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Efficiency Decomposition in Multi-Stage Network with Inputs and Desirable and Undesirable Outputs in Data Envelopment Analysis

F. Saeedi Aval Noughabi, N. Malekmohammadi[*], Sh. Razavyan
Department of Mathematics, Islamic Azad University, South Tehran Branch, Tehran, Iran

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Abstract
In the previous research, two models were suggested to evaluate the efficiency of decision-making units comprising two and three-stage network structures with desirable and undesirable intermediate measures and outputsand also the proposed models were defined with fuzzy data. In this paper, the mentioned models are improved with a multi-stage network structure. These models are then generalized to evaluate the efficiency of network decision-making units. Finally, a practical example will be presented to show the capability of the proposed model over the previous one.

Keywords: Multi-Stage Network; Data Envelopment Analysis; Desirable and Undesirable Outputs.

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[*] Corresponding author: Email: n_malekmohammadi@azad.ac.ir

1. Introduction

Data envelopment analysis (DEA) is one of the powerful techniques for calculating the efficiency of decision-making units (DMUs) in the presence of multiple inputs and outputs which have been proposed by Cooper et al. [1]. Fare and Grosskopf [2] for the first time delineated this topic, in which in some cases, decision-making units include two or multi-stage processes operating with a structure called a network DEA. Chen et al. [3] presented an efficiency model capable of determining the efficient frontier of a two-stage production process with intermediate measures. Liang et al. [4] suggested a DEA model with a two-stage process based on the game theory and decomposition approach. Kao [5] reviewed data envelopment analysis models with network structure. Ebrahimnejad [6] suggested a method for a three-stage data envelopment analysis model with an application to the banking industry.

In the presence of undesirable data, a DMU is efficient when it has more desirable outputs and less undesirable outputs/inputs. A network with undesirable data is a network with intermediate measures or output containing desirable and undesirable data. Jahanshahloo et al. [7] proposed a model with undesirable data in DEA. Yu et al. [8] suggested a network DEA model to combine operational and environmental performance in an integrated approach considering desirable and undesirable outputs. Podinovski et al. [9] presented two technologies for modeling a weak disposability in a paper entitled modeling weak disposability in data envelopment analysis under relaxed convexity assumptions

On the other side, considering the importance of fuzzy data in real-world problems, Hosseinzadeh Lotfi et al. [10] presented fuzzy data envelopment analysis models with R codes. Olfati et al. [11] proposed fuzzy stochastic undesirable two-stage data envelopment analysis models and applied them to the banking industry. Saeedi et al. ([12,13]) suggested models for the evaluation of the efficiency of the decision-making units comprising a network structure with undesirable intermediate measures in two and three stages. These models are then generalized to evaluate the efficiency of network decision-making units with triangular fuzzy data and undesirable intermediate measures

In this paper, a new model is designed to evaluate a multi-stage network with desirable and undesirable data. With numerical results, the application of the proposed model in the industry has been shown.

This paper proceeds as follows: in the second section, the three-stage network DEA model for the evaluation of the efficiency of decision-making units with undesirable data is presented. This model calculates the efficiency of stages processes under the constraints in which the overall efficiency is maintained at the same level. In the third section, the new model is presented. Finally, the numerical illustration is shown.

2. Envelopment analysis of data with three structures with undesirable data

If the decision-making unit involves two or more processes, it will be called network structure decision-making. If the network outputs or intermediate actions are undesirable, the network is called undesirable data. Assuming that there are n decision-making units (DMUs) and each DMU_j ,j=1,…,n is a multi-stage network, in , x_{kj} ,k=1,…,K, y_{sj} , s=1,…,S, $z^g_{p_1 j}$, p_1 =1,…, P_1 , $z^b_{p_2 j}$, p_2 ‘=1,…, P_2 are, respectively, inputs and desirable and undesirable outputs in the first stage. The desirable outputs of the first stage are used as the inputs of the $S_{۲۱}$ stage; in addition, x'_{dj}, d=1,…, D, are the inputs $y'^{g}_{r_1 j}$, r_1 =1,…, R_1, $y'^{b}_{r_1 j}$, r_1 =1,…, R_2.

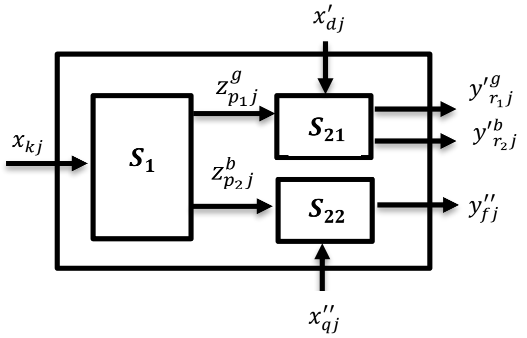

Figure 1. Third stage process with desirable and undesirable outputs

and desirable and undesirable outputs of the second stage, respectively. The undesirable outputs of the first stage are used as the inputs of the third stage; in addition, x''_{qj}, q=1, …, Q, y''_{fj}, f=1, …, F, are the inputs and outputs of the $S_{۲۲}$ stages, respectively.

2.1 Efficiency value in three Stages

Let e_1, e_{21} and e_{22} be the efficiency of DMU_o in the S_1, S_{21} and S_{22} stages, respectively, which are obtained using the SBM model (presented by Tone [12] in 2001).

$$e_1 = Min(1-\frac{1}{K}\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}})$$

$$e_{21} = Min(1-\frac{1}{P+D}(\sum_{p_1=1}^{P_1}\frac{s_{p_1}^{-2}}{Z_{p_1 o}^{g}}+\sum_{d=1}^{Q}\frac{s_d^{-4}}{x'_{do}}))$$

$$e_{22} = Min(1-\frac{1}{P+Q}(\sum_{p_2=1}^{P_2}\frac{s_{p_2}^{-2}}{Z_{p_2 o}^{b}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}))$$

Definition1: DMU_o is efficient in S_1, S_{21} and S_{22} stages, respectively, when e_1 =1, e_{21} =1 and e_{22} =1.

According to Saeedi et al. [13], it is recommended to combine these three stages with the mean weight of efficiency scores from the first, second and third stages, as follows:

$$\begin{aligned} e_o = {} & w_1(1-\frac{1}{K}\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}) \\ & +w_2(1-\frac{1}{P+D}(\sum_{p_1=1}^{P_1}\frac{s_{p_1}^{-21}}{Z_{p_1 o}^{g}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}})) \\ & +w_3(1-\frac{1}{P+Q}(\sum_{p_2=1}^{P_2}\frac{s_{p_2}^{-22}}{Z_{p_2 o}^{b}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}})) \end{aligned} \quad (1)$$

where w_1, w_2 and w_3 are, respectively, the significance or relative efficiency ratio of the first second and third stages to the overall efficiency of the DMU during the entire process. Therefore, w_1, w_2 and w_3, are defined as follows:

$$w_1 = \frac{(1-\frac{1}{K+D+Q}(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}))}{3(1-\frac{1}{K}\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}})},$$

$$w_2 = \frac{(1-\frac{1}{K+D+Q}(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}))}{3(1-\frac{1}{P+D}(\sum_{p_1=1}^{P_1}\frac{s_p^{-21}}{Z_{p_1 o}^{g}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}))}, \quad (2)$$

$$w_3 = \frac{(1-\frac{1}{K+D+Q}(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}))}{3(1-\frac{1}{P+Q}(\sum_{p_2=1}^{P_2}\frac{s_{p_2}^{-22}}{Z_{p_2 o}^{b}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}))}$$

As a result, the overall efficiency score of the three-stage process obtained with SBM

$$\begin{aligned} & e_o^* = Min(1-\frac{1}{K+D+Q}(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}})) \\ & s.t. \quad \sum_{j=1}^{n}\lambda_j x_{kj} + s_k^{-1} = x_{ko}, \qquad k=1,...,K \\ & \sum_{j=1}^{n}\lambda_j z_{p_1 j}^{g} \ge \sum_{j=1}^{n}(\rho_j+\beta_j) z_{p_1 j}^{g}, \quad p_1 = 1,...,P_1 \\ & \sum_{j=1}^{n}\lambda_j z_{p_2 j}^{b} = \sum_{j=1}^{n}\mu'_j z_{p_2 j}^{b} = z_{p_2 o}^{b}, \quad p_2 = 1,...,P \qquad (3) \\ & \sum_{j=1}^{n}\lambda_j y_{sj} - s_s^{+3} = y_{so}, \qquad s=1,...,S \\ & \sum_{j=1}^{n}(\rho_j+\beta_j) x'_{dj} + s_d^{-4} = x'_{do}, \quad d=1,...,D \\ & \sum_{j=1}^{n}\rho_j y'^{g}_{r_1 j} - s_{r_1}^{+51} = y'^{g}_{r_1 o}, \qquad r_1 = 1,...,R_1, \end{aligned}$$

$$\sum_{j=1}^{n}\rho_j y'^{b}_{r_2 j} = y'^{b}_{r_2 o}, \qquad r_2 = 1,\dots,R_2,$$
$$\sum_{j=1}^{n}\mu'_j x''_{qj} + s^{-6}_q = x''_{qo}, \qquad q = 1,\dots,Q,$$
$$\sum_{j=1}^{n}\mu'_j y''_{fj} - s^{+7}_f = y''_{fo}, \qquad f = 1,\dots,F,$$
$$\sum_{j=1}^{n}\lambda_j = 1, \sum_{j=1}^{n}\mu'_j = 1, \sum_{j=1}^{n}(\rho_j+\beta_j) = 1,$$
$$\lambda_j \ge 0, \beta_j \ge 0, \rho_j \ge 0, \mu'_j \ge 0 \quad \forall j,$$
$$s^{+51}_{r_1} \ge 0, s^{-4}_d \ge 0 \qquad \forall r_1, \forall d,$$
$$s^{+7}_f \ge 0, s^{-6}_q \ge 0 \qquad \forall f, \forall q,$$
$$s^{+3}_s \ge 0, s^{-1}_k \ge 0 \qquad \forall s, \forall k,$$

DMU_o is overall efficient if and only if the DMU is efficient at the S_1, S_{21} and S_{22} stages, respectively, when $e_1=1$, e_{21} =1and e_{22} =1If the DMU is only efficient in the S_1 or S_{21} or S_{22} stage, then the decision-making unit is not overall efficient.

2.2 Efficiency decomposition of three stages

The optimal solution obtained from Model (3), can be used to calculate the efficiency scores in the first, second and third stages. Nevertheless, Model (3) can have alternative optimal solutions; the overall efficiency decomposition in Equation (1) may not be unique. Assuming that the overall efficiency of DMU_o obtained from model (3) is e^*_o, and the overall score of the S_1 stages is maximized S_{22} و S_{21} ، by maintaining the overall efficiency e^*_o, then we have

$$e^*_1 = Min(1-\frac{1}{K}\sum_{k=1}^{K}\frac{s^{-1}_k}{x_{ko}})$$
$$s.t. \quad \sum_{j=1}^{n}\lambda_j x_{kj} + s^{-1}_k = x_{ko}, \quad k=1,\dots,K$$
$$\sum_{j=1}^{n}\lambda_j z^{g}_{p_1 j} - s^{+21}_{p_1} = z^{g}_{p_1 o}, \quad p_1 = 1,\dots,P_1$$
$$\sum_{j=1}^{n}\lambda_j z^{b}_{p_2 j} - s^{+22}_{p_2} = z^{b}_{p_2 o}, \quad p_2 = 1,\dots,P_2$$
$$\sum_{j=1}^{n}\lambda_j y_{sj} - s^{+3}_s = y_{so}, \quad s=1,\dots,S$$
$$\left(1-\frac{1}{K+D+Q}\left(\sum_{k=1}^{K}\frac{s^{-1}_k}{x_{ko}}+\sum_{d=1}^{D}\frac{s^{-4}_d}{x'_{do}}+\sum_{q=1}^{Q}\frac{s^{-6}_q}{x''_{qo}}\right)\right) = e^*_o$$
$$\sum_{j=1}^{n}\lambda_j = 1 \tag{4}$$
$$\lambda_j \ge 0, s^{+3}_s \ge 0, s^{-1}_k \ge 0, s^{-4}_d \ge 0, \forall s, \forall k, \forall d$$
$$s^{+21}_{p_1} \ge 0, s^{+22}_{p_2} \ge 0, s^{-6}_q \ge 0, \forall p_1, \forall p_2, \forall q.$$

and

$$e^*_{21} = Min(1-\frac{1}{P+D}(\sum_{p_1=1}^{P_1}\frac{s^{-2}_{p_1}}{Z^{g}_{p_1 o}}+\sum_{d=1}^{Q}\frac{s^{-4}_d}{x'_{do}}))$$
$$s.t \quad \sum_{j=1}^{n}(\rho_j+\beta_j)x'_{dj} + s^{-4}_d = x'_{do} \quad d=1,\dots,D$$
$$\sum_{j=1}^{n}(\rho_j+\beta_j)z^{g}_{pj} + s^{-21}_p = z^{g}_{po} \quad p_1 = 1,\dots,P_1$$
$$\sum_{j=1}^{n}\rho_j y'^{g}_{r_1 j} - s^{+51}_{r_1} = y'^{g}_{ro} \quad r_1 = 1,\dots,R_1$$
$$\sum_{j=1}^{n}\rho_j y'^{b}_{r_2 j} = y'^{b}_{r_2 o} \quad r_2 = 1,\dots,R_2 \tag{5}$$
$$(1-\frac{1}{K+D+Q}(\sum_{k=1}^{K}\frac{s^{-1}_k}{x_{ko}}+\sum_{d=1}^{D}\frac{s^{-4}_d}{x'_{do}}+\sum_{q=1}^{Q}\frac{s^{-6}_q}{x''_{qo}})) = e^*_o$$
$$\sum_{j=1}^{n}(\rho_j+\beta_j) = 1$$
$$\beta_j \ge 0, \rho_j \ge 0, s^{-6}_q \ge 0, s^{-1}_k \ge 0 \quad \forall j, \forall q, \forall k$$
$$s^{-21}_{p_1} \ge 0, s^{+51}_{r_1} \ge 0, s^{-4}_d \ge 0 \quad \forall p_1, \forall r_1, \forall d.$$

$$
\begin{aligned}
& e_{22}^{*}=Min(1-\frac{1}{P+Q}(\sum_{p_2=1}^{P_2}\frac{s_{p_2}^{-2}}{Z_{p_2o}^{b}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}})) \\
& s.t\ \sum_{j=1}^{n}\mu'_j x''_{qj}+s_q^{-6}=x''_{qo} \qquad q=1,...,Q \\
& \sum_{j=1}^{n}\mu'_j z_{p_2 j}^{b}+s_{p_2}^{-22}=z_{p_2o}^{b} \qquad p_2=1,...,P_2 \\
& \sum_{j=1}^{n}\mu'_j y''_{fj}-s_f^{+7}=y''_{fo} \qquad f=1,...,F \qquad (6) \\
& (1-\frac{1}{K+D+Q}(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}))=e_o^{*} \\
& \sum_{j=1}^{n}\mu'_j=1 \\
& \mu'_j\geq 0, s_k^{-1}\geq 0, s_d^{-4}\geq 0 \qquad \forall j,\forall k,\forall d \\
& s_{p_1}^{-21}\geq 0, s_f^{+7}\geq 0, s_q^{-6}\geq 0 \qquad \forall p_1,\forall f,\forall q
\end{aligned}
$$

3. Multi-stage network with desirable and undesirable input and output

Assuming that there are n decision-making units (DMU) and each DMU_j, j=1,…, n is a multi-stage network, Also, x_{kj}, k=1,…,K, y_{sj}, s=1,…,S, $z^{g}_{p_1 j}$, p_1=1,…, P_1, $z^{b}_{p_2 j}$, p_2 ،=1,…, P_2 are, respectively, inputs and desirable and undesirable outputs in the first stage. The desirable outputs of the first stage are used as the inputs of the $S_{\text{٢١}}$ stage; in addition x'_{dj}, d=1, ..., D, $y'^{g}_{r_1 j}$, r_1=1,…, R_1, $y'^{b}_{r_2 j}$, r_2 =1,…, R_2 are the inputs and desirable and undesirable outputs of the second stage, respectively. The undesirable outputs of the first stage are used as the inputs of the third stage; in addition, x''_{qj},q=1,…, Q, W_{tj} , t=1, …, T are the inputs and outputs of the $S_{\text{٢٢}}$ stages, respectively. W_{tj}, t=1, …, T. x'''_{mj}, m=1,…,M, y''_{fj}, f=1,…,F, are the inputs and outputs of the $S_{\text{٣}}$ stages, respectively.

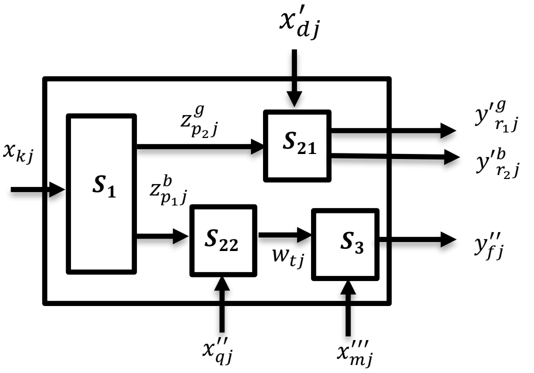

Figure 2. multi-stage network with desirable and undesirable outputs

3.1 Efficiency valuation of multi-stage networks

Let e_1 e_{22}, e_{21}, and e_3 be the efficiency of DMU_o in the S_1, S_{21}, S_{22} and S_3 stages, respectively, which are obtained using the SBM model (presented by Tone [14] in 2001).

Definition 2.1: DMU_o is efficient in S_1, S_{21}, S_{22} and S_3 stages, respectively, when e_1=1, e_{21}=1, e_{22}=1 and e_3=1.

$$
\begin{aligned}
& e_1=Min(1-\frac{1}{K}\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}) \\
& s.t.\ \sum_{j=1}^{n}\lambda_j x_{kj}+s_k^{-1}=x_{ko} \qquad k=1,...,K \\
& \sum_{j=1}^{n}\lambda_j z_{p_1 j}^{g}-s_{p_1}^{+21}=z_{p_1o}^{g} \qquad p_1=1,...,P_1 \\
& \sum_{j=1}^{n}\lambda_j z_{p_2 j}^{b}-s_{p_2}^{+22}=z_{p_2o}^{b} \qquad p_2=1,...,P_2 \qquad (7) \\
& \sum_{j=1}^{n}\lambda_j y_{sj}-s_s^{+3}=y_{so} \qquad s=1,...,S \\
& \sum_{j=1}^{n}\lambda_j=1 \\
& \lambda_j\geq 0, s_s^{+3}\geq 0, s_k^{-1}\geq 0, s_d^{-4}\geq 0 \qquad \forall s,\forall k,\forall d \\
& s_{p_1}^{+21}\geq 0, s_{p_2}^{+22}\geq 0, s_q^{-6}\geq 0 \qquad \forall p_1,\forall p_2,\forall q.
\end{aligned}
$$

$$e_{21}=Min(1-\frac{1}{P+D}(\sum_{p_1=1}^{P_1}\frac{s_{p_1}^{-2}}{Z_{p_1o}^{g}}+\sum_{d=1}^{Q}\frac{s_d^{-4}}{x_{do}^{'}}))$$

$$\begin{aligned}
s.t\ & \sum_{j=1}^{n}\mu_j x_{dj}^{'}+s_d^{-4}=x_{do}^{'} && d=1,\dots,D\\
& \sum_{j=1}^{n}\mu_j z_{p_1j}^{g}+s_{p_1}^{-21}=z_{p_1o}^{g} && p_1=1,\dots,P_1\\
& \sum_{j=1}^{n}\mu_j\theta_j y_{r_1j}^{'g}-s_{r_1}^{+51}=y_{r_1o}^{'g} && r_1=1,\dots,R_1\\
& \sum_{j=1}^{n}\mu_j\theta_j y_{r_2j}^{'}=y_{r_2o}^{'} && r_2=1,\dots,R_2\\
& \sum_{j=1}^{n}\mu_j=1\\
& \mu_j\geq 0, 0\leq\theta_j\leq 1 && j=1,\dots,n\\
& s_{p_1}^{-21}\geq 0, s_{r_1}^{+51}\geq 0, s_d^{-4}\geq 0 && \forall p_1,\forall r_1,\forall d
\end{aligned} \tag{8}$$

In order to Model (8) to the linear form, the relations (9) are applied.

$$\begin{cases}\mu_j\theta_j=\rho_j\\ \mu_j(1-\theta_j)=\beta_j\\ \rho_j+\beta_j=\mu_j\end{cases} \tag{9}$$

So:

$$e_{21}=Min(1-\frac{1}{P+D}(\sum_{p_1=1}^{P_1}\frac{s_{p_1}^{-2}}{Z_{p_1o}^{g}}+\sum_{d=1}^{Q}\frac{s_d^{-4}}{x_{do}^{'}}))$$

$$\begin{aligned}
s.t\ & \sum_{j=1}^{n}(\rho_j+\beta_j)x_{dj}^{'}+s_d^{-4}=x_{do}^{'} && d=1,\dots,D\\
& \sum_{j=1}^{n}(\rho_j+\beta_j)z_{pj}^{g}+s_p^{-21}=z_{po}^{g} && p_1=1,\dots,P_1\\
& \sum_{j=1}^{n}\rho_j y_{r_1j}^{'g}-s_{r_1}^{+51}=y_{ro}^{'g} && r_1=1,\dots,R_1\\
& \sum_{j=1}^{n}\rho_j y_{r_2j}^{\prime b}=y_{r_2o}^{\prime b} && r_2=1,\dots,R_2\\
& \sum_{j=1}^{n}(\rho_j+\beta_j)=1\\
& \beta_j\geq 0,\rho_j\geq 0, && \forall j,\\
& s_{p_1}^{-21}\geq 0, s_{r_1}^{+51}\geq 0, s_d^{-4}\geq 0 && \forall p_1,\forall r_1,\forall d.
\end{aligned} \tag{10}$$

$$e_{22}=Min(1-\frac{1}{P+Q}(\sum_{p_2=1}^{P_2}\frac{s_{p_2}^{-2}}{Z_{p_2o}^{b}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x_{qo}''}))$$

$$\begin{aligned}
s.t.\ & \sum_{j=1}^{n}\mu_j' x_{qj}''+s_q^{-6}=x_{qo}'', q=1,\dots,Q\\
& \sum_{j=1}^{n}\mu_j' z_{p_2j}^{b}+s_{p_2}^{-22}=z_{p_2o}^{b}, p_2=1,\dots,P_2\\
& \sum_{j=1}^{n}\mu_j' W_{tj}-s_f^{+8}=y_{do}, t=1,\dots,T\\
& \sum_{j=1}^{n}\mu_j'=1\\
& \mu_j'\geq 0, s_t^{+8}\geq 0\\
& \sum_{j=1}^{n}\mu_j' W_{tj}-s_f^{+8}=y_{do}, t=1,\dots,T\\
& \sum_{j=1}^{n}\mu_j'=1\\
& \mu_j'\geq 0, s_t^{+8}\geq 0, \forall j,\forall t\\
& s_{p_1}^{-22}\geq 0, s_q^{-6}\geq 0, \forall p_2,\forall q
\end{aligned} \tag{11}$$

$$e_3=Min(1-\frac{1}{T+M}(\sum_{t=1}^{T}\frac{s_t^{-10}}{W_{to}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x_{mo}'''}))$$

$$\begin{aligned}
s.t\ & \sum_{j=1}^{n}\gamma_j x_{mj}'''+s_m^{-9}=x_{mo}''', && m=1,\dots,M\\
& \sum_{j=1}^{n}\gamma_j W_{tj}+s_t^{-10}=W_{to}, && t=1,\dots,T\\
& \sum_{j=1}^{n}\gamma_j y_{fj}''-s_f^{+7}=y_{fo}'', && f=1,\dots,F\\
& \sum_{j=1}^{n}\gamma_j=1,\\
& \gamma_j\geq 0, s_t^{-4}\geq 0, && \forall j,\forall t,\\
& s_f^{+7}\geq 0, s_m^{-6}\geq 0, && \forall f,\forall m
\end{aligned} \tag{12}$$

The optimal solutions are obtained by solving Models (7), (10), (11) and (12) and we always have $0<e_1\leq 1$, $0<e_{21}\leq 1$, $0<e_{22}\leq 1$ and $0<e_3\leq 1$.

Definition2: DMU_o is efficient in S_1, S_{21} S_{22} and S_3 stages, respectively, when e_1 =1, e_{21} =1, e_{22} =1and e_3 =1

As mentioned earlier, the intermediate measures are precisely modeled by separately applying Models (7), (10), (11) and (12). Similar to Section 2.1., the mean weight of the efficiency scores from the S_1, S_{21} S_{22} and S_3 stages are calculated as follows.

$$e_o = w_1\left(1-\frac{1}{K}\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}\right) \tag{13}$$
$$+w_{21}\left(1-\frac{1}{P+D}\left(\sum_{p_1=1}^{P_1}\frac{s_{p_1}^{-21}}{Z_{p_1 o}^{g}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}\right)\right)$$
$$+w_{22}\left(1-\frac{1}{P+Q}\left(\sum_{p_2=1}^{P_2}\frac{s_{p_2}^{-22}}{Z_{p_2 o}^{b}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}\right)\right)$$
$$+w_3\left(1-\frac{1}{T+M}\left(\sum_{t=1}^{T}\frac{s_t^{-9}}{W_{to}}+\sum_{m=1}^{M}\frac{s_q^{-10}}{x'''_{mo}}\right)\right)$$

where w_1, w_{21}, w_{22} and w_3 ,respectively, the significance or relative efficiency ratio of the S_1, S_{21}, S_{22} and S_3 stages to the overall efficiency of the DMU during the entire process. Therefore, w_1, w_{21}, w_{22} and w_3, are defined as follows.

$$w_1=\frac{\left(1-\frac{1}{K+D+Q+M}\left(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}}\right)\right)}{4\left(1-\frac{1}{K}\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}\right)},$$
$$w_{21}=\frac{\left(1-\frac{1}{K+D+Q+M}\left(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}}\right)\right)}{4\left(1-\frac{1}{P+D}\left(\sum_{p_1=1}^{P_1}\frac{s_p^{-21}}{Z_{p_1 o}^{g}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}\right)\right)}, \tag{14}$$
$$w_{22}=\frac{\left(1-\frac{1}{K+D+Q+M}\left(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}}\right)\right)}{4\left(1-\frac{1}{P+Q}\left(\sum_{p_2=1}^{P_2}\frac{s_{p_2}^{-22}}{Z_{p_2 o}^{b}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}\right)\right)}$$
$$w_3=\frac{\left(1-\frac{1}{K+D+Q+M}\left(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}}\right)\right)}{4\left(1-\frac{1}{T+M}\left(\sum_{t=1}^{T}\frac{s_t^{-10}}{W_{to}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}}\right)\right)}$$

As a result, the overall efficiency score of the three-stage process obtained with SBM, DMU_o is overall efficient if and only if the DMU is efficient at the S_1, S_{21}, S_{22} and S_3 stages, respectively, when e_1 =1 , e_{21} =1, e_{22} =1 and e_3 =1 .If the DMU is only efficient in the S_1 or S_{21} or S_{22} or S_3 stage, then the decision-making unit is not overall efficient.

$$e_o^* = Min\left(\begin{array}{l}1-\frac{1}{K+D+Q+M}\\ \left(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}}\right)\end{array}\right)$$
$$s.t.\quad \sum_{j=1}^{n}\lambda_j x_{kj}+s_k^{-1}=x_{ko},\qquad k=1,\dots,K \tag{15}$$
$$\sum_{j=1}^{n}\lambda_j z_{p_1 j}^{g}\ge\sum_{j=1}^{n}(\rho_j+\beta_j)z_{p_1 j}^{g},\ p_1=1,\dots,P_1$$

$$\sum_{j=1}^{n}\lambda_j z^b_{p_2 j}=\sum_{j=1}^{n}\mu'_j z^b_{p_2 j}=z^b_{p_2 o},\ p_2=1,...,P_2$$
$$\sum_{j=1}^{n}\lambda_j y_{sj}-s_s^{+3}=y_{so},\quad s=1,...,S$$
$$\sum_{j=1}^{n}(\rho_j+\beta_j)x'_{dj}+s_d^{-4}=x'_{do},\ d=1,...,D$$
$$\sum_{j=1}^{n}\rho_j y'^g_{r_1 j}-s_{r_1}^{+51}=y'^g_{r_1 o},\quad r_1=1,...,R_1$$
$$\sum_{j=1}^{n}\rho_j y'^b_{r_2 j}=y'^b_{r_2 o},\qquad r_2=1,...,R_2$$
$$\sum_{j=1}^{n}\mu'_j x''_{qj}+s_q^{-6}=x''_{qo},\quad q=1,...,Q$$
$$\sum_{j=1}^{n}\gamma_j y''_{fj}-s_f^{+7}=y''_{fo}\qquad f=1,...,F$$
$$\sum_{j=1}^{n}\gamma_j x'''_{mj}+S_m^{-9}=x_{to}\qquad m=1,...,M$$
$$\sum_{j=1}^{n}\mu'_j W_{tj}=\sum_{j=1}^{n}\gamma_j W_{tj}=W_{to}\qquad t=1,...,T$$
$$\sum_{j=1}^{n}\lambda_j=1,\sum_{j=1}^{n}\mu'_j=1,\sum_{j=1}^{n}(\rho_j+\beta_j)=1,\sum_{j=1}^{n}\gamma_j=1$$
$$\lambda_j\ge 0,\beta_j\ge 0,\rho_j\ge 0,\mu'_j\ge 0,\quad \forall j,$$
$$s_{r_1}^{+51}\ge 0,s_d^{-4}\ge 0,\gamma_j\ge 0,\qquad \forall j,\forall r_1,\forall d,$$
$$s_f^{+7}\ge 0,s_q^{-6}\ge 0,S_m^{-9}\ge 0\qquad \forall f,\forall q,\forall m$$
$$s_s^{+3}\ge 0,s_k^{-1}\ge 0\qquad \forall s,\forall k,$$

3.2 Efficiency decomposition of multi-stage Networks

The optimal solution obtained from Model (15), can be used to calculate the efficiency scores in the first, second and third stages. Nevertheless, Model (15) can have alternative optimal solutions; the overall efficiency decomposition in Equation (13) may not be unique. Assuming that the overall efficiency of DMU_o obtained from model (15) is e_o^*, and the overall score of the S_1, S_{21}, S_{22} and S_3 stages is maximized by maintaining the overall efficiency e_o^*, then we have

$$e_1^*=Min(1-\frac{1}{K}\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}})\qquad(16)$$
$$s.t.\quad \sum_{j=1}^{n}\lambda_j x_{kj}+s_k^{-1}=x_{ko},\qquad k=1,...,K$$
$$\sum_{j=1}^{n}\lambda_j z^g_{p_1 j}-s_{p_1}^{+21}=z^g_{p_1 o},\qquad p_1=1,...,P_1$$
$$\sum_{j=1}^{n}\lambda_j z^b_{p_2 j}-s_{p_2}^{+22}=z^b_{p_2 o},\qquad p_2=1,...,P_2$$
$$\sum_{j=1}^{n}\lambda_j y_{sj}-s_s^{+3}=y_{so},\qquad s=1,...,S$$
$$1-\frac{1}{K+D+Q+M}(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}})=e_o^*$$
$$\lambda_j\ge 0,s_s^{+3}\ge 0,s_k^{-1}\ge 0,s_d^{-4}\ge 0,\forall s,\forall k,\forall d$$
$$s_{p_1}^{+21}\ge 0,s_{p_2}^{+22}\ge 0,s_q^{-6}\ge 0,\quad \forall p_1,\forall p_2,\forall q.$$

$$e_{21}^*=Min(1-\frac{1}{P+D}(\sum_{p_1=1}^{P_1}\frac{s_{p_1}^{-2}}{Z^g_{p_1 o}}+\sum_{d=1}^{Q}\frac{s_d^{-4}}{x'_{do}}))$$
$$s.t\quad \sum_{j=1}^{n}(\rho_j+\beta_j)x'_{dj}+s_d^{-4}=x'_{do}\qquad d=1,...,D$$
$$\sum_{j=1}^{n}(\rho_j+\beta_j)z^g_{pj}+s_p^{-21}=z^g_{po}\qquad p_1=1,...,P_1$$
$$\sum_{j=1}^{n}\rho_j y'^g_{r_1 j}-s_{r_1}^{+51}=y'^g_{ro}\qquad r_1=1,...,R_1$$
$$\sum_{j=1}^{n}\rho_j y'^b_{r_2 j}=y'^b_{r_2 o}\qquad r_2=1,...,R_2\qquad(17)$$
$$1-\frac{1}{K+D+Q+M}(\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}})=e_o^*$$
$$\sum_{j=1}^{n}(\rho_j+\beta_j)=1$$
$$\beta_j\ge 0,\rho_j\ge 0,s_q^{-6}\ge 0,s_k^{-1}\ge 0\quad \forall j,\forall q,\forall k$$
$$s_{p_1}^{-21}\ge 0,s_{r_1}^{+51}\ge 0,s_d^{-4}\ge 0\qquad \forall p_1,\forall r_1,\forall d.$$

$$e_{22}=Min(1-\frac{1}{P+Q}(\sum_{p_2=1}^{P_2}\frac{s_{p_2}^{-2}}{Z_{p_2o}^{b}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}))$$

$$\begin{aligned}
s.t.\quad & \sum_{j=1}^{n}\mu'_j x''_{qj}+s_q^{-6}=x''_{qo}, && q=1,\dots,Q\\
& \sum_{j=1}^{n}\mu'_j z_{p_2j}^{b}+s_{p_2}^{-22}=z_{p_2o}^{b}, && p_2=1,\dots,P_2\\
& \sum_{j=1}^{n}\mu'_j W_{tj}-s_f^{+8}=y_{do}, && t=1,\dots,T\\
& 1-\frac{1}{K+D+Q+M}\\
& (\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}})=e_o^*\\
& \sum_{j=1}^{n}\mu'_j=1 && (18)\\
& \mu'_j\ge 0, s_t^{+8}\ge 0, && \forall j,\forall t\\
& s_{p_1}^{-22}\ge 0, s_q^{-6}\ge 0, && \forall p_2,\forall q
\end{aligned}$$

$$e_3=Min(1-\frac{1}{T+M}(\sum_{t=1}^{T}\frac{s_t^{-10}}{W_{to}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}}))$$

$$\begin{aligned}
s.t.\quad & \sum_{j=1}^{n}\gamma_j x'''_{mj}+s_m^{-9}=x'''_{mo}, && m=1,\dots,M\\
& \sum_{j=1}^{n}\gamma_j W_{tj}+s_t^{-10}=W_{to}, && t=1,\dots,T\\
& \sum_{j=1}^{n}\gamma_j y''_{fj}-s_f^{+7}=y''_{fo}, && f=1,\dots,F\\
& 1-\frac{1}{K+D+Q+M}\\
& (\sum_{k=1}^{K}\frac{s_k^{-1}}{x_{ko}}+\sum_{d=1}^{D}\frac{s_d^{-4}}{x'_{do}}+\sum_{q=1}^{Q}\frac{s_q^{-6}}{x''_{qo}}+\sum_{m=1}^{M}\frac{s_m^{-9}}{x'''_{mo}})=e_o^*\\
& \sum_{j=1}^{n}\gamma_j=1, && (19)\\
& \gamma_j\ge 0, s_t^{-4}\ge 0, && \forall j,\forall t,\\
& s_f^{+7}\ge 0, s_m^{-6}\ge 0, && \forall f,\forall m
\end{aligned}$$

4. Numerical example

In this section, the furniture services production and the chipboard industries of wood lumber are shown as a multi-stage process. In the S_1 stage, the lumber, glue, and wood are introduced into the system to produce tables and furniture's coil. The coil of furniture is a desirable output of the first stage, which enters the S_{21} stage as an intermediate measure, and fabric, foam(1cm), foam (2.5 cm and 10 cm) is added to it, and the efurniture is produced. In this stage, the average furniture service and the profit are outputs.

Table 1. Input and Output S_1

Workshop (DMUj)	wood (lumber) (500 kg/m^3)	Number of nails	glue (cc)	Average Table	Average coilfs produced for a sofa service	Wood chip (kg)
1	6478	36670	23370	702	312	1200
2	9840	46125	664200	1581	527	1400
3	3250	19500	27078	783	261	730
4	5090	30528	400682	954	318	940
5	4380	23085	283450	767	289	800
6	2965	15970	129531	304	188	674
7	759	7625	55969	150	50	511
8	668	5300	89509	75	25	524
9	4221	23500	193334	620	272	777

Table 2. Input and Output S_{21} stage Network

workshop	Average coils	Fabric (m)	Foam(10cm)	Foam(2.5cm)
1	312	10639	403	816
2	527	16297	655	1242
3	261	6642	253	501
4	318	8586	343	658
5	289	7882	328	625
6	188	4978	221	402
7	50	1500	57	140
8	25	660	28	55
9	272	7316	323	578

Table 3. Input and Output S_{22} stage Network

workshop	Wood chip	additives	chipboard
1	1200	74	1232
2	1400	84	1428
3	730	40	750
4	940	69	985
5	800	50	858
6	674	29	600
7	511	25	521
8	524	24	542
9	777	53	785

Table 4. Input and Output S_3 stage Network

workshop	chiooard	additives	false ceiling
1	1232	1915	4921
2	1428	2025	5451
3	750	1845	3825
4	985	1676	3776
5	858	1748	3939
6	600	2052	3029
7	521	1438	1720
8	542	912	2762
9	785	1362	3472

Table 5. The overall efficiency scores and efficiency scores of the S_1, the S_{21}, the S_{22} and the S_3 stages

workshop	e1	e21	e22	e3	eo
1	0.72	0.838	0.983	0.991	0.994
2	1	1	1	1	1
3	1	1	0.911	1	0.995
4	0.876	0.994	0.904	0.842	0.947
5	0.833	0.945	1	0.971	0.95
6	0.795	0.951	0.913	0.774	0.923
7	1	1	0.954	1	0.999
8	1	1	1	1	1
9	1	0.937	0.873	0.937	0.914

The results of S_{21} stage of the production of this furniture are desirable output and Fabric waste (m) is an undesirable output. Some wood chips are produced in the first process, which is undesirable output data, and enters the S_{22} stage as intermediate measures, adding additives as input are added to it, and produces chipboard and the chipboard produced on the S_{22} stage as intermediate measures, mixed with polyvinyl chloride and additives as input the S_3 stage and the false ceiling is produced.

Using the Models (15), (16),(17) and (18), The system efficiency and the efficiency of the S_1, S_{21}, S_{22} and S_3 stages are obtained.

Table 5 shows the results for e_o^*, e_1 , e_{21}, e_{22} and e_3 refer to system, Process 1, Process 21, Process22 and Process3 ,respectively.

According to Table 5, the overall efficiency scores and the efficiency scores of S_1, S_{21}, S_{22} and S_3 stages of the eighth and second workshops are equal to1. So, the eighth and the second workshops are efficient. Also, the efficiency score of the S_1, S_{21} and S_3 stages of the third and seventh workshops are equal to 1. The efficiency score of the S_{22} stage and the overall efficiency score of this workshop is less than one. Therefore, this workshop is efficient in the firstly, the S21 stages and S_1, S_{21} and S_3 in the S_{22} stage and the overall workshop are inefficient. The efficiency score of the S_{22} stage of the fifth workshop are equal to 1 , and the efficiency score of the S_1, S_{21} and S_3 stage and the overall efficiency score of these workshops are less 1. Therefore, this workshop is strong efficient and the S_{22} stage is efficient, but the overall workshop are inefficient. Other workshops are inefficient in S_1, S_{21}, S_{22} and S_3 stages and the overall.

5. Conclusions

In this paper, we presented a new kind of network DEA model to evaluate the efficiency of decision-making units with undesirable and desirable intermediate measures, undesirable and desirable outputs. This model calculated the efficiency of the first, S_{21}, the S_{22} stages processes under the constraints which overall efficiency are maintained at the same level.

In the future research, we will show that, firstly these models can be extended to networks network structure with triangular fuzzy data, with undesirable and desirable intermediate measures and outputs. Secondly, the model will be solved utilizing triangular fuzzy data boundaries and will be calculated by considering α-cut levels.

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