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Assessment of two-stage processes cross-efficiency in the presence of undesirable factors

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Abstract

Cross-efficiency is a ranking technique based on the peer-evaluation that can increase the discriminating power between efficient decision-making units. This paper intends to assess the two-stage processes consisting of undesirable outputs by applying the cross-efficiency evaluation. Given undesirable outputs, the directional distance function under the weak disposability assumption is utilized. The proposed model under variable returns to scale is designed, which makes it different from the previous models. Furthermore, it can reduce the zero optimal coefficients. By measuring the inputs and outputs inefficiency, the whole system and each of its two stages rank, simultaneously. To analyze the suggested method, an application on the industrial productions of 30 regions of China is used.

Keywords: Cross-efficiency; Directional distance function; Two-stage structure; Undesirable output; Weak disposability assumption

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1. Introduction

Data envelopment analysis (DEA) is a non-parametric method to evaluate the relative efficiency of a set of homogeneous decision-making units (DMUs) with multiple inputs and outputs that, for the first time, introduced by Charnes et al. [1] and extended by Banker et al. [2]. DEA assigns an efficiency score of 1 to efficient units, and inefficient DMUs have a score of less than unity. From experience, we know that in many cases, more than one unit is efficient. It means the conventional DEA models cannot discriminate between efficient DMUs. In the last four decades, to deal with this problem, various ranking approaches have been proposed by researchers.

Cross-efficiency is one of the ranking methods initially introduced by Sexton et al. [3] and then improved by Doyle and Green [4]. Contrary to classical DEA models, this method makes it possible to evaluate the performance of each unit with the weights of others. The non-uniqueness of the optimal solution of the multiplier form is one of the weaknesses of the cross-efficiency method. Researchers have proposed ways to overcome this problem in the last two decades. For example, Liang et al. [5] utilized various objective functions to measure the cross-efficiency. After that, Liang et al. [6] examined the cross-efficiency score for each DMU using the Nash bargaining game theorem. Their essential goal was to solve the non-uniqueness problem of the optimal multipliers. They showed the best game cross-efficiency is a Nash equilibrium point. Since the assumption of variable returns to scale in data envelopment analysis models generates negative cross-efficiency values, Wu et al. [7] presented a modified model based on the model proposed by Liang et al. [6].

Ruiz [8] provided the cross-efficiency evaluation based on the directional model. Moreover, they explored the duality relations regarding the directional models and defined the cross-efficiency as a ratio. To resolve the problem of negative cross efficiency due to the assumption of variable returns to scale, Lim and Zhu [9] developed a novel method. They interpreted the relationship between variable returns to scale (VRS) and constant returns to scale models (CRS) and presented a method based on it. Cook and Zhu [10] used the Cobb-Douglas function and introduced a multiplicative model to calculate the cross-efficiency scores that are unique and have the highest value. Wu et al. [11] considered the Pareto improvement and presented a new cross-efficiency evaluation. Lin et al. [12] proposed the iterative method to simultaneously solve two problems of zero and alternative optimal coefficients. Furthermore, Wei et al. [13] presented a method to calculate the cross-efficiency utilizing the combination of the directional distance function and Nash equilibrium. Lin [14] estimated the cross-efficiency based on the domain direction measure (RDM) under the assumption of variable returns to scale while the data set contained negative values.

The aforementioned studies show that most existing works on cross-efficiency are for single-stage systems while, in reality, most systems are multiple stages and considering their internal structure is very important. Therefore, Kao and Liu [15] applied the relational model with the assumption of constant returns to scale to measure the cross-efficiency for series and parallel structures. Indeed, they used the geometric average and showed the total efficiency is obtained from the weighted sum of the efficiencies of its sub-sections. Next, Örkcü et al. [16] introduced a neutral cross-efficiency model under constant returns to scale for the basic two-stage network systems. They showed the obtained efficiency scores from their proposed model are more realistic than the suggested model by Kao and Liu [15]. Another advantage of their model is

reducing the number of zero coefficients. According to Lin's approach [14], Lin and Tu [17] provided a method to assess the cross-efficiency of series and parallel systems.
The above literature review shows that the cross-efficiency evaluation of network structures without the presence of undesirable factors has been done. Therefore, the main goal of this paper is to provide a method for assessing the two-stage network structures based on cross-efficiency such that undesirable factors are also considered. In this regard, the directional distance function is used with the weak disposability assumption. A function that allows reducing and increasing all inputs and outputs of the unit under evaluation in the direction of an appropriate vector. The proposed model is formulated with the assumption of variable returns to scale, which differentiates it from previous methods presented for two-stage structures. In addition to the above, the number of zero multipliers is also reduced in the new model and is considered a strong point for it.
The remainder of this paper is organized as follows. Section 2 briefly reviews the directional efficiency in the presence of undesirable outputs and cross-efficiency. Section 3 introduces the suggested model to evaluate the two-stage structure with undesirable intermediate measures. To describe the proposed method, a real example in the field of industrial production is used in Section 4. Conclusions appear in Section 5.

2. Preliminaries

In this section, the directional efficiency in the presence of undesirable outputs and the cross-efficiency are briefly reviewed.

2.1. Directional efficiency with undesirable outputs

Assume that there are K DMUs, where each $DMU_k : k = 1,...,K$ consumes the input vector $x_k = (x_{1k},...,x_{Ik}) \geq 0$, and produces the desirable output vector $v_k = (v_{1k},...,v_{Rk}) \geq 0$, and undesirable output vector $z_k = (z_{1k},...,z_{Mk}) \geq 0$. Consider, the production possibility set as follows:

$$T = \{(x,z,v) \mid (z,v) \text{ can be produced by } x\}$$

Definition 1. Outputs (z,v) are weakly disposable if and only if $(x,z,v) \in T$ and $0 \leq \beta \leq 1$, imply $(x, \beta z, \beta v) \in T$ (Shepherd [18]).

Kuosmanen [19] introduced the following technology under variable returns to scale satisfying weak disposability assumption:

$$T = \{(x,z,v): \qquad (1)$$

$$\sum_{k=1}^{K} \gamma_k x_{ik} \leq x_{io}, \qquad i = 1,...,I,$$

$$\sum_{k=1}^{K} \beta_k \gamma_k z_{mk} = z_{mo}, \qquad m = 1,...,M,$$

$$\sum_{k=1}^{K} \beta_k \gamma_k v_{rk} \geq v_{ro}, \qquad r = 1,...,R,$$

$$\sum_{k=1}^{K} \gamma_k = 1,$$

$$\gamma_k \geq 0, \qquad k = 1,...,K,$$

$$0 \leq \beta_k \leq 1, \qquad k = 1,...,K.\}$$

which, β_k is a distinctive abatement factor of DMU_k that resulted in the non-linearity of the mentioned technology. Therefore, using the $\gamma_k = \zeta_k + \eta_k$ whereby, $\zeta_k = (1-\beta_k)\gamma_k$, and $\eta_k = \beta_k \gamma_k$, the foregoing technology is rewritten in the following linear form:

$$
\begin{aligned}
T = \{(x, z, v) : \\
& \sum_{k=1}^{K} (\eta_k + \zeta_k) x_{ik} \le x_{ko}, \quad i = 1,...,I, \\
& \sum_{k=1}^{K} \eta_k z_{mk} = z_{mo}, \quad m = 1,...,M, \\
& \sum_{k=1}^{K} \eta_k v_{rk} \ge v_{ro}, \quad r = 1,...,R, \\
& \sum_{k=1}^{K} (\eta_k + \zeta_k) = 1, \\
& \eta_k, \zeta_k \ge 0, \quad k = 1,...,K.\}
\end{aligned} \tag{2}
$$

In the above technology, η_k and ζ_k are unknown variables. Using the manner of Färe and Grosskopf [20], the efficiency of DMU_o in the direction of the vector opposite to zero $\vec{d} = (d^{(x)}, d^{(z)}, d^{(v)})$ is evaluated as follows:

$$
\begin{aligned}
& \delta_o^* = Max \left(\sum_{i=1}^{I} \varphi_i + \sum_{m=1}^{M} \rho_m + \sum_{r=1}^{R} \theta_r \right) \\
& s.t. \\
& \sum_{k=1}^{K} (\eta_k + \zeta_k) x_{ik} \le x_{io} - \varphi_i d_i^{(x)}, \quad i = 1,...,I, \\
& \sum_{k=1}^{K} \eta_k w_{mk} = w_{mo} - \rho_m d_m^{(w)}, \quad m = 1,...,M, \\
& \sum_{k=1}^{K} \eta_k v_{rk} \ge v_{ro} + \theta_r d_r^{(v)}, \quad r = 1,...,R, \\
& \sum_{k=1}^{K} (\eta_k + \zeta_k) = 1, \\
& \varphi_i, \rho_m, \theta_r, \eta_k, \zeta_k \ge 0, \quad \forall i, m, r, k.
\end{aligned} \tag{3}
$$

The purpose of model 3 is to determine the amount of inefficiency based on the simultaneous contraction and expansion of the inputs and desirable and undesirable outputs with factors φ_i, ρ_m, and θ_r.

Definition 2. If the optimal value of the directional model (3) is equal to zero ($\delta_o^* = 0$), then DMU_o is said to be efficient; otherwise, it is not efficient.

2.2. Cross-efficiency evaluation

Suppose there are K DMUs so that each $DMU_k : k = 1,...,K$ consists of the input vector $x_k = (x_{1k},...,x_{Ik}) \ge 0$, and desirable output vector $y_k = (y_{1k},...,y_{Rk}) \ge 0$. Therefore, the efficiency of DMU_o can be calculated by the following model (Charnes et al. [1]):

$$
\begin{aligned}
& E_o^* = Max \frac{\sum_{r=1}^{R} u_r y_{ro}}{\sum_{i=1}^{I} v_i x_{io}} \\
& s.t. \\
& \frac{\sum_{r=1}^{R} u_k y_{rk}}{\sum_{i=1}^{I} v_i x_{ik}} \le 1, \quad k = 1,...,K, \\
& u_r, v_i \ge 0, \quad \forall r, i.
\end{aligned} \tag{4}
$$

Model (4) is an input-oriented fractional problem that using the Charnes-Cooper transformation is transformed into the following model (See Charnes and Cooper [21]):

$$
\begin{aligned}
& E_o^* = Max \sum_{r=1}^{R} u_r y_{ro} \\
& s.t. \\
& \sum_{i=1}^{I} v_i x_{io} = 1, \\
& \sum_{r=1}^{R} u_r y_{rk} - \sum_{i=1}^{I} v_i x_{ik} \le 0, \quad k = 1,...,K, \\
& u_r, v_i \ge 0, \quad \forall r, i.
\end{aligned} \tag{5}
$$

In the above linear model, E_o^* shows the efficiency score of DMU_o obtained with its optimal multipliers (v_{io}^*, u_{ro}^*). As a result, DMU_o is called to be efficient, if $E_o^* = 1$, and all the optimal multipliers are positive ($(v_{io}^*, u_{ro}^*) > 0$).

Also, the efficiency score of $DMU_k : k = 1,...,K, k \ne o$ with the optimal multipliers of DMU_o is defined as,

$$
E_{ok}^* = \frac{\sum_{r=1}^{R} u_{ro}^* y_{rk}}{\sum_{i=1}^{I} v_{io}^* x_{ik}} \tag{6}
$$

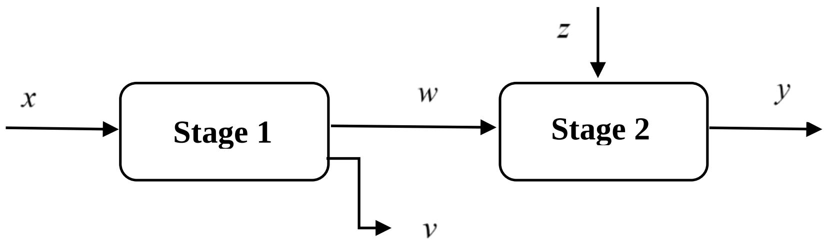

Figure 1. The two-stage network structure

According to equation (6), the value of cross-efficiency for $DMU_k : k=1,...,K$ is defined as $\bar{E}_k = \frac{1}{n}\sum_{k=1}^{K} E_{ok}^{*} (k=1,...,K)$, which is the average of all the efficiency obtained based on the optimal weights of DMU_k.

3. Cross-efficiency evaluation of the two-stage process

The main purpose of this section is to provide a proposed model for evaluating the cross-efficiency of two-stage processes including undesirable outputs.

Now, suppose there are K DMUs, and each DMU_k $(k=1,...,K)$ has a two-stage structure as can be seen in Figure 1. Stage one consumes the input vector $x_k = (x_{1k},...,x_{Nk}) \geq 0$ and produces desirable and undesirable output vectors $v_k = (v_{1k},...,v_{Mk}) \geq 0$, and $w_k = (w_{1k},...,w_{Jk}) \geq 0$, respectively. In addition to consuming the undesirable vector w_k as an input, stage two uses another input vector as $z_k = (z_{1k},...,z_{Tk}) \geq 0$. Moreover, it produces the desirable output vectors $y_k = (y_{1k},...,y_{Rk}) \geq 0$.

The linear direction distance function model for the two-stage process introduced in Figure 1 under the weak disposability assumption is as follows:

$$E_o^{*Overall} = Max\left[\sum_{n=1}^{N}\zeta_n + \sum_{m=1}^{M}\tau_m + \sum_{j=1}^{J}\gamma_j + \sum_{t=1}^{T}\rho_t + \sum_{r=1}^{R}\lambda_r\right]$$

$$s.t. \qquad (7)$$

Stage 1 constraints:

$$\sum_{k=1}^{K}(\varphi^k + \eta^k)\, x_n^k \leq x_n^o - \zeta_n d_n^{(x)}, \quad n=1,...,N,$$

$$\sum_{k=1}^{K}\varphi^k v_m^k \geq v_m^o + \tau_m d_m^{(v)}, \quad m=1,...,M,$$

$$\sum_{k=1}^{K}\varphi^k w_j^k = w_j^o - \gamma_j d_j^{(w)}, \quad j=1,...,J,$$

$$\sum_{k=1}^{K}(\varphi^k + \eta^k) = 1,$$

Stage 2 constraints:

$$\sum_{k=1}^{K}\mu^k w_j^k = w_j^o - \gamma_j d_j^{(w)}, \qquad j=1,...J,$$

$$\sum_{k=1}^{K}(\mu^k + \kappa^k)\, z_t^k \leq z_t^o - \rho_t d_t^{(z)}, \quad t=1,...,T,$$

$$\sum_{k=1}^{K}\mu^k y_r^k \geq y_r^o + \lambda_r d_r^{(y)}, \quad r=1,...,R,$$

$$\sum_{k=1}^{K}(\mu^k + \kappa^k) = 1,$$

Generic constraints:

$$\varphi^k,\ \eta^k, \mu^k,\ \kappa^k \geq 0, \qquad k=1,...,K,$$

$$\zeta_n,\ \tau_m,\ \gamma_j,\ \rho_t,\ \lambda_r \geq 0, \quad \textit{for all } n, m, j, t, r$$

The model's first and second four constraints (7) are related to the first and second components of the desired two-stage structure. The linear model (7) is written under variable returns to scale assumption. In the above formulation, $\varphi^k,\ \eta^k, \mu^k$ and κ^k are in terms of unknown variables and $\zeta_n,\ \tau_m,\ \gamma_j,\ \rho_t,\ \lambda_r \geq 0, \forall n,m,j,t,r$ are the abatement and expansion factors. The objective function model (7) measures the inefficiency score for the specific DMU_o.

In such a way that the reduction and increase of inputs, undesirable and desirable outputs are done simultaneously in the direction of the appropriate vector $\vec{d}=(d^{(x)},d^{(v)},d^{(w)},d^{(z)},d^{(y)})$.

In this manner, if the optimal value of zero obtained ($E_o^{*Overall}=0$), then the whole system is said to be efficient; otherwise, it is not efficient. Additionally, the inefficiency score of both stages for the two-stage process as Figure 1 is measured by two terms $E_o^{*Stage1}=\left[\sum_{n=1}^{N}\alpha_n^*+\sum_{m=1}^{M}\beta_m^*+\sum_{j=1}^{J}\gamma_j^*\right]$, and $E_o^{*Stage1}=\left[\sum_{j=1}^{J}\gamma_j^*+\sum_{j=1}^{T}\theta_t^*+\sum_{j=1}^{R}\varphi_r^*\right]$, which $\zeta_n^*,\tau_m^*,\gamma_j^*,\rho_t^*,\lambda_r^*\ \forall n,m,j,t,r$ represent the obtained optimal solutions of the model (7). Therefore, similarly, if the optimal value $E_o^{*Stage1}$, and $E_o^{*Stage2}$ are equal to zero, then the first and the second stage are said to be efficient; otherwise, they are inefficient.

Note that $w_j : j=1,...,J$ is an undesirable dual-role factor in the model (7). It means it plays the roles of output and input for stages 1 and 2, simultaneously. In this way, it should be decreased in both stages. For linking two stages of the mentioned network process, a common decrease factor (γ_j) for the intermediate measure of w_j.

Theorem 1. The linear model (7) is a feasible problem.

Proof. Considering $\vec{d}=(d^{(x)},d^{(v)},d^{(w)},d^{(z)},d^{(y)})$ as a directional vector, and the following solution:

$\varphi^o=\eta^o=0,\varphi^k=\eta^k=1,\ o\neq k$,

$\mu^k=\kappa^k=0 : k=1,...,K$, and

$\zeta_n=\tau_m=\gamma_j=\rho_t=\lambda_r=0$ for all n,m,j,t,r. Clearly, the model (7) is a feasible problem.

The dual to the linear model (7) is as follows:

$$\pi_o^*=\text{Min}\begin{bmatrix}\sum_{n=1}^{N}h_nx_n^o-\sum_{m=1}^{M}u_mv_m^o+2\sum_{j=1}^{J}f_jw_j^o\\+\sum_{t=1}^{T}q_tz_t^o-\sum_{r=1}^{R}g_ry_r^o+\delta_1+\delta_2\end{bmatrix}$$

$$\begin{aligned}
&s.t. && (8)\\
&\sum_{n=1}^{N}h_nx_n^k-\sum_{m=1}^{M}u_mv_m^k+\sum_{j=1}^{J}f_jw_j^k+\delta_1\geq 0, k=1,...,K,\\
&\sum_{j=1}^{J}f_jw_j^k+\sum_{t=1}^{T}q_tz_t^k-\sum_{r=1}^{R}g_ry_r^k+\delta_2\geq 0, k=1,...,K,\\
&\sum_{n=1}^{N}h_nx_n^k+\delta_1\geq 0, && k=1,...,K,\\
&\sum_{t=1}^{T}q_tz_t^k+\delta_2\geq 0, && k=1,...,K,\\
&\sum_{m=1}^{M}u_mv_m^k-\delta_1\geq 0, && k=1,...,K,\\
&\sum_{r=1}^{R}g_ry_r^k-\delta_2\geq 0, && k=1,...,K,\\
&h_nd_n^{(x)}\geq 1, && n=1,...,N,\\
&u_md_m^{(v)}\geq 1, && m=1,...,M,\\
&f_jd_j^{(w)}\geq 1, && j=1,...,J,\\
&q_td_t^{(z)}\geq 1, && t=1,...,T,\\
&g_rd_r^{(y)}\geq 1, && r=1,...,R,\\
&h_n,u_m,o_t,g_r\geq 0, && \textit{for all } n,m,t,r,\\
&f_j\forall j,\ \delta_1,\delta_2 && \textit{are free in sign.}
\end{aligned}$$

It is noteworthy that two constraints $\sum_{m=1}^{M}u_mv_m^k-\delta_1\geq 0$ and $\sum_{r=1}^{R}g_ry_r^k-\delta_2\geq 0$ in the model (8) were considered to prevent the negative cross-efficiency score. Also, the constraints $h_nd_n^{(x)}\geq 1$, $u_md_m^{(v)}\geq 1$, $f_jd_j^{(w)}\geq 1$, $q_td_t^{(z)}\geq 1$, $g_rd_r^{(y)}\geq 1$ are a guarantees to avoid zero weights. The optimal objective value of the model (8), i.e. π_o^*, is more than zero.

Let $h_n^{*(o)},u_m^{*(o)},f_j^{*(o)},q_t^{*(o)},g_r^{*(o)}\ \forall n,m,j,t,r$ be the obtained optimal weights from the model (8) for DMU_o. Therefore, the cross efficiency for the whole two-stage process of $DMU_k : k = 1,...,K$ and its components are defined as:

$$\pi_k^{*(o)Overall} = \frac{\sum_{m=1}^{M} u_m^{*(o)} v_m^k + \sum_{r=1}^{R} g_r^{*(o)} y_r^k - \delta_1^* - \delta_2^*}{\sum_{n=1}^{N} h_n^{*(o)} x_n^k + 2\sum_{j=1}^{J} f_j^{*(o)} w_j^k + \sum_{t=1}^{T} q_t^{*(o)} z_t^k} \quad (9)$$

$$\pi_k^{*(o)Stage1} = \frac{\sum_{m=1}^{M} u_m^{*(o)} v_m^k - \delta_1^*}{\sum_{n=1}^{N} h_n^{*(o)} x_n^k + \sum_{j=1}^{J} f_j^{*(o)} w_j^k} \quad (10)$$

$$\pi_k^{*(o)Stage2} = \frac{\sum_{r=1}^{R} g_r^{*(o)} y_r^k - \delta_2^*}{\sum_{j=1}^{J} f_j^{*(o)} w_j^k + \sum_{t=1}^{T} q_t^{*(o)} z_t^k} \quad (11)$$

According to the constraints, $\sum_{n=1}^{N} h_n x_n^k + \delta_1 \geq 0$, $\sum_{m=1}^{M} u_m v_m^k - \delta_1 \geq 0$, and $\sum_{n=1}^{N} h_n x_n^k - \sum_{m=1}^{M} u_m v_m^k + \sum_{j=1}^{J} f_j w_j^k + \delta_1 \geq 0$, we have $0 < \frac{\sum_{m=1}^{M} u_m^* v_m^k - \delta_1^*}{\sum_{n=1}^{N} h_n^* x_n^k + \sum_{j=1}^{J} f_j^* w_j^k} \leq 1$.

Therefore, for definition (10), we get $0 < \pi_k^{*(o)Stage1} \leq 1$ for $o,k = 1,...,K$. Similarly, regarding the second, fourth and sixth constraints of the model (8), for definition (11), we have $0 < \pi_k^{*(o)Stage2} \leq 1$ for $o,k = 1,...,K$. For the overall efficiency, according to the constraints set of the model (8), we have $0 < \pi_k^{*(o)Overall} \leq 1, \forall o,k$.

Therefore, DMU_o will be efficient in general if $\pi_k^{*(o)Overall} = 1$; otherwise, it will be inefficient. Moreover, the first and the second stages of DMU_o will be efficient if $\pi_k^{*(o)Stage1} = 1$, and $\pi_k^{*(o)Stage2} = 1$. An interesting point to note is the relationship between the overall efficiency and each of the components of a two-stage process. In other words, a two-stage process is efficient as a whole when both of its components are efficient.

So, the overall cross-efficiency for $DMU_k : k = 1,...,K$ is defined as follows:

$$\tilde{\pi}_k^{*Overall} = \frac{1}{k} \sum_{o=1}^{K} \pi_k^{*(o)Overall} \quad (12)$$

Also, the cross-efficiency value for each component of $DMU_k : k = 1,...,K$ is defined as follows:

$$\tilde{\pi}_k^{*Stage1} = \frac{1}{k} \sum_{o=1}^{K} \pi_k^{*(o)Stage1} \quad (13)$$

$$\tilde{\pi}_k^{*Stage2} = \frac{1}{k} \sum_{o=1}^{K} \pi_k^{*(o)Stage2} \quad (14)$$

According to the above definitions, $\pi_k^{*Overall}$, $\pi_k^{*Stage1}$, and $\pi_k^{*Stage2}$ are the average cross-efficiencies obtained with the weights of all decision-making units. Therefore, the cross-efficiencies defined above also belong to $(0,1]$.

4. An application to industrial production in China

Nowadays, one of the most significant problems that many countries encounter is increasing industrial pollution. This problem leads to environmental pollution and causes harmful effects on the health of people in society. Therefore, in the past decade, researchers have drawn engaged in recycling issue in the manufacturing industry. China is one of the countries that has made great efforts in this direction. Regarding the importance of the recycling process, an appropriate model that can properly evaluate such systems and detect their strength and weakness is critically important. Now, to analyze the proposed method, the performance of 30 industrial production centers in China which have a two-stage structure following Figure 2 is evaluated in this section.

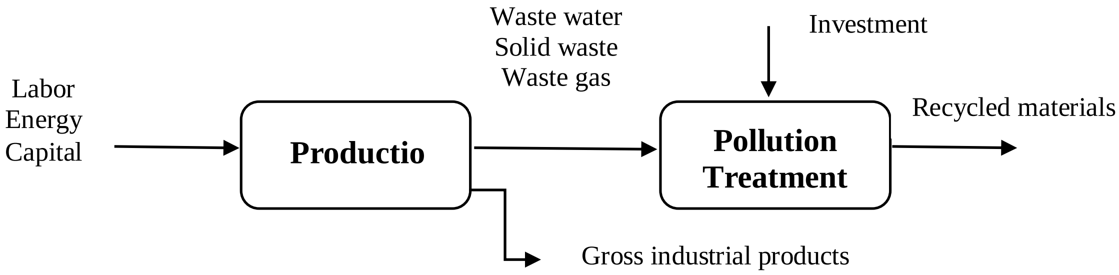

Figure 2- Structure of the industrial productions process

The inputs and outputs of each of the components of the two-stage process depict as follows:

- **Stage 1: Production stage**
  Inputs:
  Labor (x_1), Energy (x_2), Capital (x_3),
  Undesirable outputs:
  Wastewater (w_1), Solid waste (w_2), Waste gas (w_3),
  Desirable output:
  Gross industrial products (v),
- **Stage 2: Pollution treatment stage**
  Inputs:
  Investment (z), Wastewater (w_1), Solid waste (w_2), Waste gas (w_3),
  Desirable output:
  Recycled materials (y).

It should be noted that intermediate measures wastewater (w_1), waste gas (w_2) and solid waste (w_3) have a dual role (input and output).

In what follows, the input and output values of each component of industrial production processes in 31 regions of China are given in Table 1. (See Wu et al. [22]).

According to data in Table 1, the directional efficiency and the cross-efficiency for each DMU_k $(k=1,...,K)$ were evaluated by models (8), (12), (13) and (14) and the results obtained are given in Table 2. In this example, the vector of direction $\vec{d}=(x_o, v_o, w_o, z_o, y_o)$ is considered. In other words, the results were obtained in direction of reducing inputs and undesirable outputs and increasing desirable outputs.

Table 1- Data for the industrial production processes in China

DMUs	x_1	x_2	x_3	v	w_1	w_2	w_3	z	y
1	124.15	6954	22750.58	13699.84	8198	4750	1269	1.9026	3.43658
2	148.91	6818	14584.31	16751.82	19680	7686	1862	8.32203	19.26504
3	344.67	27531	24943.75	31143.29	114232	56324	31688	10.67334	107.1801
4	219.88	16808	18505.94	12471.33	49881	35190	18270	23.47653	42.63718
5	125.19	16820	14691.38	13406.11	39536	27488	16996	11.70925	27.23754
6	401.74	20947	29076.78	36219.42	71521	26955	17273	14.25687	32.80902
7	139.81	8297	10196.15	13098.35	38656	8240	4642	6.2945	39.16633
8	147.6	11234	10471.17	9535.15	38921	10111	5405	4.22225	32.34714
9	291.62	11201	27555.88	30114.41	36696	12969	2448	4.11153	17.03791
10	1153.88	25774	66134.06	92056.48	26376	31213	9064	15.52205	218.9749
11	857.58	16865	47282.79	51394.2	217426	20434	4268	11.39896	286.3867
12	264.87	9707	15930.28	18732	70971	17849	9158	4.51817	56.69216
13	411.75	9809	16058.7	21901.23	124168	13507	7487	12.84866	37.50288
14	199.16	6355	8637.45	13883.06	72526	9812	9407	5.95067	59.34731
15	931.5	34808	53761.28	83851.4	208257	43837	16038	36.4491	187.1898

16	479.27	21438	23467.42	3495.53	150406	22709	10714	12.07734	74.39088
17	294.97	15138	20894.32	21623.12	94593	13865	6813	24.24997	82.28357
18	272.44	14880	13038.95	19008.83	95605	14673	5773	13.43145	90.12068
19	1568	26908	62626.9	85824.64	187031	24092	5456	20.90697	62.42653
20	150.51	7919	8667.45	9644.13	165211	14520	6232	9.16614	51.02334
21	12.44	1359	1621.38	1381.25	5782	1360	212	0.41153	3.16232
22	146.56	7856	8099.01	9143.25	45180	10943	2837	6.83182	29.1366
23	351.67	17892	22564.76	23147.38	93444	20107	11239	7.00433	45.78465
24	80.3	8175	5960.13	4206.37	14130	10192	8188	6.51415	17.91425
25	92.6	8674	9611.09	6464.63	30926	10978	9392	10.33956	65.45546
26	151.08	8882	14688.7	11199.8	45487	13510	6892	25.22795	29.34996
27	71.34	5923	6487.35	4882.68	15325	6252	3745	13.63106	22.41208
28	20.09	2568	3053.61	1481.99	9031	3952	1783	0.97472	5.51878
29	29.04	3681	3293.16	1924.39	21977	16324	2465	2.9096	10.07503
30	60.18	8290	7911.97	5341.9	25413	9310	3914	6.67628	22.21873

Table 2- Directional Efficiency and Cross-Efficiency Results

Units	**Directional Efficiency**			**Cross-Efficiency**					
	Overall	**Stage 1**	**Stage 2**	**Overall**	**Rank**	**Stage 1**	**Rank**	**Stage 2**	**Rank**
1	0.6150	1	0.2396	0.6283	6	0.7740	6	0.1400	25
2	0.7572	1	0.2782	0.6507	5	0.9263	2	0.2163	17
3	0.4379	0.5143	0.3139	0.3613	20	0.4227	21	0.2363	14
4	0.2012	0.3073	0.1192	0.2120	29	0.2880	28	0.1153	28
5	0.3051	0.4847	0.1557	0.2700	26	0.3790	24	0.1087	30
6	0.5012	0.7577	0.1211	0.4423	13	0.6253	10	0.1103	29
7	0.8033	0.9002	0.3846	0.5560	9	0.6703	8	0.3467	7
8	0.3927	0.4313	0.3350	0.4093	16	0.4520	19	0.3020	10
9	0.7611	1	0.2254	0.7063	4	0.8990	4	0.1877	20
10	1	1	1	0.9617	1	0.9977	1	0.8753	2
11	0.8875	0.8080	1	0.7377	3	0.6390	9	0.9450	1
12	0.4609	0.5185	0.3745	0.4783	12	0.5383	14	0.3297	8
13	0.7216	0.8606	0.1735	0.4010	18	0.5383	15	0.1557	24
14	0.9042	1	0.3997	0.4910	11	0.5530	12	0.3580	6
15	0.8936	1	0.3423	0.6107	7	0.7793	5	0.3163	9
16	0.1383	0.1070	0.2320	0.1423	30	0.0783	30	0.2450	13
17	0.454	0.5216	0.3527	0.4333	14	0.5443	13	0.2833	11
18	0.7625	0.8590	0.4186	0.5107	10	0.5657	11	0.4000	4
19	0.6512	1	0.1743	0.6070	8	0.7707	7	0.1763	22
20	0.3381	0.4304	0.2237	0.2913	24	0.3370	26	0.2093	18
21	1	1	1	0.7673	2	0.9077	3	0.4397	3
22	0.3995	0.4895	0.2646	0.4183	15	0.5133	16	0.2533	12
23	0.3597	0.4538	0.2373	0.4010	17	0.4880	17	0.2083	19
24	0.2844	0.3583	0.2062	0.2493	27	0.3070	27	0.1587	23
25	0.4131	0.3768	0.4675	0.3683	19	0.3377	25	0.3900	5
26	0.3486	0.4529	0.1931	0.2810	25	0.4443	20	0.1193	27
27	0.3976	0.4794	0.2696	0.3153	23	0.4540	18	0.1847	21
28	0.4311	0.4826	0.3666	0.3550	21	0.4147	22	0.2287	15
29	0.2874	0.3509	0.2215	0.2123	28	0.2623	29	0.1357	26
30	0.3574	0.4372	0.2472	0.3287	22	0.3990	23	0.2203	16

By referring to definitions (9), (10), and (11), $DMU_k \ (k=1,...,K)$ is said to be efficient if and only if its efficiency score is equal to one. Therefore, the efficiency score of one in columns two, three and four, respectively, show which DMU is overall efficient or each of its subsections is efficient. For example, the first stage of units 1, 29, 10, 14, 15, 18, and 21 are efficient while the second stage of units 10, 11, and 21 are efficient. It should be noted that a unit is overall efficient when both its subsections are efficient. For this reason, the second column shows that only two units 10 and 21 among the others are overall efficient. Columns two and three show that unit 16 has the least efficiency score among other units while unit 4 has the least efficiency score in the fourth column.

By using the obtained optimal weights, the cross-efficiencies were calculated. The fifth to tenth columns show cross-efficiency scores and ranks for the whole two-stage structure of $DMU_k \ (k=1,...,K)$ and each of its sub-sections. The cross-efficiencies results show that unit 10 has the first rank on the whole. It means that it has the best evaluation among others. Unit 10 has the first rank in stage 1 while it has the second rank in stage 2. Unit 11 has the first rank in the second stage. In the other words, generally, unit 10 has good performance among all units. Columns six and eight show that unit 16 has the last rank while the tenth column shows that unit 5 has the last rank. It means that the mentioned units do not have good performance among 30 units.

The scattering of the cross-efficiency score for the intended two-stage process and each of its components is well in Figure 3. As we know, the overall efficiency of two-stage structures is always a value between the efficiencies of its components. The graphs in Figure 3 show this well.

The optimal weights corresponding to model (8) are provided in Table 3.

As you can see in Table 3, the optimal weight of none of the units is zero. This is one of the strengths of the proposed model that was mentioned earlier. Variables δ_1 and δ_1, which correspond to the convexity restrictions of stages 1 and 2 in the model (7), except for a few cases, have negative values. The positive or negative nature of this variable means increasing or decreasing returns to scale, which is not the subject of discussion in this research.

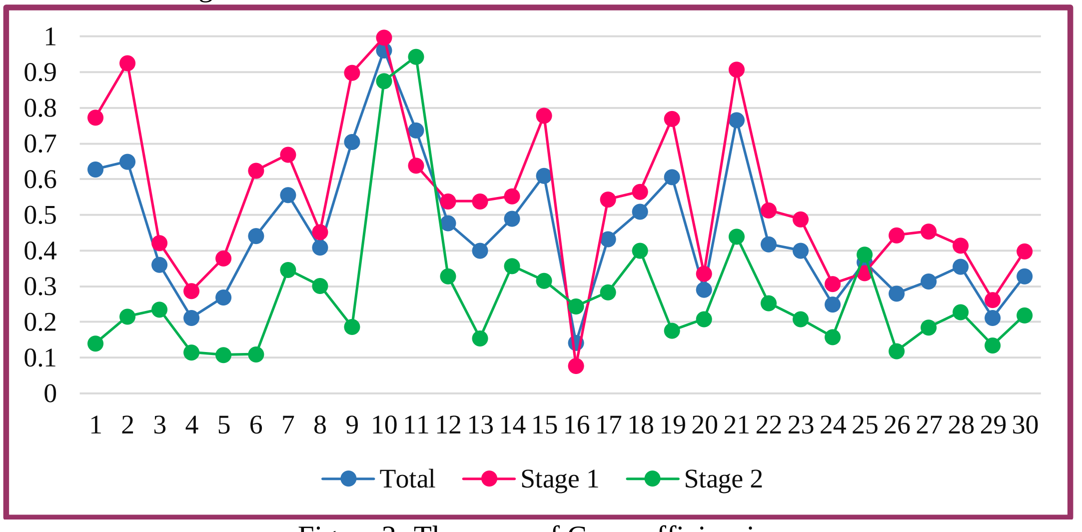

Figure 3- The score of Cross-efficiencies

Table 3- The set of optimal weights

Units	$h^{(x_1)}$	$h^{(x_2)}$	$h^{(x_3)}$	u	q	$f^{(w_1)}$	$f^{(w_2)}$	$f^{(w_3)}$	g	δ_1	δ_2
1	0.027427	0.000144	0.000044	0.000615	2.963176	0.290987	0.000198	0.000211	0.000788	-0.607900	-1.219400
2	0.019404	0.000147	0.000069	0.000439	0.120163	0.055188	0.000051	0.000130	0.000537	-0.529600	-0.049500
3	0.004341	0.000036	0.000040	0.000104	0.093691	0.011357	0.000009	0.000018	0.000032	-0.106400	-0.038600
4	0.004548	0.000059	0.000054	0.000131	0.202920	0.023454	0.000020	0.000028	0.000055	-0.210300	0.074200
5	0.009375	0.000059	0.000068	0.000206	0.377452	0.036714	0.000025	0.000036	0.000059	-0.231600	-0.155300
6	0.013929	0.000048	0.000034	0.000240	0.280768	0.030479	0.000020	0.000037	0.000058	0.332000	0.096400
7	0.038373	0.000121	0.000776	0.001135	0.158869	0.037605	0.000026	0.000121	0.000215	-0.692900	-0.065400
8	0.006775	0.000089	0.000096	0.000234	0.236841	0.038407	0.000026	0.000099	0.000185	-0.359600	-0.097500
9	0.020161	0.000089	0.000036	0.000383	0.370631	0.058693	0.000043	0.000077	0.000409	0.076500	-0.152500
10	0.000867	0.000039	0.000015	0.000064	0.064424	0.018146	0.000038	0.000032	0.000110	-0.088000	-0.026500
11	0.001166	0.000059	0.000021	0.000104	0.087727	0.016613	0.000005	0.000049	0.000420	-0.129400	-0.036100
12	0.003775	0.000103	0.000063	0.000152	0.221329	0.024814	0.000014	0.000056	0.000109	-0.259300	-0.091100
13	0.002429	0.000102	0.001028	0.000876	0.077829	0.026665	0.000008	0.000218	0.000134	-0.997300	-0.032000
14	0.005021	0.000157	0.001859	0.001402	0.168048	0.025771	0.000014	0.000102	0.000106	-1.595200	-0.069200
15	0.001074	0.000029	0.000293	0.000244	0.027436	0.007255	0.000005	0.000023	0.000062	-0.261600	-0.011300
16	0.002087	0.000047	0.000312	0.000286	0.093296	0.013443	0.000007	0.000044	0.000093	-0.318500	0.042500
17	0.003390	0.000066	0.000048	0.000135	0.041237	0.016942	0.000011	0.000072	0.000147	-0.209500	-0.017000
18	0.003671	0.000067	0.000709	0.000610	0.074452	0.018242	0.000010	0.000068	0.000173	-0.633100	-0.030600
19	0.000638	0.000037	0.000016	0.000086	0.047831	0.016019	0.000006	0.000042	0.000416	-0.084300	0.050700
20	0.006644	0.000126	0.000115	0.000239	0.200652	0.019599	0.000006	0.000069	0.000160	-0.273300	-0.082600
21	0.080386	0.000736	0.000617	0.002391	18.104836	0.948671	0.000173	0.000735	0.004717	-2.697800	-7.450700
22	0.006823	0.000127	0.000123	0.000278	0.146374	0.034321	0.000022	0.000091	0.000352	-0.402600	-0.060200
23	0.002844	0.000056	0.000044	0.000110	0.230469	0.021841	0.000011	0.000050	0.000089	-0.179800	-0.094800
24	0.012453	0.000122	0.000168	0.000370	0.409473	0.055821	0.000071	0.000098	0.000122	-0.593200	-0.168500
25	0.010799	0.000115	0.000104	0.000289	0.096716	0.027960	0.000032	0.000091	0.000106	-0.394300	-0.039800
26	0.010494	0.000113	0.000068	0.000282	0.039639	0.034072	0.000025	0.000156	0.000145	-0.393900	-0.016300
27	0.017711	0.000169	0.000154	0.000472	0.073362	0.046774	0.000065	0.000160	0.000267	-0.699700	-0.030200
28	0.049776	0.000389	0.000327	0.001121	1.843319	0.181199	0.000111	0.000253	0.000561	-1.234400	-0.758600
29	0.034435	0.000272	0.000304	0.000790	0.947830	0.099255	0.000069	0.000061	0.000406	-0.768300	-0.390100
30	0.017523	0.000121	0.000126	0.000412	0.208405	0.045007	0.000039	0.000107	0.000255	-0.445200	-0.085800

5. Conclusions

Evaluating cross-efficiency and solving its two fundamental problems, i.e. non-uniqueness of efficiency value and zero coefficients, has been the basis of various studies by researchers in recent years. This paper studies the evaluation of the two-stage network structures and their components based on cross-efficiency. For this purpose, the directional distance function under the assumption of variable returns to scale has been used, which is the point of distinction between the proposed model and the previously introduced methods. The intended two-stage system consists of undesirable factors. Therefore, for handling the desirable and undesirable outputs, the weak disposability assumption was used. Two restrictions were also considered to prevent negative cross-efficiency. The proposed model can reduce the zero coefficients which is one of its significant advantages. For further analysis, the performance of the industrial productions of 30 regions in China based on the cross-efficiency is examined.

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