Journal of Theoretical and Applied Physics (2020) 14:349–357
https://doi.org/10.1007/s40094-020-00391-4

RESEARCH

Transmission of electromagnetic waves through a nonlinear over-dense plasma slab

M. Tohfeh[1] · L. Rajaei[2] · S. Miraboutalebi[1] · L. F. Matin[1]

Received: 15 June 2020 / Accepted: 13 August 2020 / Published online: 29 August 2020

Abstract
A study on the transmission of the electromagnetic waves from a structure consisting of an over-dense plasma layer with nonlinear effects is undertaken. The applied nonlinearity is presented due to the series expansion of the polarization in the medium. The nonlinear plasma layer is supposed to be placed between two linear dielectric layers. The transparency conditions are investigated for the *p*-polarized obliquely incident waves. It is shown that the formation of the surface waves can eventuate to the transmission of the incident waves. This fact has already be seen in the case of the linear over-dense plasma, but, here, it is examined in the presence of the nonlinear effects. To determine the propitious conditions for the surface wave excitation, the exact solutions of the electromagnetic field equations are used in all regions. The transmission, the reflection, and the dissipation rates of the electromagnetic waves from the entire structure are obtained, and the effects of the main parameters on them are discussed.

Keywords Surface wave · Nonlinear effect · Over-dense plasma · Transition amplitude · Reflection amplitude

Introduction

There is a great deal of interest in the studying of the left-handed materials [1–3] today, due to their many applications in science and technology [4–9]. The high-density plasma also behaves as the left-handed material in response to the incident electromagnetic wave in frequency ω smaller than the plasma frequency $\omega_{\rm p}$, where the plasma permittivity is defined as $\varepsilon_{\rm p} = 1 - \frac{\omega_{\rm p}^2}{\omega^2}$. The anomalous transmission of the electromagnetic waves through left-handed materials takes place due to the mechanism of the surface waves excitation [10–12]. However, one of the most important initials to trigger this mechanism is to producing the evanescent waves which can simply be provided by placing dielectrics or under-dense plasma layers adjacent to the over-dense plasma [13, 14]. Also, equipping the dielectric with diffraction grating facilitates the production of the evanescent waves. Also, by applying an appropriate density profile for the plasma density, one can produce the required evanescent waves without using the dielectric layers [15].

It is well known that the susceptibility of the dielectric can generally be a function of the electric field. Hence, the nonlinear plasma property considered here comes from the expansion of the polarization in power series of the electric field E [16, 17]. The corresponding electric susceptibility χ then can be written as:

$$\chi = \chi_1 + \chi_{\rm nl}^{(1)} E + \chi_{\rm nl}^{(2)} E^2 + \cdots, \qquad (1)$$

where χ_1 is the usual linear susceptibility and $\chi_{\rm nl}^{(i)}$ denotes its ith nonlinear correction. These nonlinear corrections have significant effects on the behavior of a plasma medium. Here, we are interested to study the effect of the second-order nonlinearity on the transmission of the electromagnetic waves through an over-dense plasma layer. An investigation on the third-order nonlinearity can be found in [18].

✉ S. Miraboutalebi
s_mirabotalebi@iau-tnb.ac.ir

M. Tohfeh
m.tohfeh@gmail.com

L. Rajaei
l-rajaei@qom.ac.ir

L. F. Matin
laleh.matin@gmail.com

1 Department of Physics, North Tehran Branch, Islamic Azad University, Tehran, Iran

2 Department of Physics, Faculty of Basic Sciences, University of Qom, Qom, Iran

There exist a large number of investigations devoted to the nonlinear surface waves. Among them, the transmission of electromagnetic waves through a nonlinear layer surrounded by linear media has attracted much attention over the last four decades [19–59]. The exact solutions are available only for the Transverse Electric (TE)-polarized (*s*-polarized) waves propagating through the media with the nonlinearity. The problems deal with the Transverse Magnetic (TM) polarization (*p*-polarized) wave propagation can be solved by using the approximate methods namely the perturbation and the numerical methods. Also, there exists an integral equation method in which the eigenvalues of the problem are obtained, and the issue is reduced to the analysis of the corresponding dispersion equation [38–53]. In this method, an equivalency is established between the boundary eigenvalue problem and the dispersion equation, then the solutions can numerically be achieved from the dispersion equation. However, here, we use the exact solutions in the form of the appropriate Jacobi elliptic function [54–59], and study the transmission from the entire considered structure.

It should be noticed that the *p*-polarized wave facilitates the excitation of the surface wave and can provide the transmission of the wave from the over-dense plasma slab without absorption. The investigations show that the *p*-polarized wave can help the excitation of the surface wave and passing the wave from the over-dense plasma slab without absorption. However, in the excitation of plasma surface waves on the surface of the over-dense plasma, the absorption can be provided by applying a special condition such as entering a dissipation factor into the model. Here, it is supposed that the plasma is collisional, and therefore, we have absorption, the amount of absorption is not calculated directly and the effect of the dissipation in the transmission rate is investigated.

The organization of this article is as follows: "The model" section is devoted to introducing the model and the suppositions. In this section, the analytical solution of the governing equations of the field quantities in the nonlinear plasma layer is also presented which are expressed in terms of the Jacobi elliptic functions. In "Transmission of the incident electromagnetic waves" section, the conditions for wave transmission through the structure are studied. It is shown that the waves transmit through the structure, by the appearance of the surface waves on both sides of the over-dense plasma layer. To this end, the exact solutions of the waves of each layer are obtained in terms of the Jacobi elliptic functions and some unknown coefficients. By applying appropriate boundary conditions, the coefficients are then specified via a numerical method. Knowing the field components in the layers, we investigate the behavior of the electromagnetic waves through the structure and find the transmission, the reflection, and the dissipation rates of the waves from the entire slab. Finally, some concluding remarks are presented.

The model

To investigate the transmission of the electromagnetic waves through a nonlinear over-dense plasma medium, we consider here the structure illustrated in Fig. 1. This structure consists of a nonlinear over-dense plasma slab placed between two similar linear dielectric layers, and the entire structure is supposed to be immersed in vacuum. The electromagnetic waves in each layer of the structure are governed via the following set of the cold fluid-Maxwell equations:

$$\frac{\partial \mathbf{V}}{\partial t} = -\frac{e}{m}\mathbf{E} - \acute{\nu}\mathbf{V}, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B} = -\frac{4\pi n_0 e}{c}\mathbf{V} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{4}$$

where ions are considered to be motionless and m, e, and n_0 stand for the mass, the charge, and the density of electron, respectively. Here, c is the speed of light and $\acute{\nu}$ denotes the collision frequency. Considering the time-dependent parts of our quantities as $e^{-i\omega t}$, Eqs. (2–4) are reduced to the following equation for determining the electric field $\mathbf{E}$:

$$\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2}\varepsilon\mathbf{E}, \tag{5}$$

where $\varepsilon = 1 - \frac{\omega_p^2}{s\omega^2}$ with $\omega_p^2 = \frac{4\pi n_0 e^2}{m}$. Then, the magnetic field $\mathbf{B}$ and the velocity can be found by using:

$$\mathbf{B} = -\frac{ic}{\omega}\nabla \times \mathbf{E}, \tag{6}$$

$$\mathbf{V} = \frac{-ie}{m\omega s}\mathbf{E}. \tag{7}$$

where $s = 1 + i\frac{\acute{\nu}}{\omega}$. Considering the nonlinear effects according to Eq. (1), the electric permittivity of the plasma medium becomes:

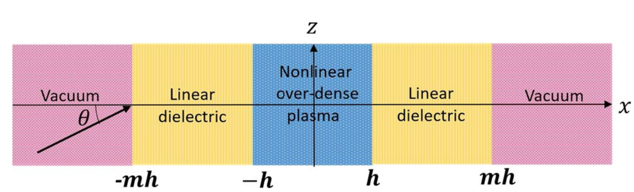


Fig. 1 The assumed structure consists of a nonlinear over-dense plasma slab which placed between two similar dielectric layers. The structure is unbounded at the y and z axis

$$\varepsilon = \varepsilon_L + \alpha E_z^2, \tag{8}$$

where ε_L is the linear part of the permittivity and α is the nonlinear constant. The electric permittivity Eq. (8) is truncated up to the second order of the electric field and ignored the first and the higher-order terms.

Let us consider a p-polarized electromagnetic wave with the electric components $\vec{E} = (E_x, 0, E_z)$ and magnetic components $\vec{B} = (0, B_y, 0)$, obliquely incident to the structure of Fig. 1. Then, the field quantities can be considered as, $\psi(x)e^{ik_z z}$ which by substituting Eq. (8) in Eq. (5), leads to the following equation for determining E_z and E_x, respectively:

$$\frac{d^2}{dx^2}E_z + \frac{\omega^2}{c^2}\varepsilon_L E_z + \frac{\omega^2}{c^2}\alpha E_z^3 = 0, \tag{9}$$

$$E_x = \frac{-ik_z \partial_x E_z}{k_z^2 - \frac{\omega^2}{c^2}(\varepsilon_L + \alpha E_z^2)}. \tag{10}$$

In deriving these equations, we assume that $E_x \ll E_z$ and ignore E_x^2 against E_z^2. Also, the term $k_z \frac{dE_x}{dx}$ is supposed to produce a neglecting effect and is omitted.

Equation (9) is a nonlinear differential equation and can be rewritten as

$$\frac{d^2}{dx^2}E_z + c_1 E_z + c_3 E_z^3 = 0, \tag{11}$$

where

$$c_1 = \frac{\omega^2}{c^2}\varepsilon_L, \quad c_3 = \frac{\omega^2}{c^2}\alpha. \tag{12}$$

Its solution is [60, 61]:

$$E_z(x) = \mathcal{A} cn(kx - \phi, f), \tag{13}$$

where cn is a kind of the Jacobi elliptic functions [62], and:

$$\begin{aligned} k^2 &= c_1 + c_3\mathcal{A}^2 = c_1(1 + \upsilon), \\ f &= \frac{c_3\mathcal{A}^2}{2(c_1 + c_3\mathcal{A}^2)} = \frac{\upsilon}{2(1+\upsilon)}. \end{aligned} \tag{14}$$

Here, $\mathcal{A}$ and ϕ are two constants determined by the boundary conditions and υ is the nonlinear factor $\upsilon = \frac{c_3}{c_1}\mathcal{A}^2$.

It is noteworthy that for a special case, specified by the condition $f = 1$ or $\upsilon = -2$ or equivalently $\mathcal{A} = -2\frac{c_1}{c_3}$, $E_z(x)$ reduces to:

$$E_z(x) = \left(\frac{-2\varepsilon_L}{\alpha}\right)^{\frac{1}{2}} \operatorname{sech}(kx - \phi), \tag{15}$$

which is in conformation with the electromagnetic field solutions acquired in the literature in the case of the semi-bounded nonlinear plasma [63].

The magnetic field is obtained from Eq. (6) as:

$$B_y = \frac{i\varepsilon}{\sin^2\theta - \varepsilon}\, \mathcal{A}\, k\, dn(kx - \phi, f) sn(kx - \phi, f), \tag{16}$$

where dn and sn are two other kinds of the Jacobi elliptic functions and θ is the incident angle.

Transmission of the incident electromagnetic waves

Let us now turn to study the conditions for the transmission and the reflection of the electromagnetic waves from the structure of Fig. 1. It has been shown that the wave transmission through a linear over-dense plasma slab is placed between two ordinary dielectric layers, can be taken place because of the excitation of the surface waves. The surface waves are known to be formed on both sides of the over-dense plasma layers simultaneously and transmit the energy of the electromagnetic waves. Here, we study these phenomena for the nonlinear over-dense plasma.

For determining the magnetic field component of the surface wave in the dielectric layers of Fig. 1, we note that Eqs. (2–4) leads to the following equation:

$$\frac{d^2 B_y}{dx^2} + \frac{\omega^2}{c^2}(\varepsilon_d - \sin^2\theta)B_y = 0, \tag{17}$$

where ε_d is the dielectrics permittivity. The general solutions of this equation are given by:

$$B_y = A_i e^{k_d x} + B_i e^{-k_d x}, \quad k_d^2 = \frac{\omega^2}{c^2}(\sin^2\theta - \varepsilon_d) \tag{18}$$

where the index i is $i = 1, 2$ in the first and the second dielectric layers of Fig. 1, placed in region $(-mh \le x \le -h)$ and $(h \le x \le mh)$, respectively. The corresponding electric field components can then be calculated by applying:

$$\mathbf{E} = i\frac{c}{\omega}\frac{\nabla \times \mathbf{B}}{\varepsilon_d}. \tag{19}$$

Also, in the vacuum regions, B_y becomes:

$$B_y = E_0 e^{\frac{i\omega}{c}\cos\theta\, x} + Re^{-\frac{i\omega}{c}\cos\theta\, x}, \quad x \le -mh \tag{20}$$

$$B_y = Te^{i\frac{\omega}{c}\cos\theta\, x}, \quad mh \le x, \tag{21}$$

where T and R are assigned to the field components of the transmitted and the reflected waves. The corresponding electric fields can then be found by substituting Eqs. (20)

and (21) in Eq. (19) with setting $\varepsilon_d = 1$. In the nonlinear over-dense plasma region, $-h \le x \le h$, the electric and magnetic field components are given by Eqs. (13) and (16), respectively.

To study the conditions for the transmission and the reflection of the electromagnetic fields from the considered structure, the eight unknown coefficients A_i, B_i, $\mathcal{A}$, ϕ, R and T should be determined. Here, matching the boundary conditions, consisting of the continuity of B_y and E_z on the four boundaries, provides eight equations for obtaining these unknown coefficients. Here, we apply the numeric methods to solve the equations. Throughout these calculations, we only consider the physical results, namely those obey the energy conservation law $\left(|\frac{T}{E_0}|^2 + |\frac{R}{E_0}|^2\right) \le 1$. The solutions are illustrated in the following figures.

Figure 2 shows the rates of the transmission, the reflection, and the dissipation amplitudes versus the incident angle θ. According to this figure, the nonlinear over-dense plasma can transmit well the incident electromagnetic wave with the transmission rate above 0.4 ($\left|\frac{T}{E_0}\right|^2 > 0.4$) for a wide range of the incident angle ($0.42 < \theta < 0.88$). The maximum transmission properties can be archived at $\theta = 0.68$ rad. It should be noticed that the anomalous transmission of the electromagnetic wave through a linear over-dense plasma slab has already been observed. In the linear case, it has been shown that there is a sharp increase in the transmission properties of the structure when the incident angle reaches its resonant values. This resonant transmission takes place due to the excitation of the surface waves at the interfaces of the linear over-dense plasma layer and the dielectrics. Here, for the nonlinear over-dense plasma, the anomalous transmission properties can also be related to the formation of the surface waves.

We plot the spatial distribution of the magnetic field $|B_y|$ throughout the structure in Fig. 3. This figure clearly shows the formation of the surface waves at the boundaries of the over-dense plasma namely at $x = -h$ and $x = h$. This figure is plotted for the conditions of the maximum transmission point of Fig. 2, at $\theta = 0.68$ rad (Table 1).

To study the effect of the nonlinear over-dense plasma widths on the transmission properties of the structure, we plot the transmission rate as a function of h in Fig. 4. It is obvious from the figure that the high transmission takes place at low widths of the over-dense plasma layer and by increasing the widths the transmission reduces.

The effect of the widths of the dielectric layers on the transmission properties of the structure is presented in Fig. 5. This figure shows the transmission rate as a function of m, where m specifies the widths of the dielectric layers by mh, as shown in Fig. 1. According to the figure, there is a maximum in the transmission properties of the structure. However, by increasing the widths of the dielectric layers, the transmission reduces.

As it is expected, by increasing the dissipation effects, the transmission rate decreases. This can be seen in Fig. 6, where the transmission rate is plotted as a function of the dissipation parameter ν for three different values of ν, where $h = 0.3$. The solid and the dotted lines are, respectively, for the transmission and the dissipation rates. By acquiring greater values of ν, the dissipation rates also grow.

The value referred to the nonlinear parameter α in literature is $\alpha = 10^{-8}$ esu, according to [17]. However, here, we study the effect of different values of α on the transmission properties of the considered structure. Figure 7 displays the transmission rate amplitude as a function of the nonlinear parameter α. The figure shows that the variation of α has no effect on the transmission rate for $10^{-12} \le \alpha \le 10^{-6}$. However, for $10^{-6} \le \alpha \le 10^{-4}$, by increasing α the transmission

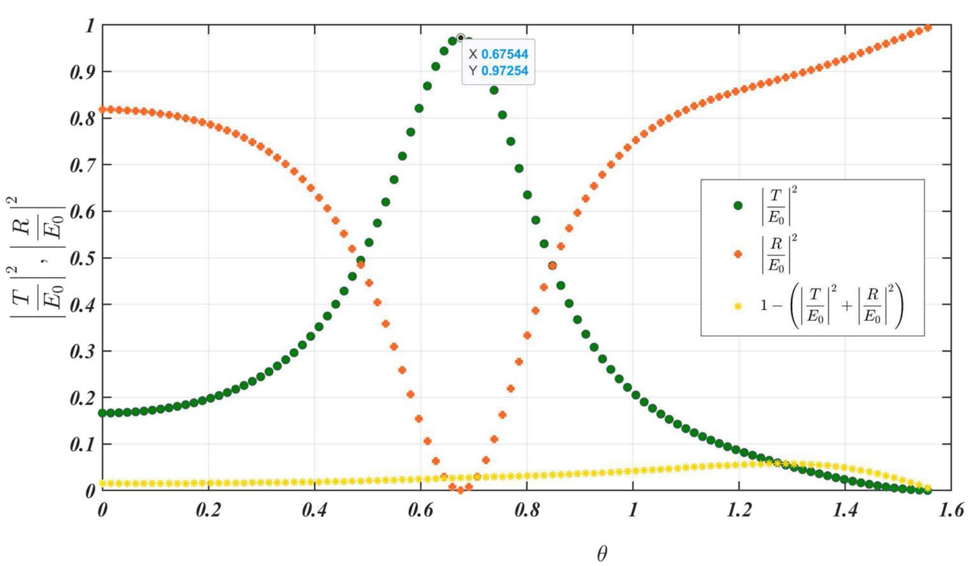


Fig. 2 The rates of transmission, reflection, and dissipation versus the incident angle θ. The structure has high transmission $\left|\frac{T}{E_0}\right|^2 \approx 0.97$ at $\theta \approx 0.68$ rad where $\left|\frac{R}{E_0}\right|^2 \approx 0$ and $1 - \left(\left|\frac{T}{E_0}\right|^2 + \left|\frac{R}{E_0}\right|^2\right) \approx 0.027$. The nonlinear plasma width is $2h = 0.4\ \frac{c}{\omega}$ and each of the linear dielectric widths are $mh = 1.76\ \frac{c}{\omega}$

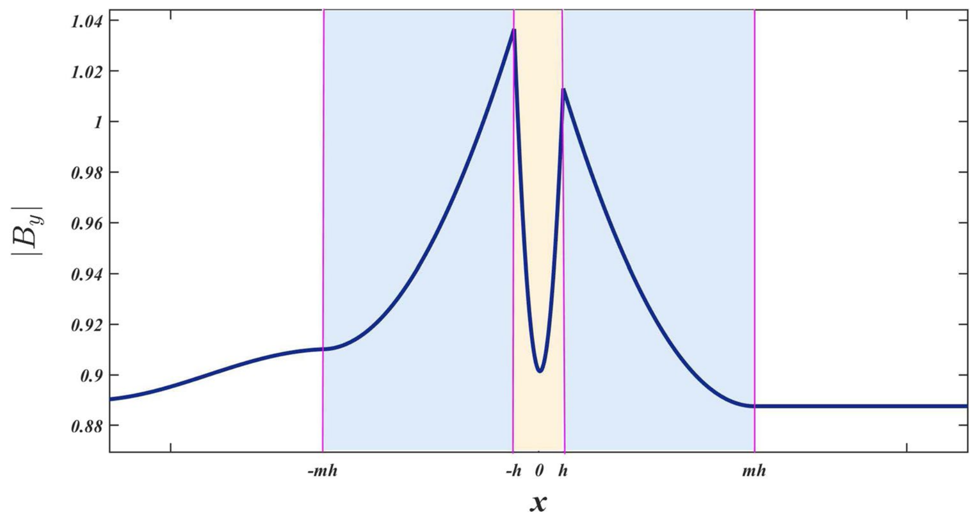


Fig. 3 The formation of the surface waves at the boundaries of the over-dense plasma slab and the two boundaries namely at $x = -h$ and $x = h$, where x is in units of $\frac{c}{\omega}$. The spatial distribution of the magnetic fields $|B_y|$ in different regions of the structure for the incident angle $\theta = 0.67544$ rad and the parameters of Fig. 2

Table 1 The parameters used in most figures

$\nu = \frac{\acute{\nu}}{\omega}$	ϵ_d	ϵ_L	α (esu)
0.02 [64]	0.3428 [11, 12]	− 2.97 [11, 12]	10^{-8} [17]

properties slightly grow. For larger values of α, namely for $\alpha > 10^{-4}$, no physical result can be found for the parameters we have considered. But by changing the dealing parameters such as E_0, θ, h, m,... the physical transmission for higher values of α exists and can be found.

To make a comparison between the transmission properties of the system consisting of the linear and the nonlinear over-dense plasma, in Fig. 8 the transmission rate $\left|\frac{T}{E_0}\right|^2$ is illustrated versus the incident angle θ. The figure is plotted for three different widths of the over-dense plasma layer. It is obvious from the figure that the nonlinear effects enhance the transmission properties of the system. This fact specifically can be seen better for larger widths of the over-dense plasma slab, where the maximum transmission rate decreases, and the distinction between the linear and the nonlinear transmission is more apparent.

Conclusion

In this paper, we have studied the nonlinear effects on the anomalous transmission of the electromagnetic waves through an over-dense plasma slab. It has been shown that a normally opaque linear over-dense plasma slab can become transparent under the resonant conditions of the formation of the surface waves. Indeed, the excited surface waves appear simultaneously on both sides of the plasma slab and transmit the energy of the incident electromagnetic waves. Here, we show that this anomalous wave transmission can also be

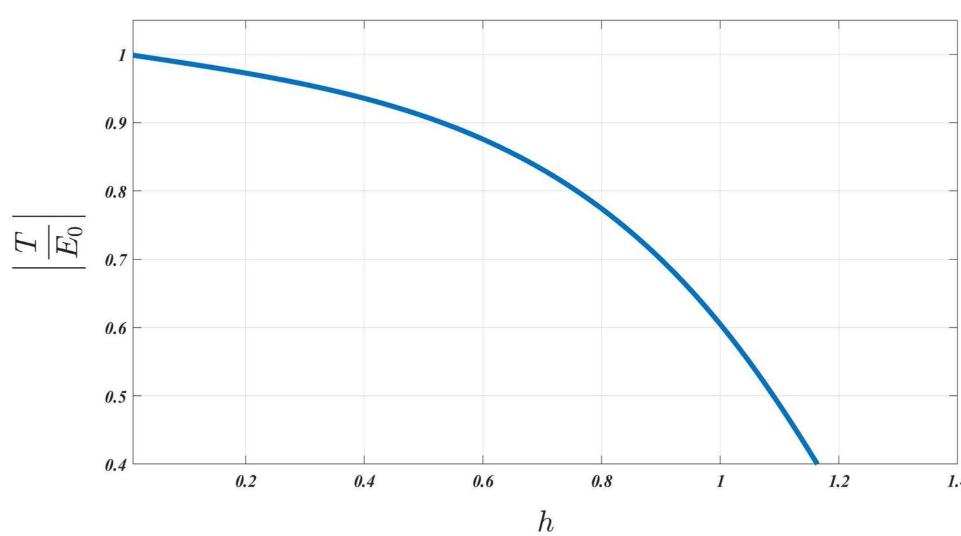


Fig. 4 The transmission rate as a function of the width parameter of the nonlinear over-dense plasma layer namely h. The incident angle is $\theta = 0.67544$ rad and $m = 8.8$

Fig. 5 The transmission rate as a function of the width parameter of the linear dielectric layer namely m. The incident angle is $\theta = 0.67544$ rad and $h = 0.2$

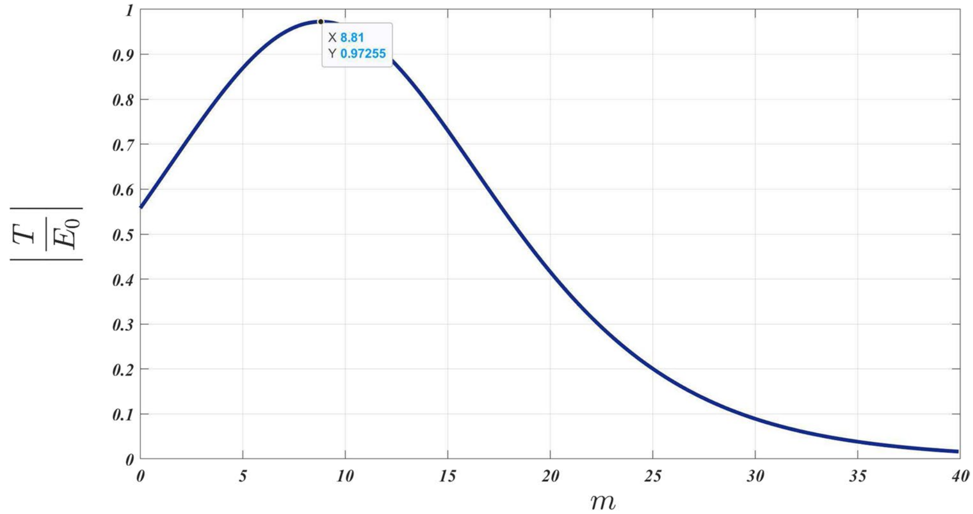


Fig. 6 The transmission rate as a function of the dissipation factor ν for three different values of ν. The solid line is the transmission rate and the dotted line, with the same color, relates the corresponding dissipation rate. The incident angle is $\theta = 0.67544$ rad and $h = 0.3$ and $m = 8.8$

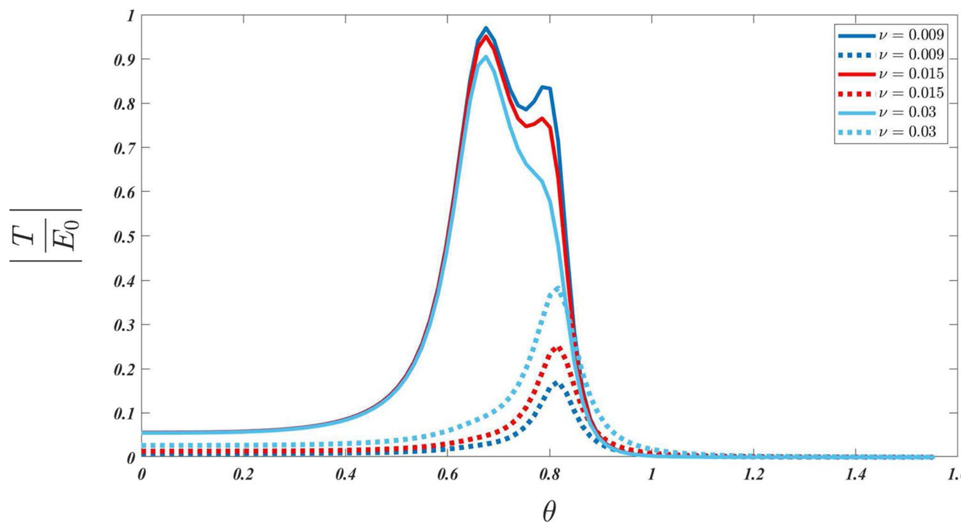
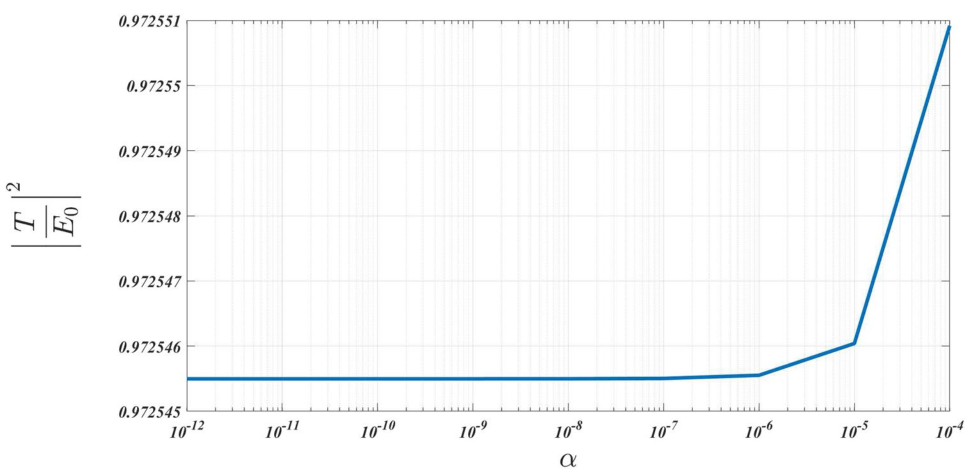


Fig. 7 The effect of the nonlinear parameter α on the transmission properties of the structure. The incident angle is $\theta = 0.67544$ rad and $h = 0.2$ and $m = 8.8$



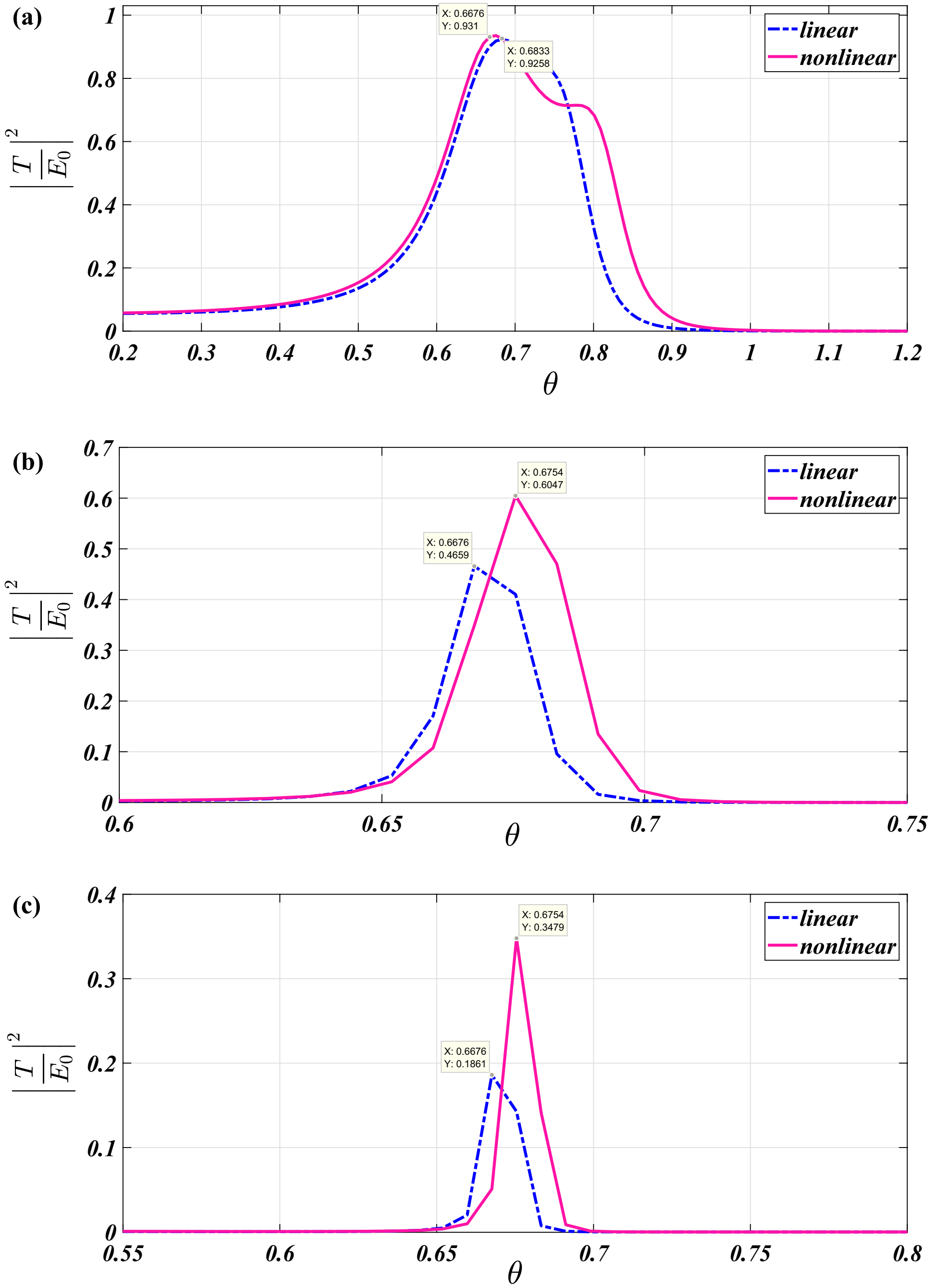


Fig. 8 The comparison between the transmission rate from the linear and the nonlinear over-dense plasma, respectively, dot-dashed and solid lines. The width of the nonlinear plasma $2h$ in unit of $(\frac{c}{\omega})$ is $2h = 0.8$, $2h = 2$ and $2h = 2.4$, respectively, in (**a**)–(**c**)

achieved for the over-dense plasma which has a nonlinear dielectric permittivity. The nonlinearity is presented due to the series expansion of the polarization of the plasma.

According to the results, it can be inferred that the considered structure is suitable for the transmission of the electromagnetic waves, and there are several conditions due to them the system shows good transitions. Also, in the nonlinear over-dense plasma, such as for the linear over-dense plasma, the anomalous transmission takes place because of the formation of the coupled surface waves. The specific form of the Jacobi elliptic function of the *cn* type is in such a way that if the period of the function is equal or proportional with the size of the nonlinear plasma layer, the surface waves can be formed. Several transmission conditions can be obtained which here some of them have been presented.

The effects of the mean parameters of the model on the transmission rates have been studied. It has been revealed that the system shows transmission properties for the small thickness of the nonlinear over-dense plasma and by increasing the plasma width, the transmission reduces and the reflection dominates. This is a feature of the transmission of the waves through the over-dense plasma mediums and has been examined for the linear over-dense plasma in the literature.

By comparing the transmission rate between the system consisting of the linear and nonlinear over-dense plasma, in the same structural conditions, it has been observed that the nonlinear effect enhances the transmission of waves through the structure. This fact can easily be seen for higher plasma width, where the transmission rates are lower.

References

1. Veselago, V.G.: The electrodynamics of substances with simultaneously negative values of permittivity and permeability. Sov. Phys. Usp. **10**, 509 (1968)
2. Pendry, J.B., Holden, A.J., Stewart, W.J., Youngs, I.: Extremely low frequency plasmons in metallic mesostructures. Phys. Rev. Lett. **76**(25), 4773 (1996)
3. Smith, D.R., Kroll, N.: Negative refractive index in left-handed materials. Phys. Rev. Lett. **85**(14), 2933 (2000)
4. Pendry, J.B.: Negative refraction makes a perfect lens. Phys. Rev. Lett. **85**(18), 3966 (2000)
5. Dragila, R., Luther-Davies, B., Vukovic, S.: High transparency of classically opaque metallic films. Phys. Rev. Lett. **55**, 1117 (1985)
6. Taya, S.A., Shabat, M.M., Khalil, H.M.: Enhancement of sensitivity in optical waveguide sensors using left-handed materials. Optik **120**(10), 504–508 (2009)
7. Rajaei, L., Miraboutalebi, S., Shokri, B.: Plasmon mechanism of over-dense plasma heating. Contrib. Plasm. Phys. **55**(4), 321–330 (2015)
8. Rajaei, L., Miraboutalebi, S., Nejati, M.: Faraday rotation of over-dense magnetized plasma. Contrib. Plasm. Phys. **55**(7), 513–521 (2015)
9. Zivari, A., Bagheri, A., Rejaei, B., Khavasi, A.: Propagation and refraction of left-handed plasmons on a semiconducting substrate covered by graphene. Opt. Lett. **43**(9), 2010–2013 (2018)
10. Dragila, R., Vukovic, S.: Surface-wave-induced high transparency of an over-dense warm plasma layer. Opt. Lett. **12**, 573 (1987)
11. Rajaei, L., Miraboutalebi, S., Shokri, B.: Transmission of electromagnetic waves through a warm over-dense plasma layer with a dissipative factor. Phys. Scr. **84**, 015506 (2011)
12. Fourkal, E., Velchev, I., Ma, C.-M., Smolyakov, A.: Evanescent wave interference and the total transparency of a warm high density palsma slab. Phys. Plasma **13**, 092113 (2006)
13. Miraboutalebi, S., Rajaei, L., Khadivi Borogeni, M.K.: Plasmon resonance coupling in cold over-dense dissipative plasma. J. Theor. Appl. Phys. **7**(1), 24–33 (2013)
14. Smolyakov, A., Sternberg, N.: Plasmon resonances, anomalous transparency, and reflectionless absorption in over-dense plasmas. Phys. Plasmas **25**(3), 031904 (2018)
15. Mirabootalebi, O., Miraboutalebi, S., Rajaei, L.: Transparency of over-dense plasma with **V** shape density profile. J. Interfaces Thin Films Low Dimens. Syst. **1**(2), 93–98 (2018)
16. Bloembergen, N.: Nonlinear Optics. W. A. Benjamin Inc., New York (1965)
17. Bertolotti, M., Sibilia, C., Fuli, I.: Magnetic rectification micro-Wiggler. Int. J. Infrared Millim. **8**(7), 723–732 (1987)
18. Boardman, A.D., Egan, P., Lederer, F., Langbein, U., Mihalache, D.: Third-order nonlinear electromagnetic TE and TM guided waves. Mod. Probl. Condens. Matter Sci. **29**, 73–287 (1991)
19. Eleonskii, V.M., Silin, V.P.: Propagation of electromagnetic waves in an inhomogeneous nonlinear medium. Zh. Eksp. Teor. Fiz. **66**, 146–153 (1974) [Sov. Phys. JETP **39**, 67–70 (1974)]
20. Achmediev, N.N., Bolter, K.O., Eleonsky, W.M.: Optical dielectric waveguide with nonlinear permittivity. Opt. Spektrosk. **53**, 906–909 (1982) [Opt. Spectrosc. (USSR) **53**, 540–542 (1982)]
21. Achmediev, N.N., Bolter, K.O., Eleonsky, W.M.: Dielectric optical waveguide with nonlinear susceptibility. Asymmetric refractive-index profiles. Opt. Spektrosk. **53**, 1097–1103 (1982) [Opt. Spectrosc. (USSR) **53**, 654–658 (1982)]
22. Langbein, U., Lederer, F., Ponath, H.-E.: A new type of nonlinear slab-guided waves. Opt. Commun. **46**, 167–169 (1983)
23. Boardman, A.D., Egan, P.: Nonlinear surface and guided polaritons of a general layered dielectric structure. J. Phys. (Paris) C5 Suppl. **45**, C5-291–C5-303 (1984)
24. Langbein, U., Lederer, F., Ponath, H.-E.: Generalized dispersion relations for nonlinear slab-guided waves. Opt. Commun. **53**, 417–420 (1985)
25. Boardman, A.D., Egan, P.: Nonlinear electromagnetic surface and guided waves: theory. In: Vukovic, S. (ed.) Surface Waves in Plasmas and Solids, pp. 3–77. World Scientific, Singapore (1986)
26. Boardman, A.D., Egan, P.: Optically nonlinear waves in thin films. IEEE J. Quantum Electron. **22**, 319–324 (1986)
27. Holland, W.R.: Nonlinear guided waves in low-index, selffocusing thin films: transverse electric case. J. Opt. Soc. Am. B **3**, 1529–1534 (1986)
28. Boardman, A.D., Twardowski, T., Shivarova, A., Stegeman, G.I.: Surface guided nonlinear TM waves in planar waveguides. Proc. Inst. Electron. Eng. **134**, 152–160 (1987)
29. Boardman, A.D., Twardowski, T.: Transverse-electric and transverse-magnetic waves in nonlinear isotropic waveguides. Phys. Rev. A **39**(5), 2481 (1989)
30. Al-Bader, S.J.: TM waves in nonlinear saturable thin films, a multilayered approach. J. Lightwave Technol. **7**(4), 717–725 (1989)
31. Hayata, K., Nagai, M., Koshiba, M.: Finite-element formulation for nonlinear dielectric slab waveguides. Electron. Commun. Jpn. (Part II: Electron.) **72**(4), 85–93 (1989)
32. Ogusu, K.: TM waves guided by nonlinear planar waveguides. IEEE Trans. Microw. Theory **37**(6), 941–946 (1989)
33. Yokota, M.: Guided transverse-magnetic waves supported by a weakly nonlinear slab waveguide. JOSA B **10**(6), 1096–1101 (1993)
34. Okafuji, S.: Propagation characteristics of TE_0 waves in three-layer optical waveguides with self focusing and self defocusing nonlinear layers. IEE Proc.-J. (Optoelectronics). **140**(2), 127–136 (1993)
35. Schürmann, H.W., Serov, V.S., Shestopalov, YuV: TE-polarized waves guided by a lossless nonlinear three-layer structure. Phys. Rev. E **58**(1), 1040 (1998)
36. Ciattoni, A., Rizza, C., Palange, E.: Extreme nonlinear electrodynamics in metamaterials with very small linear dielectric permittivity. Phys. Rev. A **81**(4), 043839 (2010)
37. Rukhlenko, I.D., Pannipitiya, A., Premaratne, M., Agrawal, G.P.: Exact dispersion relation for nonlinear plasmonic waveguides. Phys. Rev. B **84**(11), 113409 (2011)
38. Valovik, D.V., Smirnov, Y.G.: Calculation of the propagation constants of TM electromagnetic waves in a nonlinear layer. J. Commun. Technol. Electron. **53**(8), 883 (2008)
39. Valovik, D.V., Smirnov, Y.G.: A nonlinear boundary eigenvalue problem for TM-polarized electromagnetic waves in a nonlinear layer. Russ. Math. **52**(10), 60 (2008)
40. Valovik, D.V., Smirnov, Y.G.: Calculation of the propagation constants and fields of polarized electromagnetic TM waves in a nonlinear anisotropic layer. J. Commun. Technol. Electron. **54**(4), 391–398 (2009)
41. Smirnov, Y.G., Valovik, D.V.: Boundary eigenvalue problem for Maxwell equations in a nonlinear dielectric layer. Appl. Math. **1**(01), 29 (2010)

42. Valovik, D.V.: Propagation of electromagnetic waves in a nonlinear metamaterial layer. J. Commun. Technol. Electron. **56**(5), 544 (2011)
43. Valovik, D.V.: Propagation of TM waves in a layer with arbitrary nonlinearity. Comput. Math. Math. Phys. **51**(9), 1622 (2011)
44. Smirnov, Y.G., Valovik, D.V.: Nonlinear effects of electromagnetic TM wave propagation in anisotropic layer with Kerr nonlinearity. Adv. Math. Phys. (2012). https://doi.org/10.1155/2012/609765
45. Smirnov, Y.G., Valovik, D.V.: Coupled electromagnetic TE-TM wave propagation in a layer with Kerr nonlinearity. J. Math. Phys. **53**(12), 123530 (2012)
46. Smirnov, Y.G., Valovik, D.V.: Problem of nonlinear coupled electromagnetic TE–TE wave propagation. J. Math. Phys. **54**(8), 083502 (2013)
47. Valovik, D.V.: On the problem of nonlinear coupled electromagnetic transverse-electric-transverse magnetic wave propagation. J. Math. Phys. **54**(4), 042902 (2013)
48. Valovik, D.V., Zarembo, E.V.: The method of Cauchy problem for solving a nonlinear eigenvalue transmission problem for TM waves propagating in a layer with arbitrary nonlinearity. Comput. Math. Math. Phys. **53**(1), 78–92 (2013)
49. Valovik, D.V.: Integral dispersion equation method to solve a nonlinear boundary eigenvalue problem. Nonlinear Anal. Real. **20**, 52–58 (2014)
50. Smirnov, Y.G., Valovik, D.V.: Guided electromagnetic waves propagating in a plane dielectric waveguide with nonlinear permittivity. Phys. Rev. A **91**(1), 013840 (2015)
51. Smirnov, Y.G., Valovik, D.V.: On the infinitely many nonperturbative solutions in a transmission eigenvalue problem for Maxwell's equations with cubic nonlinearity. J. Math. Phys. **57**, 103504 (2016)
52. Valovik, D.V.: Novel propagation regimes for TE waves guided by a waveguide filled with Kerr medium. J. Nonlinear Opt. Phys. **25**(4), 1650051 (2016)
53. Kurseeva, V.Y., Valovik, D.V.: Theory of nonlinear guided electromagnetic waves in a plane two-layered dielectric waveguide. Math. Probl. Eng. (2017). https://doi.org/10.1155/2017/4215685
54. Boardman, A.D., Egan, P.: Optically nonlinear waves in thin films. IEEE J. Quantum Electron. **22**(2), 319–324 (1986)
55. Chen, W., Maradudin, A.A.: S-polarized guided and surface electromagnetic waves supported by a nonlinear dielectric film. JOSA B **5**(2), 529–538 (1988)
56. Maimistov, A.I., Basharov, A.M.: Non-linear waveguide structures. Nonlinear Optical Waves, pp. 491–556. Springer, Dordrecht (1999)
57. Wu, Y.D.: Analyzing multilayer optical waveguides with a localized arbitrary nonlinear guiding film. IEEE J. Quantum Electron. **40**(5), 529–540 (2004)
58. Wu, Y.D., Chen, M.H.: Method for analyzing multilayer nonlinear optical waveguide. Opt. Express **13**(20), 7982–7996 (2005)
59. Walasik, W., Rodriguez, A., Renversez, G.: Symmetric plasmonic slot waveguides with a nonlinear dielectric core: bifurcations, size effects, and higher order modes. Plasmonics **10**(1), 33–38 (2015)
60. Yuste, S.B., Bejarano, J.D.: Improvement of a Krylov-Bogoliubov method that uses Jacobi elliptic functions. J. Sound Vib. **139**(1), 151–163 (1990)
61. Meyer, K.R.: Jacobi elliptic functions from a dynamical systems point of view. Am. Math. Mon. **108**(8), 729–737 (2001)
62. Lawden, D.F.: Jacobi's Elliptic Functions. Springer, New York (1989)
63. Boardman, A.D., Shabat, M.M., Wallis, R.F.: Nonlinear magnetodynamic waves on magnetic materials. Phys. Rev. B **41**(1), 717–730 (1990)
64. Bliokh, Y.P., Felsteiner, J., Slutsker, Y.Z.: Total absorption of an electromagnetic wave by an overdense plasma. Phys. Rev. Lett. **95**(16), 165003 (2005)