introduced for the first time by Charnes et al. [1] for evaluating the relative efficiency of homogeneous decision-making units (DMU) with multiple inputs and multiple outputs. In the classical models of DAE such as CCR and BCC (see Charnes et al., [1]; Banker et al., [2]), the efficiency value of inefficient units equaled less than one and the efficiency value of efficient units equaled one which pose the problem that efficient units cannot be discriminated. There have been so many attempts to solve this problem and improve the discrimination power of decision-making units. As examples of these attempts can be mentioned ranking method developed by Andersen and Petersen [3] work of Jahanshahloo et al. [4] which provides a ranking method based on a full-inefficient frontier or ranking method by Rezai Balf et al. [5] for extreme efficient DMUs using the Tchebycheff norm. Hinojosa et al. [6] used the Shapley value of two different cooperative games in which the players are the efficient DMUs and the characteristic function represents the increase in the discriminant power of DEA contributed by each efficient DMU. Oukil [7] presented a new perspective for ranking DMUs under a DEA peer-evaluation framework. Jin et al. [8], based on regret theory, a regret-rejoice cross-efficiency evaluation (RCEE) model is developed to evaluate the cross-efficiencies of DMUs. Li et al. [9] proposed a cross-efficiency game model considering correlation coordination degree among DMUs.

Cross-efficiency method was first introduced by Sexon et al. [10]. The major problem of this method is multiple optimal solutions, i.e., multiple optimal weights for inputs and outputs of DMUs. In order to solve this problem, Doyle and Green [11] suggested a collection of formulas known as aggressive and benevolent formulas, Wang and Chin [12] also proposed a neutral model, Wang et al. [13] suggested four neutral models based on ideal and anti-ideal DMUs. Jahanshahloo et al. [14] also proposed choosing symmetric weights as a second goal in evaluating cross-efficiency.

In addition to the mentioned problem, i.e., multiple optimal solutions which were present even after modifying cross-efficiency model, there was the problem of obtaining zero values for some of the optimal weights of inputs and outputs. Due to obtaining zero values for some weights, parts of information related to the input and output components which can be influential in decision-making units ranking are eliminated which decreased the validity of the cross-efficiency method. In this field a number of attempts have been presented in order to solve this problem. For instance, through presenting neutral weight determining models, Wang et al. [13] were successful in decreasing the number of zero weights. Also, Nuria Ramón et al. [15] recommend a method for selecting weights.

In this paper, we suggest a method which in addition to solving the presence of multiple optimal solutions and obtaining zero optimal weights, it can be helpful in fair distribution of weights.

The paper is organized as follows: In Section 2, as a literature of DEA, the input oriented CCR multiple models is illustrated. In Section 3, a model is provided for recognizing the presence of the zero weights and in Section 4, a method for selecting optimal weights with fair distribution among them is suggested. In Section 5, the implementation of the cross-efficiency method using fair distribution of the weights is stated and the idea is explained through a numerical example. Finally, at the end of the paper, a summary of the paper along with conclusion are presented.

2. The input oriented CCR multiple model

Assume that DMU_j , $(j = 1,\ldots,n)$ is n homogenous decision-making units. If DMU_j through utilizing input vector $x_j = (x_{1j},\ldots,x_{mj})$ product the output vector $y_j = (y_{1j},\ldots,y_{sj})$, the relative efficiency of the $DMUp$ $(p \in \{1,\ldots,n\})$ is obtained using linear fractional programming problem as it is shown below:

$$
\begin{aligned}
Max \quad & \theta_P = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}} \qquad (1)\\
s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,\cdots,n\\
& v_i \ge 0, \quad i = 1,\cdots,m\\
& u_r \ge 0, \quad r = 1,\cdots,s
\end{aligned}
$$

In this model $u_r(r = 1,\ldots,s)$ and $v_i(i = 1,\ldots,m)$ are the weights of input and output components respectively. By using Charnes and Cooper transformation [16] the model (1) can be transformed into the following linear programming problem known as the input oriented CCR multiple models as follows:

$$
\begin{aligned}
\text{Max} \quad & \theta_P = \sum_{r=1}^{s} u_r y_{rp} \qquad (2)\\
s.t \quad & \sum_{i=1}^{m} v_i x_{ip} = 1,\\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \; j = 1,\cdots,n\\
& v_i \ge 0, \quad i = 1,\cdots,m\\
& u_r \ge 0, \quad r = 1,\cdots,s
\end{aligned}
$$

Assume that θ_p^* is the optimal value of the model (2), if $\theta_p^* < 1$, then DMU_p is inefficient. When $\theta_p^* = 1$, if there are optimal solutions like $u_r^*(r = 1,\ldots,s)$ and $v_i^*(i = 1,\ldots,m)$ such that all the variables are positive, then DMU_p is a strong efficient and is on the strong frontier, else if there are no such optimal solution, then DMU_p is weak efficient. In other words, if DMU_p is the strong efficient with optimal value 1 for the model (2) in evaluating DMU_p , certainly an optimal solution can be found from the model (2) and consequently from the model (1) in a manner that all the weights will be positive. Note that even for inefficient DMUs which the optimal value of the model (2) for them is less than one, if the corresponding image or project units are located on the strong efficiency frontier, such optimal positive solution is obtained. The order of the image unit for DMU_p is the $(\sum_{i=1}^{n} \lambda_j^* x_j, \sum_{i=1}^{n} \lambda_j^* y_j)$, where $(\theta_p^*, \lambda^*) = \left(\theta_p^*, (\lambda_1^*,\ldots,\lambda_n^*)\right)$ is the optimal solution for dual of model (2) known as CCR model in the envelopment form. A model will be proposed in the next section in order to recognize whether there is such solution or not.

3. A model for identifying optimal solutions with minimum number of zero

As it was mentioned in the previous section, if DMU_p is a strong efficient or an inefficient with an image on the strong frontier, it is possible to obtain optimal weights for DMU_p which altogether are positive and the number of zero weights reaches too minimum. In these regards, the following model is suggested:

$$\begin{aligned} & Max \quad \min\{v_1,\ldots,v_m,u_1,\ldots,u_s\} \\ & s.t. \quad \sum_{r=1}^{s} u_r y_{rp} = \theta_p^*, \qquad (3) \\ & \qquad \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j=1,\cdots,n \\ & \qquad u_r \ge 0, \quad r=1,\cdots,s, \\ & \qquad v_i \ge 0, \quad i=1,\cdots,m \end{aligned}$$

In this model, θ_p^* is the optimal value of the model (2) in evaluating the DMU_p. The nonlinear model (3) through defining $z = min\{u_1,\ldots,u_s,v_1,\ldots,v_m\}$ is transformed into linear model as follows:

$$\begin{aligned} & Max \quad z \qquad (4) \\ & s.t. \quad \sum_{r=1}^{s} u_r y_{rp} = \theta_p^*, \\ & \qquad \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j=1,\cdots,n \\ & \qquad z \le u_r, z \le v_i, \qquad r=1,\cdots,s, i=1,\cdots,m \\ & \qquad u_r \ge 0, \quad r=1,\cdots,s, \\ & \qquad v_i \ge 0, \quad i=1,\cdots,m \end{aligned}$$

If z^* be the optimal value of the model (4), the following situations will happen:

a) If $z^* > 0$, depend on whether $\theta_p^* = 1$ or $\theta_p^* < 1$, then DMU_p will be a strong efficient and inefficient respectively but through utilizing the model (4) optimal weights for the model (2) are found which altogether are positive and none of them are zero.

b) If $z^* = 0$, again depend on whether $\theta_p^* = 1$ or $\theta_p^* < 1$, then DMU_p will be a strong efficient and inefficient respectively but $z^* = 0$ shows that among all optimal weights of the model (2) a zero weight will certainly be there.

Pay attention that even in the condition (b), in which among optimal weights of the model (2) zero weight is certainly present, the solution which the model (4) presents, is the optimal solution for the model (2) with less zeros.

As it was stated in the first section, together with obtaining optimal weights with less zeros, we were also looking for restricting multiple optimal solutions. To this end and other strategic aims, the idea of fair distribution of weights have been utilized in a way which is presented in the following section.

4. Fair distribution of the weights

The model presented in this section from optimal weights in the model (2) is looking for solutions which contain a smaller number of zeros and are fairly distributed among input and output components. Fair distribution of weights means weights are selected in such way that by considering the scale of input and output components they have minimum distance from each other. In this article, these weights are used with name of fair weights. In order to get fair weights with less zeros, the model (5) named as F.W (Fair Weight) is proposed below:

$$\max_{\substack{i=1,\ldots,m-1\\ r=1,\ldots,s-1}}\{|\frac{u_{r+1}}{y_{r+1p}}-\frac{u_r}{y_{rp}}|,|\frac{v_{i+1}}{x_{i+1p}}-\frac{v_i}{x_{ip}}|\}$$
$$\sum_{r=1}^{s}u_r y_{rp}=\theta_p^*,\quad \sum_{i=1}^{m}v_i x_{ip}=1, \tag{5}$$
$$\sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}\le 0,\quad j=1,\cdots,n$$
$$z^*\le v_i,\quad i=1,\cdots,m$$
$$z^*\le u_r,\quad r=1,\cdots,s$$
$$v_i\ge 0,\quad i=1,\cdots,m$$
$$u_r\ge 0,\quad r=1,\cdots,s$$

Where θ_p^* is the optimal value of the model (2) and z^* is the optimal value of the model (4). For transforming the model (5) into a linear model the β variable is defined as follows:

$$\beta=\max\{|\frac{u_{r+1}}{y_{r+1p}}-\frac{u_r}{y_{rp}}|,|\frac{v_{i+1}}{x_{i+1p}}-\frac{v_i}{x_{ip}}|\} \tag{6}$$

According to the new β variable, the model (5) can be transformed into linear model as the model (7):

$$\text{Min}\quad \beta \tag{7}$$
$$s.t\quad \sum_{r=1}^{s}u_r y_{rp}=\theta_p^*,$$
$$\sum_{i=1}^{m}v_i x_{ip}=1,$$
$$\sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}\le 0,\ j=1,\cdots,n$$
$$z^*\le v_i,\quad i=1,\cdots,m$$
$$z^*\le u_r,\quad r=1,\cdots,s$$
$$\frac{u_{r+1}}{y_{r+1p}}-\frac{u_r}{y_{rp}}\le\beta\quad r=1,\cdots,s-1$$
$$\frac{u_{r+1}}{y_{r+1p}}-\frac{u_r}{y_{rp}}\ge-\beta\quad r=1,\cdots,s-1$$
$$\frac{v_{i+1}}{x_{i+1p}}-\frac{v_i}{x_{ip}}\le\beta\quad i=1,\cdots,m$$
$$\frac{v_{i+1}}{x_{i+1p}}-\frac{v_i}{x_{ip}}\ge-\beta\quad i=1,\cdots,m$$
$$v_i\ge 0,\quad i=1,\cdots,m$$
$$u_r\ge 0,\quad r=1,\cdots,s$$

5. Implementing cross-efficiency method using fair weights.

In this section the implementation of cross-efficiency method using fair weight is illustrated. The whole process undertaken in this part is the same as the usual cross-efficiency method but the optimal weights of the model (2) for calculating cross-efficiency is defined by the model (6).

Therefore if i be the arbitrary member of $\{1,\ldots,n\}$, in order to calculate the table of cross-efficiency, the model (6) for DMU_i is solved and the optimal solution (u_i^*,v_i^*) is obtained. Now, through utilizing the obtained optimal solution the cross-efficiency of DMU_i is calculated using the relation (8):

$$\theta_{ij} = \frac{U_i^* Y_j}{V_i^* X_j} \quad j = 1, \ldots, n \tag{8}$$

If we reiterate the mentioned process for each $i \in \{1, \ldots, n\}$, finally θ_p^* will be the index for ranking DMU_p as the relation (9):

$$\theta_p = \frac{1}{n} \sum_{i=1}^{n} \theta_{ip} \tag{9}$$

In order to elucidate the effect of utilizing the fair weights in the cross-efficiency ranking method, the following numerical example is provided. This numerical example has been used in the work of Wong et al. [18].

6. Numerical example

Consider seven faculties (DMUs) at a university with three inputs and three outputs which the input and output data along with CCR efficiency are presented in Table 1. The objective is the ranking of these units.

Table 1: The related data to 7 decision-making units

DMU	I1	I2	I3	O1	O2	O3	CCR efficiency
1	12	400	20	60	35	17	1
2	19	750	70	139	41	40	1
3	42	750	70	225	68	75	1
4	15	600	100	90	12	17	089197
5	45	2000	250	253	145	130	1
6	19	730	50	132	45	45	1
7	41	2350	600	305	159	97	1

Table 2 shows the optimal weights obtained from the model (2) and Table 3 shows the optimal weights obtained from the model (6) for corresponding input and output components of each DMU.

Table 2: Optimal weights obtained from the model (2)

DMU	V1	V2	V3	U1	U2	U3
1	0	5.8750	0	0	4.5429	0
2	0	3.1333	0	1.8428	0.6212	0
3	0	0.8784	3.7657	1.3556	0	0
4	2.8868	0.1478	0	2.7780	0	0
5	0.9177	0.0968	0	0.4899	0	0.5936
6	2.1979	0.2318	0	1.1734	0	1.4218
7	0.9837	0.1037	0	0.5252	0	0.6363

Table 3: Optimal weights obtained from the model (6)

DMU	V1	V2	V3	U1	U2	U3
1	1.9630	1.6930	7.8103	1.6930	2.0240	1.6930
2	1.9803	0.3760	0.3760	1.7277	0.3760	0.3760
3	0.2904	0.2904	4.6589	0.6689	0.2904	0.6627
4	2.8868	0.1478	0	2.7780	0	0
5	0.5976	0.3174	0.3174	0.3174	0.4598	0.3174
6	0.9417	1.6867	0.9417	0.9417	0.9417	0.9417
7	0.7102	0.1765	0.1765	0.4176	0.4507	0.1765

Considering Table 2 and Table 3, for optimal weights obtained from the model (2), 19 weights took zero value while in the model (6) just 3 optimal weights took zero value. Through utilizing optimal weights related to table 2 and table 3, the tables of cross-efficiency related to them are presented in Tables 4 and 5.

Table 4: Cross-efficiency resulted from the optimal weights in the model (2)

DMU	1	2	3	4	5	6	7
1	1	0.6248	0.5181	0.2286	0.8286	0.7045	0.7733
2	0.9361	1	0.8152	0.7383	0.7857	1	0.7864
3	0.9696	0.8584	1	0.4695	0.4854	1	0.2919
4	0.6874	1	0.7349	0.8197	0.7649	0.9506	1
5	0.6662	0.9703	0.7666	0.6721	1	1	1
6	0.6662	0.9703	0.7666	0.6721	1	1	1
7	0.6662	0.9703	0.7666	0.6721	1	1	1

Table 5: Cross-efficiency resulted from the optimal weights in the model (6)

DMU	1	2	3	4	5	6	7
1	1	0.8375	0.8654	0.3802	0.7738	1	0.4509
2	0.7803	1	0.7749	0.7174	0.8759	1	0.9327
3	1	0.7691	1	0.3229	0.5980	1	0.2788
4	0.6874	1	0.7349	0.8197	0.7649	0.9506	1
5	0.9160	0.9238	0.7699	0.5099	1	1	0.8598
6	0.9053	0.9196	0.7944	0.5237	0.9321	1	0.7417
7	0.9074	0.9577	0.7570	0.5795	1	1	1

According to the usual cross-efficiency and through utilizing fair weights, the ranking index and the rank of decision-making units will be as follows in Table 6:

Table 6: Ranking DMUs according to the conventional cross-efficiency and fair weights

DMU		1	2	3	4	5	6	7
Conventional cross-efficiency	Ranking index	0.7988	0.9135	0.7665	0.6104	0.8378	0.9507	0.8359
	Ranking	5	2	6	7	3	1	4
Cross-efficiency with fair weights	Ranking index	0.8852	09154	0.8138	0.5505	0.8493	0.9929	0.7520
	Ranking	3	2	5	7	4	1	6

7. Conclusion

Cross-efficiency method faced with the problems of selecting zero optimal weights in calculating cross-efficiency and also multiple optimal weights. In this paper, a method based on fair weights was proposed in order to resolve the mentioned problems. In other words, through selecting optimal weights of multiple CCR model in a fair way and with a smaller number of zeros these problems were somehow solved. It should be mentioned that main idea proposed in the model (6) is not just confined to the cross-efficiency method and it can also be used in the most of methods and discussions which have to do with multiple optimal weights.

References

[1] Charnes A, Cooper WW, Rhodes E. Measuring the efficiency of decision-making units. European journal of operational research. 1978 Nov 1;2(6):429-44.

[2] Banker RD, Charnes A, Cooper WW. Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management science. 1984 Sep;30(9):1078-92.

[3] Andersen P, Petersen NC. A procedure for ranking efficient units in data envelopment analysis. Management science. 1993 Oct;39(10):1261-4.

[4] Jahanshahloo GR, Afzalinejad M. A ranking method based on a full-inefficient frontier. Applied Mathematical Modelling. 2006 Mar 1;30(3):248-60.

[5] Balf FR, Rezai HZ, Jahanshahloo GR, Lotfi FH. Ranking efficient DMUs using the Tchebycheff norm. Applied Mathematical Modelling. 2012 Jan 1;36(1):46-56.

[6] Hinojosa MA, Lozano S, Borrero DV, Mármol AM. Ranking efficient DMUs using cooperative game theory. Expert Systems with Applications. 2017 Sep 1; 80:273-83.

[7] Oukil A. Ranking via composite weighting schemes under a DEA cross-evaluation framework. Computers & Industrial Engineering. 2018 Mar 1; 117:217-24.

[8] Jin F, Cai Y, Pedrycz W, Liu J. Efficiency evaluation with regret-rejoice cross-efficiency DEA models under the distributed linguistic environment. Computers & Industrial Engineering. 2022 Jul 1; 169:108281.

[9] Li M, Wang YM, Lin J. Global correlation coordination model for ranking decision-making units based on cross-efficiency game. Computers & Industrial Engineering. 2023 Nov 1; 185:109649.

[10] Sexton TR, Silkman RH, Hogan AJ. Data envelopment analysis: Critique and extensions. New directions for program evaluation. 1986 Dec;1986(32):73-105.

[11] Doyle J, Green R. Efficiency and cross-efficiency in DEA: Derivations, meanings and uses. Journal of the operational research society. 1994 May 1; 45:567-78.

[12] Wang YM, Chin KS. A neutral DEA model for cross-efficiency evaluation and its extension. Expert Systems with applications. 2010 May 1;37(5):3666-75.

[13] Wang YM, Chin KS, Luo Y. Cross-efficiency evaluation based on ideal and anti-ideal decision-making units. Expert systems with applications. 2011 Aug 1;38(8):10312-9.

[14] Jahanshahloo GR, Lotfi FH, Jafari Y, Maddahi RJ. Selecting symmetric weights as a secondary goal in DEA cross-efficiency evaluation. Applied mathematical modelling. 2011 Jan 1;35(1):544-9.

[15] Ramón N, Ruiz JL, Sirvent I. Reducing differences between profiles of weights: A “peer-restricted” cross-efficiency evaluation. Omega. 2011 Dec 1;39(6):634-41.

[16] Charnes A, Cooper WW. Programming with linear fractional functionals. Naval Research logistics quarterly. 1962 Sep;9(3-4):181-6.