

# Thermodynamics of charged rotating solutions in Brans–Dicke gravity with Born–Infeld field

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**Abstract** We derive new exact charged rotating solutions of  $(n + 1)$ -dimensional Brans–Dicke theory in the presence of Born–Infeld field and investigated their properties. Because of the coupling between scalar field and curvature, the field equations cannot to be solved directly. Using a new conformal transformation, which transforms the Einstein–dilaton–Born–Infeld gravity Lagrangian to the Brans–Dicke–Born–Infeld gravity one, the field equations are solved. We also compute temperature, charge, mass, electric potential, and entropy; entropy, however, does not obey the area law. These quantities are invariant under conformal transformation and satisfy the first law of thermodynamics.

**Keywords** Brans–Dicke · Black holes · Born–Infeld · Thermodynamics

## Introduction

Brans–Dicke theory [1] is one of the most important alternative theories of gravity to modify Einstein’s theory to incorporate Mach’s principle into the theory of gravitation. In this theory, gravity is described through a metric tensor  $g_{\mu\nu}$  and a scalar field  $\Phi$ , which replace Newton’s gravitational constant. This theory involves a dimensionless parameter  $\omega$  which represents the strength of coupling between scalar field and curvature. Recently, this theory

has obtained more attention as it arises as the low energy limit of many theories of quantum gravity such as the supersymmetric string theory or the Kaluza–Klein theory [2]. Besides, recent discoveries show that the universe is accelerating [3–5], so scalar-tensor theories, including Brans–Dicke theory, can be used to explain some features of dark energy (the cause of the acceleration). Cylindrically symmetric solutions are one of the most studied solutions of various gravity theories. For instance, the cylindrical symmetry is used in the study of gravitational waves, cosmological models, and gravitational collapse of non-spherical matter distributions. The first black hole solutions of Brans–Dicke theory in four dimensions were obtained by Brans [6]. Four-dimensional static cylindrical vacuum solutions of Brans–Dicke theory were obtained in [7, 8]. Static solutions of Brans–Dicke–Maxwell theory were presented in [9]. Recently, higher dimensional cylindrically symmetric solutions were investigated by many authors [10–12]. Charged rotating solutions in  $(n + 1)$ -dimensions for an arbitrary value of  $\omega$  were presented in [19], but charged rotating solutions for an arbitrary value of  $\omega$  in the presence of Born–Infeld field have not been constructed. In this paper, we will obtain the  $(n + 1)$ -dimensional charged rotating solutions in Brans–Dicke–Born–Infeld theory and investigate their properties.

The organization of this paper is as follows. In Sect. 2, we introduce the action of Brans–Dicke theory and dilaton gravity in the presence of Born–Infeld field and obtain field equations and conformal transformations between them. In Sect. 3, a new charged rotating solution in  $(n + 1)$ -dimensions with Liouville-type potential is constructed. In Sect. 4, temperature, charge, electric potential, and entropy are obtained. In Sect. 5, by calculating the Euclidean action method, we obtain the conserved quantities and study the

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first law of thermodynamics. We finish our paper with some concluding remarks.

### The action and field equations

The gravitation action in  $(n + 1)$ -dimensions for the Brans–Dicke theory with scalar field  $\Phi$  coupled to a Born–Infeld field can be written as

$$I_{BD} = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left( \Phi R - \frac{\omega}{\Phi} (\nabla\Phi)^2 - V(\Phi) + L(F) \right), \tag{1}$$

where  $R$  is the Ricci scalar,  $\omega$  is the coupling constant,  $\Phi$  denotes the BD scalar field, and  $V(\Phi)$  is a potential for the scalar field  $\Phi$ . The last term in Eq. (1) is the Born–Infeld term and is given by the following:

$$L(F) = 2\gamma \left( 1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{\gamma}} \right). \tag{2}$$

Here,  $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$  is the electromagnetic field tensor and  $A_\mu$  is the electromagnetic vector potential.  $\gamma$  is called the Born–Infeld parameter with dimension of mass. Notice that when  $\gamma \rightarrow \infty$ ,  $L(F)$  reduces to Maxwell electrodynamics

$$L(F) = -F_{\mu\nu}F^{\mu\nu}. \tag{3}$$

In the limit  $\gamma \rightarrow 0$ ,  $L(F) \rightarrow 0$ . It is convenient to set

$$L(F) = 2\gamma L(Y), \tag{4}$$

where

$$L(Y) = 1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{\gamma}}. \tag{5}$$

Varying the action (1) with respect to the gravitational field  $g_{\mu\nu}$ , the scalar field  $\Phi$  and the electromagnetic field  $A_\mu$  give the following field equations:

$$\begin{aligned} G_{\mu\nu} &= \frac{\omega}{\Phi^2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\nabla\Phi)^2 \right) \\ &\quad - \frac{g_{\mu\nu}}{2\Phi} [V(\Phi) - 2\nabla^2\Phi + L(Y)] \\ &\quad + \frac{1}{\Phi} [\nabla_\mu \nabla_\nu \Phi - 4\partial_Y L(Y) F_{\mu\lambda} F_\nu^\lambda], \end{aligned} \tag{6}$$

$$\begin{aligned} \nabla^2\Phi &= -\frac{4\partial_Y L(Y)}{[(n-1)\omega + n]} F^2 + \frac{(n+1)}{[(n-1)\omega + n]} \left[ L(Y) - \frac{V(\Phi)}{2} \right] \\ &\quad + \frac{(n-1)}{2[(n-1)\omega + n]} \left[ \Phi \frac{dV(\Phi)}{d\Phi} \right], \end{aligned} \tag{7}$$

$$\nabla_\mu (\partial_Y L(Y) F^{\mu\nu}) = 0, \tag{8}$$

where  $G_{\mu\nu}$  and  $\nabla_\mu$  are, respectively, the Einstein tensor and covariant differentiation corresponding to the metric  $g_{\mu\nu}$ . Because the right-hand side of Eq. (6) appears as second derivatives of the scalar field  $\Phi$ , so we cannot solve it directly and use a suitable conformal transformation such as:

$$\begin{aligned} g_{\mu\nu} &= \Omega^2 \tilde{g}_{\mu\nu}, \\ \Phi &= \exp\left(\frac{4\alpha\tilde{\Phi}}{n-3}\right), \end{aligned} \tag{9}$$

$$F_{\mu\nu} = \tilde{F}_{\mu\nu},$$

where  $\Omega = \Phi^{-\frac{1}{n-1}}$ . Using this conformal transformation, the action (1) becomes

$$\begin{aligned} \tilde{I} &= -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-\tilde{g}} \left( \tilde{R} - \frac{16\alpha^2}{(n-3)^2} \right. \\ &\quad \left. \left( \omega + \frac{n}{n-1} \right) (\nabla\tilde{\Phi})^2 - \tilde{V}(\tilde{\Phi}) + \tilde{L}(\tilde{F}, \tilde{\Phi}) \right), \end{aligned} \tag{10}$$

this action equals the action of Einstein–dilaton gravity coupled to a Born–Infeld field, which has been studied in [13]:

$$\begin{aligned} \tilde{I}_{ED} &= -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-\tilde{g}} \left( \tilde{R} - \frac{4}{n-1} (\nabla\tilde{\Phi})^2 \right. \\ &\quad \left. - \tilde{V}(\tilde{\Phi}) + \tilde{L}(\tilde{F}, \tilde{\Phi}) \right), \end{aligned} \tag{11}$$

provided that

$$\omega = \frac{(n-3)^2}{4(n-1)\alpha^2} - \frac{n}{n-1}, \tag{12}$$

$$\gamma = \exp\left(\frac{8\alpha\tilde{\Phi}}{n-3}\right) \tilde{\gamma}.$$

In action (11),  $\tilde{R}$  and  $\tilde{\nabla}$  are, respectively, the Ricci scalar and covariant differentiation corresponding to the metric  $\tilde{g}_{\mu\nu}$ , and  $\alpha$  is a constant which determines the strength of coupling between the scalar and electromagnetic field. The transformed Born–Infeld field  $\tilde{L}(\tilde{F}, \tilde{\Phi})$  and  $\tilde{V}(\tilde{\Phi})$  are given by

$$\tilde{L}(\tilde{F}, \tilde{\Phi}) = 2\tilde{\gamma} e^{4\alpha\tilde{\Phi}/(n-1)} \left( 1 - \sqrt{1 + \frac{e^{-8\alpha\tilde{\Phi}/(n-1)} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}}{\tilde{\gamma}}} \right), \tag{13}$$

$$\tilde{V}(\tilde{\Phi}) = V(\Phi) \Phi^{-(n+1)/(n-1)} = \exp\left(\frac{4(n+1)\alpha\tilde{\Phi}}{(n-1)(n-3)}\right), \tag{14}$$

for convenience, we set

$$\tilde{L}(\tilde{F}, \tilde{\Phi}) = 2\tilde{\gamma} e^{4\alpha\tilde{\Phi}/(n-1)} (1 - \sqrt{1 + \tilde{Y}}), \tag{15}$$

where

$$\tilde{Y} = \frac{e^{-8\alpha\tilde{\Phi}/(n-1)} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}}{\tilde{\gamma}}. \tag{16}$$

Varying the action (11) with respect to  $\tilde{g}_{\mu\nu}$ ,  $\tilde{\Phi}$  and  $\tilde{F}_{\mu\nu}$ , the equations of motion are obtained as

$$\begin{aligned} \tilde{R}_{\mu\nu} = & \frac{4}{n-1} \left( \tilde{\nabla}_\mu \tilde{\Phi} \tilde{\nabla}_\nu \tilde{\Phi} + \frac{1}{4} \tilde{V} \tilde{g}_{\mu\nu} \right) \\ & - 4e^{-4\alpha\tilde{\Phi}/(n-1)} \tilde{\partial}_{\tilde{Y}} \tilde{L}(\tilde{Y}) \tilde{F}_{\mu\eta} \tilde{F}_\nu^\eta \\ & + \frac{2\tilde{\gamma}}{n-1} e^{4\alpha\tilde{\Phi}/(n-1)} \left[ 2\tilde{Y} \tilde{\partial}_{\tilde{Y}} \tilde{L}(\tilde{Y}) - \tilde{L}(\tilde{Y}) \right] \tilde{g}_{\mu\nu}, \end{aligned} \tag{17}$$

$$\tilde{\nabla}^2 \tilde{\Phi} = \frac{n-1}{8} \frac{\partial \tilde{V}}{\partial \tilde{\Phi}} + \tilde{\gamma} \alpha e^{4\alpha\tilde{\Phi}/(n-1)} \left[ 2\tilde{Y} \tilde{\partial}_{\tilde{Y}} \tilde{L}(\tilde{Y}) - \tilde{L}(\tilde{Y}) \right], \tag{18}$$

$$\tilde{\nabla}_\mu \left[ e^{-4\alpha\tilde{\Phi}/(n-1)} \tilde{\partial}_{\tilde{Y}} \tilde{L}(\tilde{Y}) \tilde{F}^{\mu\nu} \right] = 0. \tag{19}$$

In the present work, we wish to find charge rotating solutions of Eqs. (6)–(8) with potential  $V(\Phi)$ . Therefore, we use conformal transformation (9) and solutions of Eqs. (17)–(19).

### Charge rotating solutions in $(n + 1)$ -dimensions

The solutions of the field Eqs. (17) and (18) have been obtained by many authors. Here, we want to obtain the charge rotating solutions in  $(n + 1)$ -dimensional of Brans–Dicke theory with Born–Infeld field. The  $(n + 1)$ -dimensional charge rotating solution to the field equations (17)–(19) has been obtained by [13] for a Liouville-type potential:

$$\tilde{V}(\tilde{\Phi}) = 2\Lambda \exp\left(\frac{4\alpha\tilde{\Phi}}{n-1}\right). \tag{20}$$

By applying the conformal transformation (9), the potential  $\tilde{V}(\tilde{\Phi})$  becomes  $V(\Phi) = 2\Lambda\Phi^2$ . The metric for this solution was written as [13]:

$$\begin{aligned} ds^2 = & -F(r) \left( \Xi dt - \sum_{i=1}^k a_i d\varphi_i \right)^2 \\ & + \frac{r^2}{l^4} R^2(r) \sum_{i=1}^k (a_i dt - \Xi l^2 d\varphi_i)^2 \\ & - \frac{r^2}{l^2} R^2(r) \sum_{i=1}^k (a_i d\varphi_j - a_j d\varphi_i)^2 + \frac{dr^2}{F(r)} + \frac{r^2}{l^2} R^2(r) dX^2, \end{aligned} \tag{21}$$

where  $a_i$ s are  $k$  rotation parameters and

$$\Xi^2 = 1 + \sum_{i=1}^k \frac{a_i^2}{l^2}.$$

In metric (21),  $F(r)$  and  $R(r)$  are functions of  $r$  which should be determined. The modified Maxwell equation (19) for the metric (21) can be integrated, where all of the components are zero except

$$\begin{aligned} \tilde{F}_{tr} = & \frac{q\Xi e^{4\alpha\tilde{\Phi}/(n-1)}}{(rR)^{n-1} \sqrt{1 + \frac{2q^2}{\tilde{\gamma}(rR)^{2n-2}}}}, \\ \tilde{F}_{\varphi r} = & -\frac{a_i}{\Xi} \tilde{F}_{tr}. \end{aligned} \tag{22}$$

It can be showed that  $F(r)$ ,  $R(r)$ , and  $\tilde{\Phi}(r)$  have solutions of the form [13]:

$$\begin{aligned} F(r) = & \frac{2\Lambda(\alpha^2 + 1)^2 c^{2\beta}}{(n-1)(\alpha^2 - n)} r^{2(1-\beta)} - \frac{m}{r^{(n-1)(1-\beta)-1}} \\ & - \frac{2\tilde{\gamma}(\alpha^2 + 1)^2 c^{2\beta} r^{2(1-\beta)}}{(n-1)(\alpha^2 - n)} \\ & \times \left( 1 - {}_2F_1\left(-\frac{1}{2}, \left[\frac{\alpha^2 - 1}{2(n-1)} - \frac{1}{2}\right], \left[\frac{\alpha^2 - 1}{2(n-1)} + \frac{1}{2}\right]; \right. \right. \\ & \left. \left. \frac{-2q^2}{\tilde{\gamma}c^{2\beta(n-1)} r^{2(n-1)(1-\beta)}} \right) \right), \end{aligned} \tag{23}$$

$$R(r) = \exp\left(\frac{2\alpha\tilde{\Phi}}{n-1}\right) = \left(\frac{c}{r}\right)^\beta, \tag{24}$$

$$\tilde{\Phi}(r) = \frac{(n-1)\alpha}{2(1+\alpha^2)} \ln\left(\frac{c}{r}\right), \tag{25}$$

where  $c$  and  $m$  are integration constants, and  $\beta = \alpha^2/(\alpha^2 + 1)$ . One may note that in the limit  $\gamma \rightarrow \infty$ , these solutions reduce to the solutions presented in [14]. In the absence of dilaton field ( $\alpha = \beta = 0$ ), these solutions reduce to the charged rotating black brane solutions of Einstein–Born–Infeld theory [15]. Using the conformal transformation (9), the  $(n + 1)$ -dimensional charged rotating solutions of BD theory in the presence of Born–Infeld field can be obtained as follows:

$$\begin{aligned} ds^2 = & -A(r) \left( \Xi dt - \sum_{i=1}^k a_i d\varphi_i \right)^2 \\ & + \frac{r^2}{l^4} H^2(r) \sum_{i=1}^k (a_i dt - \Xi l^2 d\varphi_i)^2 \\ & - \frac{r^2}{l^2} H^2(r) \sum_{i=1}^k (a_i d\varphi_j - a_j d\varphi_i)^2 + \frac{dr^2}{B(r)} + \frac{r^2}{l^2} H^2(r) dX^2, \end{aligned} \tag{26}$$

where  $A(r)$ ,  $B(r)$ ,  $H(r)$ , and  $\Phi(r)$  are

$$A(r) = \frac{2\Lambda(\alpha^2 + 1)^2 c^{2\beta(\frac{n-5}{n-3})}}{(n-1)(\alpha^2 - n)} r^{2(1-\frac{\beta(n-5)}{n-3})} - \frac{mc^{(\frac{4\beta}{n-3})}}{r^{(n-2)}} r^{\beta((n-1)+\frac{4}{n-3})} + \frac{2\gamma(\alpha^2 + 1)^2 c^{-4\beta(\frac{n-4}{n-3})}}{(n-1)(\alpha^2 - n)r^{-2(1+\frac{\beta(n-4)}{n-3})}} \times \left( 1 - {}_2F_1\left(-\frac{1}{2}, \left[\frac{\alpha^2 - 1}{2(n-1)} - \frac{1}{2}\right], \left[\frac{\alpha^2 - 1}{2(n-1)} + \frac{1}{2}\right]; \frac{-2q^2 r^{2(n-1)(\beta-\frac{n-1}{n-3})}}{\gamma c^{\frac{2\beta(n-1)(n-5)}{(n-3)}}}\right) \right), \tag{27}$$

$$B(r) = \frac{2\Lambda(\alpha^2 + 1)^2 c^{2\beta(\frac{n-1}{n-3})}}{(n-1)(\alpha^2 - n)} r^{2(1-\frac{\beta(n-1)}{n-3})} - \frac{mc^{(\frac{2\beta}{n-3})}}{r^{(n-2)}} r^{\beta((n-1)-\frac{4}{n-3})} + \frac{2\gamma(\alpha^2 + 1)^2 c^{-4\beta(\frac{2n-1}{n-3})}}{(n-1)(\alpha^2 - n)r^{-2((1-\beta)-2\beta\frac{n-1}{n-3})}} \times \left( 1 - {}_2F_1\left(-\frac{1}{2}, \left[\frac{\alpha^2 - 1}{2(n-1)} - \frac{1}{2}\right], \left[\frac{\alpha^2 - 1}{2(n-1)} + \frac{1}{2}\right]; \frac{-2q^2 c^{2(n-1)(\beta-\frac{n-1}{n-3})}}{\gamma r^{\frac{2\beta(n-1)(n-5)}{(n-3)}}}\right) \right), \tag{28}$$

$$H(r) = \left(\frac{c}{r}\right)^{\frac{(n-5)\beta}{n-3}}, \tag{29}$$

$$\Phi(r) = \left(\frac{c}{r}\right)^{\frac{2(n-1)\beta}{n-3}}. \tag{30}$$

In the above solutions,  $c$  and  $m$  are integration constants. We obtain for the electromagnetic fields

$$F_{tr} = \frac{q\Xi c^{(3-n)\beta}}{r^{2+(n-3)(1-\beta)}} \frac{1}{\sqrt{1 + \frac{2q^2 c^{2\beta(n-3)}}{\gamma r^{2(n-6)(1-\beta)+4}}}}, \tag{31}$$

$$F_{\varphi r} = -\frac{a_i}{\Xi} F_{tr}.$$

Let us notice that as  $r \rightarrow \infty$ , the electromagnetic fields (31) become zero and, in the large  $\gamma$  limit, reduce to the Brans–Dicke–Maxwell theory [19]:

$$F_{tr} = \frac{q\Xi c^{(3-n)\beta}}{r^{2+(n-3)(1-\beta)}}, \tag{32}$$

$$F_{\varphi r} = -\frac{a_i}{\Xi} F_{tr}.$$

Using the fact that  ${}_2F_1(a, b, c; z)$  converges for  $|z| < 1$ , so we can study the behavior of the  $A(r)$  in the limiting case where  $r \rightarrow \infty$ . The Function  $A(r)$  for large  $r$  is given by

$$A(r) = \frac{2\Lambda(\alpha^2 + 1)^2 c^{2\beta(\frac{n-5}{n-3})}}{(n-1)(\alpha^2 - n)} r^{2(1-\frac{\beta(n-5)}{n-3})} - \frac{mc^{(\frac{4\beta}{n-3})}}{r^{(n-2)}} r^{\beta((n-1)+\frac{4}{n-3})} + \frac{2q^2(\alpha^2 + 1)^2 c^{-2\beta((n-2)+\frac{2}{n-3})} r^{2((1-\beta)+\frac{2\beta}{n-3})}}{(n-1)(\alpha^2 + n - 2)r^{2((n-2)(1-\beta)-\frac{2\beta}{n-3})}} - \frac{q^4(\alpha^2 + 1)^2 c^{-4\beta(\frac{n-4}{n-3})}}{\gamma(n-1)^2(\alpha^2 + 3n - 4)r^{2((2n-3)(1-\beta)-2\beta\frac{n-4}{n-3})}}. \tag{33}$$

Note that in the limit  $\gamma \rightarrow \infty$  and  $\alpha = 0$ , it has the form of the asymptotically AdS black holes. The last term in the above equation is due to the Born–Infeld field in the large  $\gamma$  limit. We can see from Eq. (27) that the solution is well-defined except for  $\alpha = \sqrt{n}$ . Therefore, we investigate two cases  $\alpha > \sqrt{n}$  and  $\alpha < \sqrt{n}$  separately. When  $\alpha > \sqrt{n}$ , as  $r \rightarrow \infty$ , the second term is dominant term, and therefore, the spacetime has a horizon for positive values of the mass parameter. In the second case, where  $\alpha < \sqrt{n}$ , as  $r \rightarrow \infty$ , the first term is dominant term, and therefore, there exists a horizon provided that  $\Lambda > 0$ . If  $\Lambda < 0$ , it is possible to have horizon depending on the different values of the parameters  $m$ ,  $q$ , and  $\alpha$ . Because exponential terms appear in (26), it is not straightforward task to find the location of horizons for an arbitrary value of  $\alpha$ . However, we can obtain some information by studying the temperature of the horizons.

### Thermodynamic quantities

#### Temperature

By taking  $t \rightarrow it$  and  $a_i \rightarrow ia_i$ , we define the Euclidean section of the metric (26). The regularity of the metric at  $r_+$  requires that we must identify  $\tau \rightarrow \tau + \beta_+$  and  $\phi_i \sim \phi_i + \beta_+ \Omega_i$ , where  $\beta_+$  and  $\Omega_i$  are the inverse temperature and the angular velocity of the outer horizon [16]

$$\Omega_i = \frac{a_i}{\Xi r^2}. \tag{34}$$

The Killing horizon is a null hypersurface whose null generators are tangent to a Killing field. It is easy to see that the Killing vector:

$$\chi = \frac{\partial}{\partial t} + \sum_{i=1}^k \Omega_i \frac{\partial}{\partial \phi_i}, \tag{35}$$

is the null generator of the event horizon. The temperature on the outer horizon  $r_+$ , is defined through the use of definition of surface gravity  $\kappa$ ,

$$T_+ = \frac{\kappa}{2\pi}, \tag{36}$$

where the surface gravity  $\kappa = \frac{2\pi}{\beta_+}$  is given by

$$\kappa = \sqrt{-\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)} = \frac{1}{2\Xi} \left(\frac{dF(r)}{dr}\right)_{r_+}. \tag{37}$$

Then, it is a matter of calculation to show that

$$\begin{aligned} \frac{1}{\beta_+} = T_+ = & \frac{(n - \alpha^2)m}{4\pi\Xi(\alpha^2 + 1)} r_+^{(n-1)(\beta-1)} \\ & - \frac{q^2(\alpha^2 + 1)c^{2(2-n)\beta}}{\pi\Xi(\alpha^2 + n - 2)} r_+^{2(2-n)(1-\beta)-1} \\ & \times {}_2F_1\left(\frac{1}{2}, \left[\frac{\alpha^2-1}{2(n-1)} + \frac{1}{2}\right], \left[\frac{\alpha^2-1}{2(n-1)} + \frac{3}{2}\right]; \right. \\ & \left. - \frac{2q^2 r_+^{2(n-1)(\beta-1)}}{\gamma c^{2(n-1)(\beta-1) - \frac{2\beta}{n-3}}}\right), \end{aligned} \tag{38}$$

which shows that the temperature of the solution is invariant under the conformal transformation (9). This result concludes from this point that the conformal parameter is regular at the horizon. There is also an extreme value for the mass parameter in which the temperature of the black hole is zero. Using the fact that  $F(r_+) = 0$ , it is easy to show that

$$\begin{aligned} m_{\text{ext}} = & \frac{4q^2(\alpha^2 + 1)c^{2(2-n)\beta}}{(n - \alpha^2)(\alpha^2 + n - 2)} r_+^{(3-n)(1-\beta)-1} \\ & \times {}_2F_1\left(\frac{1}{2}, \left[\frac{\alpha^2-1}{2(n-1)} + \frac{1}{2}\right], \left[\frac{\alpha^2-1}{2(n-1)} + \frac{3}{2}\right]; \right. \\ & \left. - \frac{2q^2 r_+^{2(n-1)(\beta-1)}}{\gamma c^{2(n-1)(\beta-1) - \frac{2\beta}{n-3}}}\right). \end{aligned} \tag{39}$$

Depending on the value of  $m$ , there are three cases to consider separately. In the first case, where  $m > m_{\text{ext}}$ , the metric of (26) has two inner and outer horizons ( $r_-$  and  $r_+$ ). In the case of  $m = m_{\text{ext}}$ , we have an extreme black brane and a naked singularity if  $m < m_{\text{ext}}$ . It is notable to mention that in the absence of scalar field, where  $(\alpha = 0)$  and  $\gamma \rightarrow \infty$ ,  $m_{\text{ext}}$  reduces to the equation obtained in [17, 18], while in the limiting case, where  $\gamma \rightarrow \infty$ ,  $m_{\text{ext}}$  reduces to the result in [19].

### Charge and electric potential

Here, we want to calculate the electric charge and potential of the solutions. We should consider the projections of the electromagnetic field tensor on special hypersurfaces to determine the electric field. The normal to such hypersurfaces is

$$u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{V^i}{N}, \tag{40}$$

where  $N$  and  $V^i$  are the lapse and shift function and the electric field is  $E^\mu = g^{\mu\alpha} F_{\alpha j} u^j$ . The electric charge per unit volume,  $Q$ , can be obtained by calculating the flux of the electric field at infinity, obtaining

$$Q = \frac{\Xi q}{4\pi l^{n-2}}. \tag{41}$$

By comparing the charge (41) with the charge of black brane solutions of Einstein–Born–Infeld theory obtained in [13], we notice that the charge  $Q$  is invariant under the conformal transformation (9). The electric potential,  $U$ , measured at infinity with respect to the horizon is defined by the following [20, 21]:

$$U = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+}. \tag{42}$$

Here,  $\chi$  is the null generators of the horizon. The vector potential  $A_\mu$  corresponding to electromagnetic field (31) can be obtained as

$$\begin{aligned} A_\mu = & \frac{qc^{(3-n)\beta}}{\Delta r^\Delta} {}_2F_1\left(\frac{1}{2}, \left[\frac{\alpha^2-1}{2(n-1)} + \frac{1}{2}\right], \left[\frac{\alpha^2-1}{2(n-1)} + \frac{3}{2}\right]; \right. \\ & \left. - \frac{2q^2}{\gamma c^{2\beta \frac{(n-1)(n-5)}{n-3}} r_+^{2(n-1)(n-3-\beta(n-5))}}\right) \times (\Xi \delta_\mu^t - a_i \delta_\mu^i), \end{aligned} \tag{43}$$

where  $\Delta = (1 - \beta)(n - 3) + 1$ . Therefore, using (42) and (43), the electric potential can be obtained as

$$\begin{aligned} U = & \frac{qc^{(3-n)\beta}}{\Xi \Delta r_+^\Delta} {}_2F_1\left(\frac{1}{2}, \left[\frac{\alpha^2-1}{2(n-1)} + \frac{1}{2}\right], \left[\frac{\alpha^2-1}{2(n-1)} + \frac{3}{2}\right]; \right. \\ & \left. - \frac{2q^2}{\gamma c^{2\beta \frac{(n-1)(n-5)}{n-3}} r_+^{2(n-1)(n-3-\beta(n-5))}}\right). \end{aligned} \tag{44}$$

### Entropy

For a charged rotating black hole, the first law of thermodynamics takes the form

$$T dS = dM - \sum_{i=1}^k \Omega_i dJ_i + U dQ, \tag{45}$$

where  $T$  and  $S$  are the horizon temperature and entropy,  $M$ ,  $J$ ,  $U$ , and  $Q$  are the mass, angular momentum, electric potential, and charge measured at infinity, and  $\Omega_H$  is the angular velocity of the horizon. The first law of thermodynamics connects the quantities  $M$ ,  $J$ ,  $U$ , and  $Q$  measured at infinity with the local quantities  $S$ ,  $T$ ,  $A$ , and  $\Omega_i$  on the horizon. As studied in [22], in alternative theories of gravity, the first law of thermodynamics (45) still holds true, but the area law for the entropy  $S = \frac{A}{4G}$  is no longer valid in these theories which has been known since 1980s [23–27]. Some numerical studies showed that the area law during the collapse of dust to black holes in alternative theories of gravity is violated [28–30]. Black hole entropy in Brans–Dicke theory was studied by Kang [31]. Kang noticed that the problem is in the expression of the black hole entropy, which is not one quarter of the area. The relation for the entropy is

$$S = \frac{1}{4} \int_{\Sigma} d^{n-1}x \sqrt{g^{(n-1)}} \Phi = \frac{A\Phi}{4}, \tag{46}$$

where  $\Phi$  is the Brans–Dicke scalar and  $g^{(n-1)}$  is the determinant of the metric  $g_{\mu\nu}$  to the horizon surface  $\Sigma$ . This expression can be obtained by the replacement of the Newton constant  $G$  with the inverse of Brans–Dicke scalar  $\Phi$ . In the Einstein frame, the gravitational coupling is a constant, but in Brans–Dicke frame matter couples to the scalar field. Massive test particles which follow time-like geodesics of the metric  $g_{\mu\nu}$ , do not follow geodesics of the rescaled metric  $\tilde{g}_{\mu\nu}$ . Null geodesics remain unchanged under conformal transformation, and also null vectors and all forms of conformally invariant matter. Therefore, a black hole event horizon which is a null surface remains unchanged. The area of an event horizon is not a null surface. Therefore, the change in the entropy formula can be obtained as the change in the area due to the conformal transformation of the metric  $g_{\mu\nu}$ . In fact,  $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$ , and the relation between areas in two frames is

$$\tilde{A} = \int_{\Sigma} d^{n-1}x \sqrt{\tilde{g}^{(n-1)}} = \int_{\Sigma} d^{n-1}x \Omega^{-(n-1)} \sqrt{g^{(n-1)}} = \Phi A, \tag{47}$$

where  $\Omega = \Phi^{\frac{-1}{n-1}}$  according to (9). Therefore, the entropy in the Brans–Dicke and Einstein theory becomes equal

$$S = \frac{A\Phi}{4} = \frac{\tilde{A}}{4} = \tilde{S}. \tag{48}$$

Denoting the volume of the hypersurface boundary at constant  $t$  and  $r$  by  $V_{n-1}$ , it is easy to show that the entropy per unit volume is

$$S = \frac{\Xi c^{(n-1)\beta}}{4l^{n-2}} r_+^{(n-1)(1-\beta)}. \tag{49}$$

It is notable to mention that the equality between black hole entropies in the Brans–Dicke and Einstein theory is not confined to scalar-tensor gravity but is valid in all theories with action  $\int d^{n+1}x \sqrt{g} f(R_{\mu\nu}, g_{\mu\nu}, \varphi, \nabla_{\mu}\varphi)$  [32].

### Action and conserved quantities

In this section, we obtain the action and the thermodynamic quantities of our solutions. In general, the action (1) does not have a well-defined variational principle as well as is divergent when evaluated on the solution. By variation of the action (1), one encounters a total derivative which gives rise to a surface integral involving the normal derivative of  $\delta g_{\mu\nu}$ . These normal derivative terms are removed by the variation of the surface term

$$I_b = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-h} \Theta \Phi. \tag{50}$$

Here,  $h_{ab}$  is the determinant of the boundary metric and  $\Theta$  is the trace of extrinsic curvature  $\Theta^{ab}$  of the boundary. In general, the action  $I_{BD} + I_b$  is divergent when evaluated on the solutions. For asymptotically (A)dS solutions of Einstein gravity, one can remove these divergences through the use of counterterm method. In this method, we add a finite number of surface integral to leave the action finite [33]. In this paper, our solutions have zero curvature boundary, and therefore, all the counterterms containing the curvature invariants of the boundary are zero. Thus, there exists one counterterm as

$$I_c = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-h} \left( \frac{(n-1)\Phi V(\Phi)}{\alpha^2 - n} \right)^{\frac{1}{2}}, \tag{51}$$

it is notable that the counterterm has the same form as in the case of asymptotically AdS solutions with zero curvature boundary, where  $\alpha \rightarrow 0$  ( $\Phi = 1$ ). The total finite action can be written as

$$I_{\text{tot}} = I_{BD} + I_b + I_c. \tag{52}$$

Using Eqs. (1), (50), and (51), we obtain the Euclidean action per unit volume  $V_{n-1}$  as

$$\begin{aligned} \frac{I_{\text{tot}}}{\beta_+} = & -\frac{n(\alpha^4 - 1)mc^{(n+1)\beta}}{16\pi l^n (\alpha^2 - n)} r_+^{(n+1)(1-\beta)-1} \\ & - \frac{2\gamma(\alpha^4 - 1)c^{\beta\left(\frac{n^2-6n+1}{n-3}\right)} r_+^{n-4\beta\left(\frac{n^2-6n+1}{n-3}\right)}}{16\pi l^{n-2}(n-1)(\alpha^2 - n)} \\ & \times \left( 1 - {}_2F_1\left(-\frac{1}{2}, \left[\frac{\alpha^2 - 1}{2(n-1)} - \frac{1}{2}\right], \left[\frac{\alpha^2 - 1}{2(n-1)} + \frac{1}{2}\right]; \right. \right. \\ & \left. \left. \frac{-2q^2}{\gamma c^{2\beta\left(\frac{n-1)(n-5)}{n-3}\right)} r_+^{2(n-1)(1-\frac{\beta n-5}{n-3})}} \right) \right). \end{aligned} \tag{53}$$

According to Refs. [34–36], we can calculate the entropy, the mass and angular momentum through the relation

$$I_{\text{tot}} = \beta_+ M - S - \beta_+ (\Omega_i J_i - QU), \tag{54}$$

by comparing Eqs. (53) and (54), we can easily find that

$$S = \frac{\Xi c^{(n-1)\beta}}{4l^{n-2}} r_+^{(n-1)(1-\beta)}, \tag{55}$$

$$M = \frac{c^{(n-1)\beta}}{16\pi l^{n-2}} \left( \frac{(n - \alpha^2)\Xi^2 + \alpha^2 - 1}{1 + \alpha^2} \right) m, \tag{56}$$

$$J_i = \frac{c^{(n-1)\beta}}{16\pi l^{n-2}} \left( \frac{n - \alpha^2}{1 + \alpha^2} \right) \Xi m a_i. \tag{57}$$

Comparing the thermodynamic quantities calculated in this section with those obtained in the previous sections, we



find that they are invariant under the conformal transformation. When  $a_i = 0$ , the angular momentum vanishes, and so,  $a_i$  is the  $i$ th rotational parameter of the spacetime. Black hole entropy typically satisfies the so-called area law of the entropy [37, 38], but it does not follow the area law in Brans–Dicke theory [39–45]. Nevertheless, the entropy remains invariant under conformal transformations. It is notable to mention that the thermodynamic quantities calculated above satisfy the first law of thermodynamics:

$$dM = TdS + \sum_{i=1}^k \Omega_i dJ_i + UdQ. \quad (58)$$

### Closing remarks

We presented the  $(n + 1)$ -dimensional BD-BI action coupled to a scalar field  $\Phi$  and obtained the field equations by varying this action with respect to the gravitational field  $g_{\mu\nu}$ , the dilaton field  $\Phi$ , and the gauge field  $A_\mu$ . In the special case of the linear electrodynamics where we have  $\mathcal{L}(Y) = -\frac{1}{2}Y$ , the system of equations reduced to the equations of Brans–Dicke–Maxwell theory [19]. Because of the coupling between the scalar field and curvature, solving the field equations is complicated. Therefore, to solve field equations, we present new conformal transformations. In this paper, using these conformal transformations, we obtained the charged rotating solutions of  $(n + 1)$ -dimensional ( $n \geq 4$ ) BD-BI equations in the presence of a potential and studied their properties. These solutions are neither asymptotically flat nor  $(A)dS$ . In the particular case  $\gamma \rightarrow \infty$ , these solutions reduce to the  $(n + 1)$ -dimensional charged rotating dilaton black brane in Brans–Dicke theory with quadratic scalar field potential [19]. These solutions are ill-defined for  $\alpha = \sqrt{n}$  (corresponding to  $\omega = \frac{-3(n+3)}{4n}$ ), so we investigated two cases  $\alpha > \sqrt{n}$  and  $\alpha < \sqrt{n}$  separately. We also computed the entropy, temperature, charge, mass, and electric potential, and found that these quantities are invariant under conformal transformations and satisfy the first law of thermodynamics. We show that the entropy does not obey the area law, but remain invariant under the conformal transformation. In addition, we found that when we have ( $\alpha = 0$ ), the scalar field becomes a constant ( $\Phi = 1$ ) and the BD theory degenerates into the Einstein theory of gravitation.

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