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Undesirable factors in stochastic cross-efficiency evaluation

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Abstract

Cross-efficiency evaluation in Data envelopment analysis (DEA) has been accepted as a useful tool for performance evaluation and ranking of decision making units. In this paper using Undesirable Multiple Form (UMF) model with specific risk of α, a new stochastic model called Expected Ranking Criterion is introduced using statistical techniques for efficiency evaluation decision making units (DMU). Another issue in applying cross-efficiency DEA models is considering stochastic in input and output variables. Also, the non-uniqueness of optimal weights in this evaluation has reduced the usefulness of this powerful method. As a result, it is recommended that secondary goals be introduced in cross-efficiency evaluation. In this paper, the cross-efficiency model is modified to deal with stochastic data by applying chance-constrained approach.

Keywords: Undesirable Multiple Form (UMF); stochastic cross-efficiency evaluation; expected ranking criterion; ranking priority

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1. Introduction

Cross-efficiency evaluation method was employed for the first time by Sexton [1] to address such concerns in DEA modeling by providing more flexible and realistic weighting approach which is called peer-evaluation method. Successful applications of cross-efficiency in DEA literature shows the advantage of applying this method in the applications. The idea of using peer-evaluation instead of self-evaluation in cross-efficiency method provided an extension to the theory of DEA by promoting the power of individual voices in the process peer valuation [2]. This method is then improved in Doyle [3]. Efficiency among DMUs with weight schemes was introduced by Anderson [4].

In spite of merits of DEA cross-efficiency evaluation and its wide applications, it still has some short falls. For each DMU, the fact that DEA optimal weights are not unique may reduce the usefulness of cross-efficiency evaluation [1]. As Doyle [3] mentioned, multiple optimal weights acquired from classic DEA models can be used as secondary goals for better discrimination among DMUs. Evaluation of DEA cross-efficiency on the basis of Pareto improvement model was proposed by Wu [5]. During recent years, the research concerning cross-efficiency evaluation has developed fast. Some of the significant studies in this field are as follows: Javier Alcaraz [6], Oral [7], Soltanifar [8], Li [9].

In DEA, DMUs are evaluated using envelopment or multiplier models, via running n times for each DMU to obtain the relative efficiency of the whole DMUs. Traditional DEA models do not deal with imprecise data and presume that all input and output are exactly known. However, this assumption may not always be true, especially given the weakness in DEA models in dealing with stochastic variations in inputs and outputs. Land [10] formulated a chance-constrained envelopment model, when inputs and outputs are normally distributed but the probabilistic chance constraints in the model are individually imposed on the outputs. Although this formulation allows for dealing with dependencies among the different DMUs (decision making units), it does not incorporate the statistical dependencies among the outputs of the DMU to be considered. To cope with this, Olesen [11] presented a chance-constrained multiplier model, in which the chance constraints are individually imposed on each DMU so that its outputs are stochastically dependent. Although both models have some advantages, it is reasonable to think that a model having the two features (stochastic dependences among the DMUs and among the outputs of the considered DMU) will provide a greater flexibility and reality in practical analysis.

The overall attitude toward performance evaluation is to reduce inputs and increase outputs. The CCR and BCC models are based on this. But it should be noted that organizations are not always looking to maximize outputs and minimize inputs. Because outputs and inputs can be desirable or undesirable. For example, the number of defective goods, or the amount of pollution and waste, or the release of CO_2 in the production process is undesirable, which should be reduced. Accordingly, models with undesirable inputs/outputs should be taken into account. In the event that the input and output values of DMUs are definitive, Mandal [12] show that in assessing the energy efficiency if the undesirable outputs are ignored then the biased results are observed in efficiency scores.

In addition, Chen [13] use a stochastic network DEA model for Chinese airline efficiency under CO_2 emissions and flight delays. Izadikhah [14] propose a chance-constrained two-stage DEA model in the presence of undersiable factores to evaluate the sustainability of supply chains. Jin [15] compare APEC countries in terms of efficiency in Gross Domestic Product (GDP) considering undesirable stochastic input and output (i.e. CO_2 production) with given risk. Wu [16] compare several provinces in China considering wasting water, emission of toxic gases, and the production of useless solid material as undesirable stochastic outputs with given error. Liu [17] proposed a multi-attribute decision making based on stochastic DEA cross-efficiency with ordinal variable and Its application to evaluation of banks' sustainable development by considering undesirable outputs with weak disposability. Ren [18] proposed a measuring the energy and carbon emission efficiency of regional transportation systems in China by chance-constrained DEA models.

This article is organized as follows. In the following section, we first summarize the Undesirable Multiple Form (UMF) model. Then, in the third section, we obtain a stochastic model by taking into account the error α. Besides, we propose a model under the mean rating criterion. In order to prevent the possible elimination of desirable and undesirable outputs a new two-objective model is presented. In the fourth section, considering the uniqueness of the stochastic cross-efficiency solutions, we propose an aggressive stochastic cross-efficiency evaluation. Finally, in Section 5, applicability of the proposed models is examined using an Example and Discussion. In Section 6 contains conclusions and suggestions for future research.

2. Undesirable Multiple Form (UMF) model

In this section, we first briefly introduce an Undesirable Multiple Form (UMF) model. Assume that there are no decision units for evaluation, each DMU_j has m inputs and s is the number of desirable outputs and k is the number of undesirable outputs. Let x_{id} (j = 1, ..., m), y_{rj} (r = 1, ..., s) and z_{pj} (p = 1, ..., k) be the inputs, desirable outputs and undesirable outputs, respectively. Kousmanen [19] introduced the following linear programming model for evaluating $DMU_d \quad d \in \{1, \dots, n\}$

$$
\begin{aligned}
&\text{Min } \theta \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} (\omega_j + \mu_j)\, x_{id} \leq \theta\, x_{id} \quad i = 1, \dots, m \\
&\textstyle\sum_{j=1}^{n} \omega_j y_{rj} \geq y_{rd} \quad r = 1, \dots, s \quad (1) \\
&\textstyle\sum_{j=1}^{n} \omega_j z_{pj} = z_{pd} \quad p = 1, \dots, k \\
&\textstyle\sum_{j=1}^{n} (\omega_j + \mu_j) = 1 \\
&\omega_j \geq 0 \quad \mu_j \geq 0 \quad j = 1, \dots, n
\end{aligned}
$$

Where ω_j, $\mu_j \geq 0$ (for $j = 1, \dots, n$) are structural variables for the production possibility set (PPS). This model seeks for decreasing the inputs of DMU_d by rate of θ. The model (1) is an input oriented model with an optimal value of $\theta^* \in (0, 1]$. If $\theta^* = 1$, we say that DMU_d is technical efficient. If we write the dual of model (1), the following model (2) is obtained that we name it as Undesirable Multiple Form (UMF):

$$
\begin{aligned}
&E^*_{dd} = max \textstyle\sum_{r=1}^{s} u_{rd} y_{rd} + \sum_{p=1}^{k} w_{pd} z_{pd} + \alpha_d \\
&\text{s.t.} \\
&\textstyle\sum_{i=1}^{m} v_{id} x_{id} = 1 \\
&\textstyle\sum_{i=1}^{m} v_{id} x_{ij} - \sum_{r=1}^{s} u_{rd} y_{rj} - \\
&\textstyle\sum_{p=1}^{k} w_{pd} z_{pj} - \alpha_d \geq 0 \quad j = 1, \dots, n \quad (2) \\
&\textstyle\sum_{i=1}^{m} v_{id} x_{ij} - \alpha_d \geq 0 \quad j = 1, \dots, n \\
&v_{id} \geq 0 \quad i = 1, \dots, m \\
&u_{rd} \geq 0 \quad r = 1, \dots, s \\
&w_{pd} \; free \quad p = 1, \dots, k \\
&\alpha_d \quad \text{free}
\end{aligned}
$$

In model (2) v_{id} (i = 1, ..., m), u_{rd} (r = 1, ..., s), w_{pd} (p = 1, ..., k) are weights of inputs and outputs and undesirable outputs, respectively. Given the existence of free variables α_d and w_{pd}, the model can produce the-often-hidden negative efficiency. This efficiency shows up when cross efficiency is computed. de Mello [20] showed that this can be corrected with creating non-negative constraints in the model. Following de Mello [20] we modify model (3) as follows:

$E^*_{dd} = max \sum_{r=1}^{s} u_{rd} y_{rd} + \sum_{p=1}^{k} w_{pd} z_{pd} + \alpha_d$

s.t.

$\sum_{i=1}^{m} v_{id} x_{id} = 1$

$\sum_{i=1}^{m} v_{id} x_{ij} - \sum_{r=1}^{s} u_{rd} y_{rj} - \sum_{p=1}^{k} w_{pd} z_{pj} - \alpha_d \geq 0 \quad j = 1, \dots, n$ (3)

$\sum_{i=1}^{m} v_{id} x_{ij} - \alpha_d \geq 0 \quad j = 1, \dots, n$

$\sum_{r=1}^{s} u_{rd} y_{rj} + \sum_{p=1}^{k} w_{pd} z_{pj} + \alpha_d \geq 0 \quad j = 1, \dots, n$

$v_{id} \geq 0 \quad i = 1, \dots, m$

$u_{rd} \geq 0 \quad r = 1, \dots, s$

w_{pd} $free$ $p = 1, \dots, k$

α_d free

3. The stochastic model of the proposed model

Assume that random variables $\tilde{x}_{ij}, \tilde{y}_{rj}, \tilde{z}_{pj}$ are inputs, desirable outputs and undesirable outputs, respectively , each of which has normal distribution.That is,

$$\tilde{x}_{ij} \sim N\left(\bar{x}_{ij}, (\sigma_{ij}{}^{x})^2\right) \qquad \forall\, i,j$$

$$\tilde{y}_{rj} \sim N\left(\bar{y}_{rj}, (\sigma_{rj}{}^{y})^2\right) \qquad \forall\, r,j$$

$$\tilde{z}_{pj} \sim N\left(\bar{z}_{pj}, (\sigma_{pj}{}^{z})^2\right) \qquad \forall\, p,j$$

Besides for each DMU, there is a variance covariance matrix Σ that it's the main diameter shows the variance of variables and the other components of Σ represent the covariance of the variables. Therefore, taking into account the error value α (confidence coefficient (1-α)), the stochastic form of the model is as follows.

$E^*_{dd} = max\, E\left(\sum_{r=1}^{s} u_{rd} \tilde{y}_{rd} + \sum_{p=1}^{k} w_{pd} \tilde{z}_{pd} + \alpha_d\right)$

$s.t.$

$E\left(\sum_{i=1}^{m} v_{id} \tilde{x}_{id}\right) = 1$ (4)

$P\left(\sum_{i=1}^{m} v_{id} \tilde{x}_{ij} - \sum_{r=1}^{s} u_{rd} \tilde{y}_{rj} - \sum_{p=1}^{k} w_{pd} \tilde{z}_{pj} - \alpha_d \geq 0\right) \geq 1 - \alpha \quad j = 1, \dots, n$

$P\left(\sum_{i=1}^{m} v_{id} \tilde{x}_{ij} - \alpha_d \geq 0\right) \geq 1 - \alpha \quad j = 1, \dots, n$

$P\left(\sum_{r=1}^{s} u_{rd} \tilde{y}_{rj} + \sum_{p=1}^{k} w_{pd} \tilde{z}_{pj} + \alpha_d \geq 0\right) \geq 1 - \alpha \quad j = 1, \dots, n$

$v_{id} \geq 0 \quad i = 1, \dots, m$

$u_{rd} \geq 0 \quad r = 1, \dots, s$

w_{pd} $free$ $p = 1, \dots, k$

α_d free

We need to transform the model (4) into non-stochastic quadratic form to calculate the optimal solution. The stochastic version of UMF envelopment model is formulated as follow; Cooper [21]:

$E^*_{dd} = max \sum_{r=1}^{s} u_{rd} \bar{y}_{rd} + \sum_{p=1}^{k} w_{pd} \bar{z}_{pd} + \alpha_d$

s.t.

$\sum_{i=1}^{m} v_{id} \bar{x}_{id} = 1$ (5)

$\sum_{i=1}^{m} v_{id} \bar{x}_{ij} - \sum_{r=1}^{s} u_{rd} \bar{y}_{rj} - \sum_{p=1}^{k} w_{pd} \bar{z}_{pj} - \alpha_d + \sigma_{1j} \Phi^{-1}(\alpha) \geq 0 \quad j = 1, \dots, n$

$\sigma_{1j}{}^2 = \sum_{i=1}^{m} \sum_{k=1}^{m} v_{id} v_{kd}\, cov(\tilde{x}_{ij}, \tilde{x}_{kj}) + \sum_{r=1}^{s} \sum_{k=1}^{s} u_{rd} u_{kd}\, cov(\tilde{y}_{rj}, \tilde{y}_{kj}) + \sum_{p=1}^{k} \sum_{q=1}^{k} w_{pd} w_{qd}\, cov(\tilde{z}_{pj}, \tilde{z}_{qj}) - 2 \sum_{i=1}^{m} \sum_{r=1}^{s} v_{id} u_{rd} cov(\tilde{x}_{ij}, \tilde{y}_{rj}) - 2 \sum_{i=1}^{m} \sum_{p=1}^{k} v_{id} w_{pd} cov(\tilde{x}_{ij}, \tilde{z}_{pj}) + 2 \sum_{r=1}^{s} \sum_{p=1}^{k} u_{rd} w_{pd} cov(\tilde{y}_{rj}, \tilde{z}_{pj}) \quad j = 1, \dots, n$

$\sum_{i=1}^{m} v_{id} \bar{x}_{ij} + \sigma_{2j} \Phi^{-1}(\alpha) \geq 0 \quad j = 1, \dots, n$

$\sigma_{2j}{}^2 = \sum_{i=1}^{m} \sum_{k=1}^{m} v_{id} v_{kd}\, cov(\tilde{x}_{ij}, \tilde{x}_{kj}) \quad j = 1, \dots, n$

$\sum_{r=1}^{s} u_{rd} \bar{y}_{rj} + \sum_{p=1}^{k} w_{pd} \bar{z}_{pj} + \alpha_d + \sigma_{3j} \Phi^{-1}(\alpha) \geq 0 \quad j = 1, \dots, n$

$\sigma_{3j}{}^2 = \sum_{r=1}^{s} \sum_{k=1}^{s} u_{rd} u_{kd}\, cov(\tilde{y}_{rj}, \tilde{y}_{kj}) + \sum_{p=1}^{k} \sum_{q=1}^{k} w_{pd} w_{qd}\, cov(\tilde{z}_{pj}, \tilde{z}_{qj}) + 2 \sum_{r=1}^{s} \sum_{p=1}^{k} u_{rd} w_{pd} cov(\tilde{y}_{rj}, \tilde{z}_{pj}) \quad j = 1, \dots, n$

$v_{id} \geq 0 \quad i = 1, \dots, m$

$u_{rd} \geq 0 \quad r = 1, \dots, s$

w_{pd} $free$ $p = 1, \dots, k$

α_d free

$\sigma_{1j} \geq 0 \quad j = 1, \dots, n$

$\sigma_{2j} \geq 0 \quad j = 1, \dots, n$

$\sigma_{3j} \geq 0 \quad j = 1, \dots, n$

In which $\phi(\alpha)$ is the standard normal distribution function and $\phi^{-1}(\alpha)$ is the inverse of the standard normal distribution function in the value of α. $\sigma_{1j}{}^2$, $\sigma_{2j}{}^2$ and $\sigma_{3j}{}^2$, are the variance of the first, second and third constraints of model (4).

Definition 1: The feasible solution $(v^*_{id}, u^*_{rd}, w^*_{pd}, \alpha^*_d)$ is an optimal solution of model (5) if
$\sum_{r=1}^{s} u^*_{rd}\bar{y}_{rd} + \sum_{p=1}^{k} w^*_{pd}\bar{z}_{pd} + \alpha^*_d \geq \sum_{r=1}^{s} u_{rd}\bar{y}_{rd} + \sum_{p=1}^{k} w_{pd}\bar{z}_{pd} + \alpha_d$
For any feasible solution $(v_{id}, u_{rd}, w_{pd}, \alpha_d)$.

Expected raking criterion: The higher E^*_{dd} means the DMU is more efficient and higher rank of the DMUj.

From the optimal solution of model (5), we obtain stochastic efficiency score, but for better discrimination, we take into account the stochastic cross- efficiency. If the optimal solution of model (5) is $(v^*_{id}, u^*_{rd}, w^*_{pd}, \alpha^*_d)$ then, stochastic cross-efficiency of DMU_j based on DMU_d is defined as below

$$E^*_{dj} = \frac{\sum_{r=1}^{s} u^*_{rd}\bar{y}_{rj} + \sum_{p=1}^{k} w^*_{pd}\bar{z}_{pj} + \alpha^*_d}{\sum_{i=1}^{m} v^*_{id}\bar{x}_{ij}} \quad (6)$$

Therefore, the stochastic cross-efficiency of each DMU_j is equal to the mean of E^*_{dj} ,that is,

$$E^*_j = \frac{1}{n}\sum_{d=1}^{n} E^*_{dj} \quad (7)$$

This value is a new stochastic efficiency score for each DMU.

4. The stochastic priority ranking model

Given that the optimal solution of the models may not be unique, the obtained stochastic cross- efficiency scores are some extent arbitrarily. To solve this problem, a new model is proposed to rank DMUs. The proposed model not only maintains the stochastic efficiency value, but also increases stochastic cross-efficiency.

$$R^*_{pd} = \min \sum_{j=1}^{n} z_j$$

s.t.

$$\sum_{i=1}^{m} v_{id}\bar{x}_{id} = 1 \quad (8)$$

$$\sum_{r=1}^{s} u_{rd}\bar{y}_{rd} + \sum_{p=1}^{k} w_{pd}\bar{z}_{pd} + \alpha_d = E^*_{dd}$$

$$\sum_{i=1}^{m} v_{id}\bar{x}_{ij} - \sum_{r=1}^{s} u_{rd}\bar{y}_{rj} - \sum_{p=1}^{k} w_{pd}\bar{z}_{pj} - \alpha_d + \sigma_{1j}\Phi^{-1}(\alpha) \geq 0 \quad j = 1, \dots, n$$

$$\sigma_{1j}^2 = \sum_{i=1}^{m}\sum_{k=1}^{m} v_{id}v_{kd}\,\mathrm{cov}(\tilde{x}_{ij}, \tilde{x}_{kj}) + \sum_{r=1}^{s}\sum_{k=1}^{s} u_{rd}u_{kd}\,\mathrm{cov}(\tilde{y}_{rj}, \tilde{y}_{kj}) + \sum_{p=1}^{k}\sum_{q=1}^{k} w_{pd}w_{qd}\,\mathrm{cov}(\tilde{z}_{pj}, \tilde{z}_{qj}) - 2\sum_{i=1}^{m}\sum_{r=1}^{s} v_{id}u_{rd}\mathrm{cov}(\tilde{x}_{ij}, \tilde{y}_{rj}) - 2\sum_{i=1}^{m}\sum_{p=1}^{k} v_{id}w_{pd}\mathrm{cov}(\tilde{x}_{ij}, \tilde{z}_{pj}) + 2\sum_{r=1}^{s}\sum_{p=1}^{k} u_{rd}w_{pd}\mathrm{cov}(\tilde{y}_{rj}, \tilde{z}_{pj}) \quad j = 1, \dots, n$$

$$\sum_{i=1}^{m} v_{id}\bar{x}_{ij} + \sigma_{2j}\Phi^{-1}(\alpha) \geq 0 \quad j = 1, \dots, n$$

$$\sigma_{2j}^2 = \sum_{i=1}^{m}\sum_{k=1}^{m} v_{id}v_{kd}\,\mathrm{cov}(\tilde{x}_{ij}, \tilde{x}_{kj}) \quad j = 1, \dots, n$$

$$\sum_{r=1}^{s} u_{rd}\bar{y}_{rj} + \sum_{p=1}^{k} w_{pd}\bar{z}_{pj} + \alpha_d + \sigma_{3j}\Phi^{-1}(\alpha) \geq 0 \quad j = 1, \dots, n$$

$$\sigma_{3j}^2 = \sum_{r=1}^{s}\sum_{k=1}^{s} u_{rd}u_{kd}\,\mathrm{cov}(\tilde{y}_{rj}, \tilde{y}_{kj}) + \sum_{p=1}^{k}\sum_{q=1}^{k} w_{pd}w_{qd}\,\mathrm{cov}(\tilde{z}_{pj}, \tilde{z}_{qj}) + 2\sum_{r=1}^{s}\sum_{p=1}^{k} u_{rd}w_{pd}\mathrm{cov}(\tilde{y}_{rj}, \tilde{z}_{pj}) \quad j = 1, \dots, n$$

$$E^*_{dd}\sum_{i=1}^{m} v_{id}\bar{x}_{ij} - \sum_{r=1}^{s} u_{rd}\bar{y}_{rj} - \sum_{p=1}^{k} w_{pd}\bar{z}_{pj} - \alpha_d + s_j = 0 \quad j = 1, \dots, n$$

$$s_j \leq M \times z_j \quad j = 1, \dots, n$$

$$z_j \in \{0,1\} \quad j = 1, \dots, n$$

$$s_j \text{ free} \quad j = 1, \dots, n$$
$$v_{id} \geq 0 \quad i = 1, \dots, m$$
$$u_{rd} \geq 0 \quad r = 1, \dots, s$$
$$w_{pd} \text{ free} \quad p = 1, \dots, k$$
$$\alpha_d \text{ free}$$
$$\sigma_{1j} \geq 0 \quad j = 1, \dots, n$$
$$\sigma_{2j} \geq 0 \quad j = 1, \dots, n$$
$$\sigma_{3j} \geq 0 \quad j = 1, \dots, n$$

The main idea of model (8) is that the first and second constraints ensure that the stochastic efficiency DMU_d is maintained. In the seventh constraint, it is easy to see that

$$E^*_{dd}\sum_{i=1}^{m} v_{id}\bar{x}_{ij} - \sum_{r=1}^{s} u_{rd}\bar{y}_{rj} - \sum_{p=1}^{k} w_{pd}\bar{z}_{pj} - \alpha_d = -s_j$$

If $s_j>0$, then

$$E^*_{dd} < \frac{\sum_{r=1}^{s} u_{rd}\bar{y}_{rj} + \sum_{p=1}^{k} w_{pd}\bar{z}_{pj} + \alpha_d}{\sum_{i=1}^{m} v_{id}\bar{x}_{ij}} = E^*_{dj}$$

This means that the cross-efficiency of DMU_j based on DMU_d is greater than the stochastic efficiency of DMU_d. If $s^*_j<0$

then $E^*_{dd}>E^*_{dj}$, that is, the stochastic cross- efficiency of DMU_jbased on $DMU_d$is smaller than the stochastic efficiency of DMU_d.
In the eighth constraint, the M value is defined as the largest positive value. If $s^*_j>0$, then the value of z^*_j will be one, and if $s^*_j<0$, then the value z^*_j is zero. And since the objective function is the minimum type, then the most values of z^*_j is zero. Thus, $s^*_j<0$ is further observed. In this case, the stochastic cross- efficiency of DMUj is lower than the DMU_d.
In model (8), the optimal solution may not be unique, so we consider below aggressive model which not only maintains stochastic efficiency but also minimizes the cross-efficiency score of other DMUs:
In model (8), the optimal solution may not be unique, so we consider the following aggressive model which not only maintains stochastic efficiency but also minimizes the cross-efficiency score of other DMUs:

$$
\begin{aligned}
& max \sum_{j=1}^{n} \Psi_j \\
& \text{s.t} \\
& \sum_{i=1}^{m} v_{id}\bar{x}_{id} = 1 \qquad (9) \\
& \sum_{r=1}^{s} u_{rd}\bar{y}_{rd} + \sum_{p=1}^{k} w_{pd}\bar{z}_{pd} + \alpha_d = E^*_{dd} \\
& \sum_{i=1}^{m} v_{id}\bar{x}_{ij} - \sum_{r=1}^{s} u_{rd}\bar{y}_{rj} - \sum_{p=1}^{k} w_{pd}\bar{z}_{pj} - \alpha_d - \Psi_j = 0 \qquad j = 1, \dots, n \\
& E^*_{dd} \sum_{i=1}^{m} v_{id}\bar{x}_{ij} - \sum_{r=1}^{s} u_{rd}\bar{y}_{rj} - \sum_{p=1}^{k} w_{pd}\bar{z}_{pj} - \alpha_d + s_j = 0 \qquad j = 1, \dots, n \\
& \sum_{j=1}^{n} z_j = R^*_{pd} \\
& s_j \le M \times z_j \qquad j = 1, \dots, n \\
& z_j \in \{0,1\} \qquad j = 1, \dots, n \\
& s_j \text{ free} \qquad j = 1, \dots, n \\
& v_{id} \ge 0 \qquad i = 1, \dots, m \\
& u_{rd} \ge 0 \qquad r = 1, \dots, s \\
& w_{pd} \text{ free} \quad p = 1, \dots, k \\
& \alpha_d \quad \text{free}
\end{aligned}
$$

The model (9) increases the efficiency deviation of other $DMUs$ in order to improve cross-efficiency of DMU_d while maintaining the stochastic efficiency of E^*_{dd}. In the model (9), R^*_{pd} value is the same as obtained in model (8).

Therefore, if $(v^*_{id}, u^*_{rd}, w^*_{pd}, \alpha^*_d)$ is an optimal solution of the model (11), then stochastic cross-efficiency of DMU_j based on DMU_d is as follows:

$$\theta'_{dj} = \frac{\sum_{r=1}^{s} u^*_{rd}\bar{y}_{rj} + \sum_{p=1}^{k} w^*_{pd}\bar{z}_{pj} + \alpha^*_d}{\sum_{i=1}^{m} v^*_{id}\bar{x}_{ij}} \qquad (10)$$

Therefore, for each DMU_j, the mean stochastic cross-efficiency is equal to

$$\theta_j = \frac{1}{n}\sum_{d=1}^{n} \theta'_{dj} \qquad (11)$$

which is a new stochastic cross-efficiency score for DMU_j $(j = 1, \dots, n)$.

5. Numerical Example and Discussion

In this section, we illustrate the proposed models for 15 DMUs. Any DMUs has two inputs and two outputs, one of which is undesirable. The first input variable $(\tilde{x}_1)$ and the second input variable $(\tilde{x}_2)$. Also, the first output variable $(\tilde{y}_1)$ is desirable, and the second output $(\tilde{z}_1)$ is undesirable. Suppose that the mean values and standard deviation and covariance variance matrix were estimated. Tables 1and 2 show the estimated parameters of data and their distribution.

Table 1: The estimated parameters of inputs normal distributions

DMU	$\tilde{x}_1$	$\tilde{x}_2$
1	N(23.6,5.8)	N(22.4,4.3)
2	N(32.6,4.3)	N(72,3.6)
3	N(6,2.5)	N(23.2,3.3)
4	N(12.6,5.3)	N(19.4,4.13)
5	N(37.4,3.3)	N(36.8,4.27)
6	N(7.4,4.3)	N(45.4,2.3)
7	N(21.8,2. 3)	N(15.6,8.6)
8	N(32.6,2.3)	N(37,2.5)
9	N(17.2,8.7)	N(32.8,6.2)
10	N(13.2,1.2)	N(47.2,2.12)
11	N(19.4,5.3)	N(42.4,4.12)
12	N(7.8,2.7)	N(32,13)
13	N(16.8,3.8)	N(23.4,4.13)
14	N(12.2,5.6)	N(47.8,3.17)
15	N(73.2,14.2)	N(17,18.5)

Table 2: The estimated parameters of outputs normal distributions

DMU	$\tilde{y}_1$	$\tilde{z}_1$
1	N(41.72,3.8)	N(2.08,0.01)
2	N(23,3.5)	N(2.14,0.03)
3	N(95.4,286.3)	N(3.22,0.02)
4	N(17.6,2.8)	N(5.03,0.07)
5	N(33,3.5)	N(3.6,0.05)
6	N(44.4,6.8)	N(2.9,0.03)
7	N(15.2,0.7)	N(2.54,1 3)
8	N(42.4,7.8)	N(1.44,0. 03)
9	N(32.4,5.8)	N(1.94,0.02)
10	N(83,31)	N(2.54,0.04)
11	N(35.4,5.9)	N(3.96,0.05)
12	N(41.6,1.8)	N(6.18,0.07)
13	N(31.6,1.9)	N(2.78,0.03)
14	N(23,4.5)	N(3.18,0.04)
15	N(84.4,28.3)	N(2.212,0.03)

For each DMU we estimated a variance-covariance symmetric matrix. For instance, Table 3 shows this matrix for DMU_1.

Table 3: The variance-covariance symmetric matrix for DMU_1

covariance	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{y}_1$	$\tilde{z}_1$
$\tilde{x}_1$	2.3	2.01	3.55	0.44
$\tilde{x}_2$	2.01	1.4	2.55	0.06
$\tilde{y}_1$	3.55	2.55	4.9	0.025
$\tilde{z}_1$	0.44	0.06	0.025	0.61

For example, in Table 4, when $\alpha = 0.05$, stochastic cross-efficiency are derived from model (7) and aggressive model (13). As we see the stochastic cross-efficiency ranking from models (7) and (11) change in some DMUs.

Table 4: Stochastic cross-efficiency scores of the DMUs ($\alpha = 0.05$)

	Using model (7)		Using model (11)	
DMU	$E^*{}_d$	Rank	θ_j	Rank
1	0.257306	15	0.240489	15
2	0.491782	9	0.426745	9
3	0.704516	2	0.680328	2
4	0.576423	4	0.500328	5
5	0.737885	1	0.689765	1
6	0.381304	10	0.365998	10
7	0.293780	14	0.242265	14
8	0.494318	8	0.474606	8
9	0.324968	13	0.306799	12
10	0.371087	11	0.364414	11
11	0.534212	7	0.497253	6
12	0.560378	5	0.514563	4
13	0.346360	12	0.297978	13
14	0.536418	6	0.483297	7
15	0.670433	3	0.634228	3

6- Conclusions and suggestions for further research

The purpose of this paper was to propose a model based on which we can calculate the efficiency of DMUs in the presence of stochastic inputs / outputs, as well as in case of undesirable outputs. Using stochastic UMF model and statistical techniques and normal distribution, stochastic models were proposed for calculating stochastic efficiency and based on this we defined the mean ranking criterion`. Then the stochastic cross-efficiency for ranking in DEA was proposed based on the stochastic constrained and the mean value of the objective function. Finally, we implemented the proposed models for 32 energy-producing plants. For future research, new models can be found for other models in DEA, as well as for other distributions such as Normal skew and Weibull and Riley. Similarly, models can be extended to determine the coverage of fuzzy and hybrid data.

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