Journal of Theoretical and Applied Physics (2020) 14:171–179
https://doi.org/10.1007/s40094-020-00367-4

RESEARCH

Higher-curvature corrections to holographic mutual information

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Received: 29 July 2019 / Accepted: 28 January 2020 / Published online: 15 February 2020

Abstract
In this paper, we study some non-local measurements of quantum correlations in extended gravities with higher-order curvature terms, including conformal gravity. Precisely, we consider higher-curvature correction on holographic mutual information in conformal gravity. There is in fact one deformation in the states because of the higher-curvature corrections. Here by making use of the holographic methods, we study the deformation in the holographic mutual information due to the higher-curvature terms. We also address the change in the quantum phase transition due to these deformations.

Keywords Holographic mutual information · Higher-curvature correction

Introduction

Theories of gravity with higher-order curvature terms are of interest because of some reasons. For example, exploring a suitable theory in four dimensions which have a well-formulated quantized version is one of the motivations. It is known that by adding quadratic curvature terms to the Einstein gravity, one obtains a perturbatively renormalizable theory. One of the most known higher-order gravity theories is the conformal gravity which is invariant under conformal transformations, and in four dimensions, the corresponding action is given by the square of the Weyl tensor. The conformal theory has unique Lagrangian and also leads to a renormalizable theory [1]. The equations of motion are of fourth order, and despite its renormalizability, the theory contains massive spin-2 and the massless graviton where the kinetic term of the massive modes has the wrong sign, so they are ghostlike [2, 3] (see also [4–7]).

There are some deep relations between conformal gravity and Einstein gravity in four dimensions. It was shown that the renormalized shell action of Einstein gravity in an asymptotically hyperbolic Einstein spaces when evaluated on an Einstein solution is indeed given by the action of conformal gravity [8]. Moreover, it is claimed that the physical content of Einstein and conformal theories would be the same after removing the ghost [9].

In this paper, we are interested in studying the quantum correlation between two systems in four-dimensional conformal gravity. In other words, we want to address the effect of deformation in the states because of the higher-curvature corrections. Corresponding measure is the mutual information. As a matter of fact, the entanglement entropy is a measure of storing quantum information in a quantum state where it is indeed a remarkable tool in studying quantum systems. Entanglement entropy has indeed been deduced from the first principles of quantum mechanics. However, when we want to evaluate the amount of the correlation between two systems, the mutual information is mostly used. In order to compute the mutual information, one should find the corresponding expression for the entanglement entropy.

In four dimensions in Ref. [10], the holographic entanglement entropy of conformal gravity was studied in details, and in this paper, we will consider the *n*-partite information in this theory. There is in fact one kind of deformation in the states of conformal field theory in this model: the higher-curvature terms which could address the low-energy quantum excitation corrections. Precisely, in this paper, we are interested in the effect of the low-energy quantum excitation on the quantum correlation between two systems. To do so, we use the holographic methods which powered by the Anti-de Sitter/conformal field theory (AdS/CFT) correspondence. This correspondence is in fact a relationship between strongly correlated many-body systems and the classical dynamics of gravity, noting that in one higher dimension,

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the gravity is dealing with classically. In the reverse side, on the other hand, this correspondence may improve our knowledge of geometry and quantum gravity in the reverse side. This correspondence also covers topics related to the condensed matter theory [11–14]. In this paper, by making use of the holographic method, we investigate the effect of including higher-order curvature terms in the gravitational action on holographic mutual information. In the model considered in this work, the gravitational action is given by four-dimensional Weyl-square tensor term, and we are going to study the change of holographic mutual information for strip entangling region due to these corrections.

In four dimensions, the action of the conformal gravity is given by

$$\begin{aligned}\mathcal{I} &= \frac{-\kappa}{32\pi}\int d^4x\sqrt{-g}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\\ &= \frac{-\kappa}{32\pi}\int d^4x\sqrt{-g}\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2R_{\mu\nu}R^{\mu\nu}+\frac{1}{3}R^2\Big),\end{aligned} \tag{1}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and κ is a dimensionless coupling constant. The equations of motion upon varying the metric is called the Bach equation given by

$$2\nabla_\mu\nabla_\nu C^{\mu\nu}{}_{\rho\sigma}-C^{\mu\nu}{}_{\rho\sigma}R_{\rho\sigma}=0; \tag{2}$$

the above equations admit black hole solutions (see Ref. [15]) which will be discussed later. The aim is to consider holographic mutual information in this theory, and as mentioned to do so first, one should obtain the corresponding expression for the holographic entanglement entropy. We should emphasize that in the Einstein's theory of gravity, in order to compute holographic entanglement entropy, one should use Ryu and Takayanagi proposal [16], where in the boundary for a given entangling region, the entanglement entropy is related to the minimal surface in the bulk whose boundary coincides with the boundary of the entangling region. In the extended version of this proposal for time-dependent geometries, one should use the extremal surface [17]. However, it is important to mention that this proposal only works for the Einstein gravity. So that in order to explore higher-derivative theories, this proposal should be replaced by some other recipes [18–22]. Some related works in this subject can also be found, for example, in [23, 24] and references therein.

Here, we will follow the proposal of [19] and review the procedure of computing the most general form of holographic entanglement entropy in higher-derivative theories which will be the subject of Sect. 3. In Sect. 4 we compute the holographic entanglement entropy (HEE) for strip entangling region. In Sect. 5, we consider tripartite information and its sign in conformal gravity. Finally, the subject is concluded in the last section.

Holographic entanglement entropy in higher-order theories

Identifying the behaviors of correlation functions of local operators becomes an important issue if one wants to study a given field theory. Entanglement entropy (or geometric entropy) similar to other non-local quantities, e.g., Wilson loop and correlation functions, is an important non-local measure of different degrees of freedom in a quantum mechanical system [25]. Entanglement entropy measures how a given quantum system is entangled or strongly correlated. It is defined as the Von Neumann entropy when we trace out degrees of freedom inside a d-dimensional space-like sub-manifold in a given $d+1$ dimensional quantum field theory. To define entanglement entropy in its spatial (or geometric) description, let us divide a constant time slice into two spatial regions A and B where they are complement to each other. Therefore, the entanglement entropy is the entropy for an observer sitting in region A in a way that there is no accessibility to the region B as the information is lost by the tracing out in region B. By this definition, the corresponding total Hilbert space can be written in a specific partitioning as $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$. By tracing out the degrees of freedom that live in B, the reduced density matrix for region A can be computed as $\rho_A=\mathrm{Tr}_B\rho$ where ρ is the total density matrix. The entanglement entropy is then given by $S=-\mathrm{Tr}\ \rho_A\log\rho_A$, which is indeed the von Neumann formula.

The entanglement entropy is directly related to the degrees of freedom, and for a local d-dimensional quantum field theory, it follows the area law that results in a UV-divergent theory. The coefficient of the most divergent term is proportional to the area of the entangling surface, and this is indeed the area law which is due to the infinite correlations between degrees of freedom near the boundary of entangling surface. It is worth to mention that computing the entanglement entropy in the context of field theory is indeed a difficult task; however, the AdS/CFT correspondence provides a rather simple geometric structure to find the entanglement entropy. Interestingly, making use of the Ryu–Takayanagi proposal, one can define entanglement entropy in terms of the minimal area of codimension-two hypersurface in the bulk. This elegant proposal shows the power of AdS/CFT techniques in computing some quantities in strongly coupled field theories. On the other hand, for actions with higher-derivative terms, for example, in the case of squared-curvature terms with the following action

$$\begin{aligned}\mathcal{I}=\frac{1}{16\pi G_N}\int_M d^{d+1}x\sqrt{-g}\big[&R-2\Lambda+aR^2+bR_{\mu\nu}R^{\mu\nu}\\ &+cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big],\end{aligned} \tag{3}$$

one should use other proposal. Note that in the above equations G_N stands for Newton's constant and the cosmological constant is $\Lambda = -\frac{d(d-1)}{2L_{\mathrm{AdS}}}$. It was shown that in this case the holographic entanglement entropy is given by [19],

$$S = \frac{A(\Sigma)}{4G_N} + \frac{1}{4G_N}\int_\Sigma \sqrt{\sigma} d^{d-1}x\Big[2aR + b\Big(R_{\mu\nu}n_i^\mu n_i^\nu - \frac{1}{2}\sum_i \big(Tr\mathcal{K}^{(i)}\big)^2\Big) + 2c\Big(R_{\mu\nu\alpha\beta}n_i^\mu n_i^\alpha n_j^\nu n_j^\beta - \sum_i \mathcal{K}^{(i)}_{\mu\nu}\mathcal{K}^{\mu\nu}_{(i)}\Big)\Big]. \tag{4}$$

In the above formula, $n_i\,(i=1,2)$ are the orthogonal normal vectors on the codimension-two hypersurface and Σ and $\mathcal{K}^{(i)}_{\mu\nu}$ are the extrinsic curvature tensors on Σ defined as

$$\mathcal{K}^{(i)}_{\mu\nu} = h^\lambda_\mu h^\rho_\nu (n_i)_{\lambda;\rho}, \quad h^\lambda_\mu = \delta^\lambda_\mu + \xi\sum_i (n_i)_\mu (n_i)^\lambda, \tag{5}$$

where ξ is $+1$ for time-like and -1 for space-like vectors. It is noted that the first term in (4) is just the Ryu–Takayngi formula which appears for Einstein gravity part.

Corresponding equations of motion of (3) are given by

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\big(aR^2 + bR_{\alpha\beta}R^{\alpha\beta} \\ &\quad + cR_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma}\big) + 2aR_{\mu\nu}R - 4cR_\mu{}^\alpha R_{\nu\alpha} \\ &\quad + (2b+4c)R^{\alpha\beta}R_{\mu\alpha\nu\beta} + 2cR_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} \\ &\quad + \Big(2a + \frac{b}{2}\Big)g_{\mu\nu}\nabla_\alpha\nabla^\alpha R + (b+4c)\nabla_\alpha\nabla^\alpha R_{\mu\nu} \\ &\quad - (2a+b+2c)\nabla_\nu\nabla_\mu R = 0. \end{aligned} \tag{6}$$

There is a homogeneous and isotropic black brane solution where in four dimensions can be written as follows

$$ds^2 = \frac{L^2}{\rho^2}\Big(-f(\rho)dt^2 + \frac{1}{f(\rho)}d\rho^2 + dx_1{}^2 + dx_2{}^2\Big), \tag{7}$$

where L stands for the curvature. We will return to this solution later. In what follows, we use the conformal gravity and compute the holographic entanglement entropy, and also we find the expression for the mutual information in this theory.

Holographic entanglement entropy: conformal gravity

As mentioned, the pure conformal gravity can indeed be obtained by (1). It was shown that the theory admits black brane solution (7) where one has

$$f(\rho) = 1 - \frac{a}{3}\rho \pm \sqrt{ma}\rho^2 - m\rho^3 \tag{8}$$

where a and m are the two parameters of the solutions. Note that by setting $a=0$ in the above solutions, one gets AdS black hole (brane) solutions of the Einstein gravity in four dimensions. Now the aim is to compute the holographic entanglement entropy for this theory. We use a strip as the entangling region on the boundary.

Strip entangling region

Now let us use a strip as an entangling region in the boundary and find the related holographic entanglement entropy. To do so, we define the following strip:

$$-\frac{\ell}{2} < x_1 \equiv x < \frac{\ell}{2}, \quad -\frac{H}{2} < x_2 < \frac{H}{2}, \tag{9}$$

where ℓ is the length of the strip and H plays an infrared regulator distance along the entangling surface and we assume $H \gg \ell$ and. One can use the following parametrization for the codimension-two hypersurface in a constant time slice as

$$x_1 = x(\rho),$$

so that the corresponding induced metric becomes

$$ds^2_{\text{ind}} = \frac{L^2}{\rho^2}\big[\big(x'(\rho)^2 + f(\rho)^{-1}\big)d\rho^2 + dx_1{}^2 + dx_2{}^2\big], \tag{10}$$

note that in the above formula the *prime* is the derivative with respect to ρ. For two hypersurfaces which are given by $t=0$ and $x_1 - x(\rho) = 0$, one can find the orthogonal normal vectors as follows

$$\begin{aligned} &\Sigma_1 : t = 0 \quad n_1 = \Big\{\tfrac{\sqrt{f}L}{\rho}, 0, 0, 0\Big\}, \\ &\Sigma_2 : x_1 - x(\rho) = 0 \quad n_2 = \Bigg\{0, -\frac{x'L}{\rho\sqrt{fx'^2+1}}, \frac{L}{\rho\sqrt{fx'^2+1}}, 0\Bigg\}. \end{aligned} \tag{11}$$

And the extrinsic curvatures of the hypersurface are

$$\mathcal{K}^{(1)}_{\mu\nu} = 0, \quad \mathcal{K}^{(2)}_{\mu\nu} = L\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_1 f^{-1} & C_1 x' & 0 \\ 0 & C_1 x' & C_1 f x'^2 & 0 \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \tag{12}$$

where

$$C_1 = \frac{2\left(1+fx'^2\right)fx' - \rho\left(f'x' + 2fx''\right)}{2\rho^2(1+fx'^2)^{5/2}},$$
$$C_2 = \frac{fx'}{\rho^2\sqrt{1+fx'^2}}. \tag{13}$$

Thus for this region, the entanglement entropy of (4) becomes

$$S = \frac{HL}{8G_N}\int \mathrm{d}\rho \frac{\sqrt{x'^2+f^{-1}}}{\rho^3}\left(s_1+s_2\right), \tag{14}$$

and this is the entropy functional which should be minimized. Note that in (14) s_1 and s_2 are given by

$$\begin{aligned} s_1 &= -8(10a+2b+c)f + (32a+7b+4c)\rho f' \\ &\quad - (4a+b)\rho^2 f'' + \frac{\left[(3b+4c)\rho f' - (b+4c)\rho^2 f''\right]fx'^2}{\left(1+fx'^2\right)}, \\ s_2 &= -\rho^4\Big[b\left(2C_2 + C_1\left(1+fx'^2\right)\right)^2 \\ &\quad + 4c\left(2{C_2}^2 + {C_1}^2\left(1+fx'^2\right)^2\right)\Big]. \end{aligned} \tag{15}$$

We should emphasize that the above entanglement functional has been obtained for general higher-derivative action (3), though for our case in conformal gravity one should set $a=-\frac{1}{3}$, $b=2$ and $c=-1$. Now with these parameters, one can compute the holographic entanglement entropy. Before going to details, it is important to mention that in the final expression of the entanglement entropy, neither L nor the radial coordinate ρ appeared explicitly. This is an interesting observation leads to the fact that the resultant entanglement entropy does not have UV-divergent terms as long as the integrand does not diverge at $\rho=0$, for more details see [10].

Holographic entanglement entropy

In this section, we find the corrections to the entanglement entropy for conformal gravity. This is followed by minimizing the entropy functional (14) in order to find the profile of the hypersurface which has been parametrized by $x(\rho)$. It is noted that $x(\rho)$ is supposed to be a smooth differentiable function with the condition $x(0)=\ell/2$. To proceed, one may consider the entropy functional as a one dimensional action in which the corresponding Lagrangian is independent of $x(\rho)$ which leads to a conservation law. In other words, let us write (14) as $S=\int \mathrm{d}\rho\mathcal{L}$, and thus the equation of motion becomes

$$\frac{\partial}{\partial\rho}\left(\frac{\partial\mathcal{L}}{\partial x''}\right) - \left(\frac{\partial\mathcal{L}}{\partial x'}\right) = C, \quad \text{with} \quad \frac{\partial\mathcal{L}}{\partial x} = 0, \tag{16}$$

where C is a constant which can be fixed by imposing the condition that at the turning point ρ_t of the hypersurface in the bulk one has $x'(\rho_t)\to\infty$. After minimizing the functional and using the condition of the hypersurface turning point, one gets the following conserved quantity along the radial profile

$$x'\frac{1+f\left(x'^2-2\lambda\right)}{f\left(f^{-1}+x'^2\right)^{3/2}} = \frac{\rho^3}{{\rho_t}^3}. \tag{17}$$

In principle, the above equation allows us to find $x'(\rho)$. In general, it is a difficult task to solve (17) to find a proper profile since it is a cubic equation for $x'(\rho)$. However, in some special cases, the semianalytic solutions might be obtained. In the following, we will develop the behavior of HEE of a conformal field theory whose states are in fact under the excitation of the higher-curvature terms due to the conformal gravity. Up to the leading order and after making use of the following expression

$$\frac{\ell}{2} = \int_0^{\rho_t} x'(\rho)\,\mathrm{d}\rho, \tag{18}$$

one obtains

$$\ell = \frac{2\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\rho_t + \frac{m\pi}{8}\rho_t^4 + \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{6\Gamma(\frac{1}{4})}a\rho_t^2 + \mathcal{O}(m^2,a^2,ma) \tag{19}$$

which can be inverted to find the turning point of the proposed hypersurface in the bulk as follows

$$\rho_t = \frac{\Gamma\left(\frac{1}{4}\right)}{2\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}\ell - \frac{\Gamma\left(\frac{1}{4}\right)^5}{256\pi^{3/2}\Gamma\left(\frac{3}{4}\right)^5}m\ell^4 - \frac{\Gamma\left(\frac{1}{4}\right)^2}{48\pi\Gamma\left(\frac{3}{4}\right)^2}a\ell^2 + \mathcal{O}\left(m^2,a^2,ma\right). \tag{20}$$

Gathering all the results, one finally gets the holographic entanglement entropy for a strip of length ℓ for a black hole solution in conformal gravity as follows

$$S(\ell) = \frac{2H}{G_N}\left(\frac{2\pi\Gamma(\frac{3}{4})^2}{\Gamma(\frac{1}{4})^2}\frac{1}{\ell} - \frac{\Gamma(\frac{1}{4})^2}{64\Gamma(\frac{3}{4})^2}m\ell^2\right) + \mathcal{O}\left(m^2,a^2,ma\right). \tag{21}$$

To end up this section, we would like to mention that the above expression has no UV-divergent terms (for more details see Ref. [10]). In what follows, we will consider the quantum correlation for two and more systems.

Holographic *n*-partite information

In the context of the quantum information, if one interests in evaluating the amount of correlations between two or more subsystems, mutual and n-partite information are useful quantities namely these are regarded as criterion that indicates the amount of shared information, or more precisely the correlation, between the entangling regions [26]. Concerning the peculiarities of holographic mutual and tripartite information, it is worth investigating the effect of higher-order terms, e.g., conformal gravity, on these quantities which is the main task of this section.

First let us consider two separated systems, e.g., A_1 and A_2, and we want to identify the amount of entanglement (or information) that these two systems can share. So that as mentioned the mutual information would be a proper measure which is given by [27].

$$I(A_1, A_2) = S(A_1) + S(A_2) - S(A_1 \cup A_2), \tag{22}$$

where $S(A_i)$'s are the entanglement entropy of the region A_i and $S(A_1 \cup A_2)$ stands for the entanglement entropy for the union of two entangling regions.Actually, there are two options in computing the minimum area for two systems, precisely as Fig. 1 shows, and depending on the separation of two systems, one can use S_{dis} or S_{con} for the entanglement entropy of the union part. For the union of two strips with the same length ℓ separated by distance h, there are two different configurations which are the disconnected and connected ones, and the one with minimum area should be chosen; these configurations are schematically shown in Fig. 1. Therefore, due to the transition of $S(A_1 \cup A_2)$ from $S_{\text{dis.}} = 2S(\ell)$ to $S_{\text{con.}} = S(2\ell + h) + S(h)$ and vice versa, there is a phase transition in the mutual information. In other words, holographic mutual information vanishes or takes a finite value depending on the values of the entangling regions lengths and their separation. In other words, one has

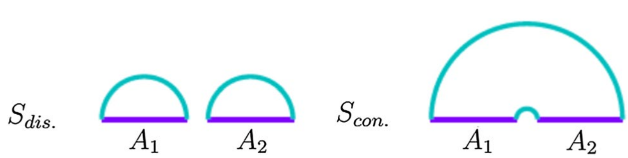


Fig. 1 Schematic representation of two different configurations for computing the entanglement entropy of union of regions

$$I(A_1, A_2) = \begin{cases} 2S(\ell) - S(h) - S(h + 2\ell), & 0 < \frac{h}{\ell} < r_1 \\ 0, & r_1 \leq \frac{h}{\ell}, \end{cases} \tag{23}$$

and it can be shown that there is a critical value of distance, say as r_1, where the mutual information undergoes a phase transition. Interestingly, in the context of the quantum information, it is shown that mutual information undergoes a first order phase transition due to a discontinuity in its first derivative [28]. Therefore, via the holographic methods such phase transition gets a rather simple explanation, e.g., Depending on the value of h/ℓ the corresponding minimal configurations, i.e., Ryu–Takayanagi surfaces, may change from one to another.

Now the aim is to compute the effect of higher-curvature term on the mutual information. To do so first, let us obtain the corrections to the mutual information.

$$I(A_1, A_2) = \frac{2H}{G_N}\left[\frac{2\pi\Gamma(\frac{3}{4})^2}{\Gamma(\frac{1}{4})^2}\left(\frac{2}{\ell} - \frac{1}{h} - \frac{1}{2\ell + h}\right) + \frac{\Gamma(\frac{1}{4})^2}{16\Gamma(\frac{3}{4})^2} mh\ell\right]. \tag{24}$$

The last term in Eq. (24) appears due to the effect of the conformal gravity.

This study can be generalized for three and more systems with topological order, where the tripartite and n-partite information might be utilized as a quantity to characterize entanglement in states of the system. Actually, this quantity could measure the amount of information or correlations (both classical and quantum) between the systems. More generally for a subsystem consisting of n disjoint regions A_i, $i = 1, \ldots, n$, the n-partite information is defined as follows [29]

$$I^{[n]}(A_{\{i\}}) = \sum_{i=1}^{n} S(A_i) - \sum_{i<j}^{n} S(A_i \cup A_j) + \sum_{i<j<k}^{n} S(A_i \cup A_j \cup A_k) - \cdots - (-1)^n S(A_1 \cup A_2 \cup \cdots \cup A_n), \tag{25}$$

where again $S(A_i \cup A_j \cdots)$ is the entanglement entropy of the region $A_i \cup A_j \ldots$ with the rest of the system. It is worth to mention that this definition gives us 1-partite information and 2-partite information which are indeed the entanglement entropy and mutual information, respectively. One may use the definition of the mutual information to write the above equation as follows

$$I^{[n]}(A_{\{i\}}) = \sum_{i=2}^{n} I^{[2]}(A_1, A_i) - \sum_{i=2<j}^{n} I^{[2]}(A_1, A_i \cup A_j) + \sum_{i=2<j<k}^{n} I^{[2]}(A_1, A_i \cup A_j \cup A_k) - \cdots + (-1)^n I^{[2]}(A_1, A_2 \cup A_2 \cdots \cup A_n). \tag{26}$$

It is worth mentioning that although the mutual information is always non-negative, the n-partite information $I^{[n]}$ could have either signs. In the literature of information theory for a subsystem consisting of n disjoint regions, one may define another quantity which, indeed, is a direct generalization of mutual information (known as multi-partite entanglement) defined as (see for example [25])

$$J^{[n]}(A_{\{i\}}) = \sum_{i}^{n} S(A_i) - S(A_1 \cup A_2 \cup \cdots \cup A_n). \tag{27}$$

In terms of the mutual information, it may be recast into the following form

$$J^{[n]}(A_{\{i\}}) = I^{[2]}(A_1, A_2) + I^{[2]}(A_1 \cup A_2, A_3) + \cdots + I^{[2]}(A_1 \cup A_2 \cdots \cup A_{n-1}, A_n). \tag{28}$$

where $S(A_1, A_2, A_3)$ is the entanglement entropy for the union of three subsystems. To compute the holographic tripartite information by pursuing the Ryu–Takayanagi proposal of finding the minimal surface, the union terms of $S(A_i \cup A_j)$ and $S(A_1 \cup A_2 \cup A_3)$ should be considered in more details. In Fig. 2, we have plotted all possible diagrams of the union of three regions.

Thus, $S(A_i \cup A_j)$ and $S(A_1 \cup A_2 \cup A_3)$ are given by the minimum among the possible diagrams. More generally, for three strips with entangling lengths ℓ_1, ℓ_2 and ℓ_3 with separations h_1 and h_2, there are, in fact, 11 possible configurations for unions of regions:

$$S(A_1 \cup A_2)\begin{cases} S(\ell_1) + S(\ell_2) & \equiv S_1 \\ S(\ell_1 + h_1 + \ell_2) + S(h_1) & \equiv S_2 \end{cases},$$
$$S(A_1 \cup A_3)\begin{cases} S(\ell_1) + S(\ell_3) & \equiv S_3 \\ S(\ell_1 + h_1 + \ell_3 + h_2 + \ell_3) + S(h_1 + \ell_2 + h_2) & \equiv S_4 \end{cases},$$
$$S(A_2 \cup A_3)\begin{cases} S(\ell_2) + S(\ell_3) & \equiv S_5 \\ S(\ell_2 + h_2 + \ell_3) + S(h_2) & \equiv S_6 \end{cases}, \tag{30}$$

and also one has

$$S(A_1 \cup A_2 \cup A_3)\begin{cases} S(\ell_1) + S(\ell_2) + S(\ell_3) & \equiv S_7 \\ S(\ell_1) + S(\ell_2 + h_2 + \ell_3) + S(h_2) & \equiv S_8 \\ S(\ell_1 + h_1 + \ell_2) + S(h_1) + S(\ell_3) & \equiv S_9 \\ S(\ell_1 + h_1 + \ell_2 + h_2 + \ell_3) + S(h_1 + \ell_2 + h_2) + S(\ell_2) & \equiv S_{10} \\ S(\ell_1 + h_1 + \ell_2 + h_2 + \ell_3) + S(h_1) + S(h_2) & \equiv S_{11} \end{cases}. \tag{31}$$

Note that this quantity is finite for a system with n disjoint regions and is zero for n un-correlated regions. In particular, the tripartite information was first introduced as the topological entropy and defined by

$$I^{[3]}(A_1, A_2, A_3) = S(A_1) + S(A_2) + S(A_3) - S(A_1 \cup A_2) - S(A_1 \cup A_3) - S(A_2 \cup A_3) + S(A_1 \cup A_2 \cup A_3), \tag{29}$$

From these possible configurations, the minimum expression should be used in each case so the holographic tripartite information can be written as follows

$$I^{[3]}(A_1, A_2, A_3) = S(\ell_1) + S(\ell_2) + S(\ell_3) - \min\{S_1, S_2\} - \min\{S_3, S_4\} - \min\{S_5, S_6\} + \min\{S_7, S_8, S_9, S_{10}, S_{11}\}. \tag{32}$$

For our case in hand, for three strips with the same length and separation, the above relations reduce to

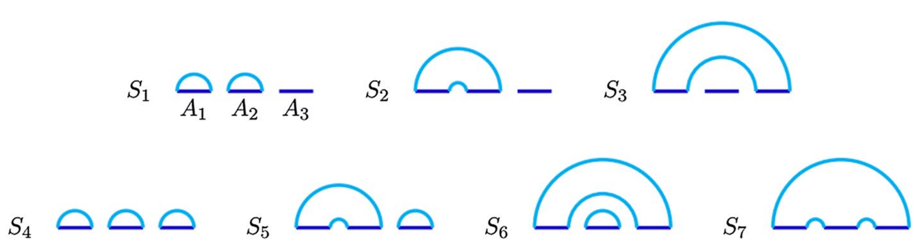


Fig. 2 Schematic representation of competing configurations in the computation of $S(A_i \cup A_j)$ and $S(A_1 \cup A_2 \cup A_3)$

$$S(A_i \cup A_j)\begin{cases} 2S(\ell) \equiv S_1 \\ S(2\ell + h) + S(h) \equiv S_2 \\ S(3\ell + 2h) + S(\ell + 2h) \equiv S_3 \end{cases}$$

$$S(A_1 \cup A_2 \cup A_3)\begin{cases} 3S(\ell) \equiv S_4 \\ S(3\ell + 2h) + S(\ell + 2h) + S(\ell) \equiv S_5 \\ S(2\ell + h) + S(\ell) + S(h) \equiv S_6 \\ S(3\ell + 2h) + 2S(h) \equiv S_7 \end{cases}$$

Therefore, one can write

$$I^{[3]}(A_1, A_2, A_3) = 3S(\ell) - 2\min\{S_1, S_2\} - \min\{S_1, S_3\} + \min\{S_4, S_5, S_6, S_7\}. \tag{33}$$

Therefore, the holographic tripartite information for three entangling regions with the same length ℓ separated by distance h is given by

$$I^{[3]}(A_1, A_2, A_3) = \begin{cases} S(\ell) - 2S(h + 2\ell) + S(2h + 3\ell), & 0 < \frac{h}{\ell} < r_1 \\ 2S(h) - 3S(\ell) + S(2h + 3\ell), & r_1 \leq \frac{h}{\ell} < r_2 \\ 0, & r_2 \leq \frac{h}{\ell} \end{cases} \tag{34}$$

which is the same as mutual information case, there are two critical values of separations where the tripartite information changes its value (Fig. 3).

Conclusion

In this paper, we studied the effect of higher-order derivative terms on mutual and tripartite information. In principle, higher-order theories of gravity including the conformal gravity are interesting because such theories could provide an effective description of quantum corrections and also probe the finite coupling effects via making such corrections to the Einstein gravity theory in the bulk space. We used conformal gravity theory as an example of higher-order derivative in four-dimensional space-time and for a strip entangling region, and obtained the corrections to holographic mutual information as well as holographic tripartite information. As mentioned for two disjointed systems, the mutual information is usually used as a measure of quantum entanglement that two systems can share. The mutual information can also be utilized as a useful probe to address certain phase transitions and critical behavior in these theories.

The main result of this paper can be described as follows: It was shown that the Ryu–Takayanagi entropies obey the inequality

$$I^{[3]}(A_1, A_2, A_3) \leq 0 \tag{35}$$

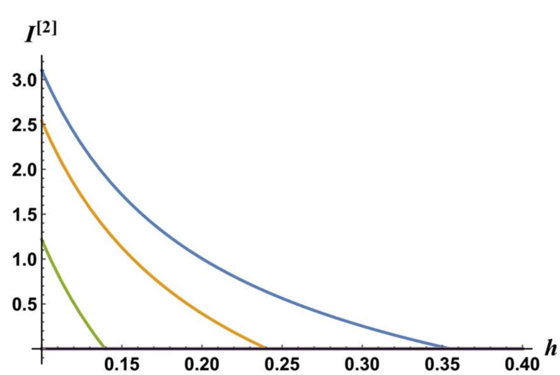
for any regions A_1, A_2, A_3 in the boundary field theory [35]. On the other hand, Ryu–Takayanagi formula applies only to field theories dual to Einstein gravity. We considered the effect of higher-curvature corrections to the bulk gravitational action (conformal gravity), and showed that for our case namely for a strip again one has the above inequality. From the definition of the tripartite information, one can write the tripartite information in terms of the mutual information as

$$I^{[3]}(A_1, A_2, A_3) = I(A_1, A_2) + I(A_1, A_3) - I(A_1, A_2 \cup A_3). \tag{36}$$

Having noted that the tripartite information is negative, one gets

$$I(A_1, A_2) + I(A_1, A_3) \leq I(A_1, A_2 \cup A_3), \tag{37}$$

in the context of quantum information theory, for any measure of the information say as $F(A_i)$, the inequality of the form $F(A_1, A_2) + F(A_1, A_3) \leq F(A_1, A_2 \cup A_3)$ is known as monogamy relation which implies that the holographic mutual information is monogamous. This features are



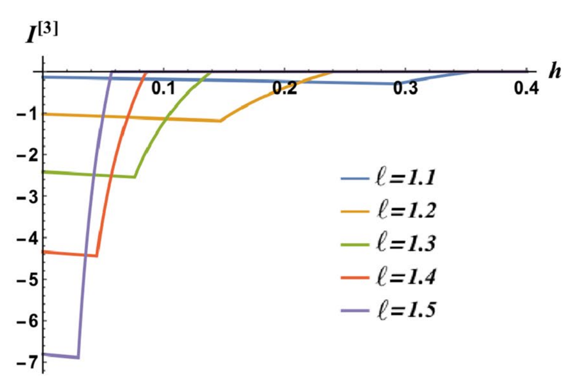


Fig. 3 Numerical results for holographic mutual information (left plot) and tripartite information (right plot) as a function of the separation distance: for $\ell = 1.1, \ldots, 1.5$. Note in all cases $\mathrm{I}(\ell_1, \ell_2)$ is positive where as tripartite information remains negative

characteristic of measures of quantum entanglement. In the context of quantum information theory, the monogamy property is related to the security of quantum cryptography, since, unlike classical correlation, quantum entanglement is not a shareable resource. In other words, entangled correlations between A_1 and A_2 cannot be shared with a third system A_3 without spoiling the original entanglement [29]. We showed that adding higher-order terms at the gravity side does not break the monogamy feature of the information at the field theory side.

In this direction, considering time-dependent backgrounds and check how the sign of these quantities changes during the thermalization process can be a challenging work. We leave further investigations to future works.

Acknowledgements The author would like to thank Mohsen Alishahiha, for his helpful comments and discussions. MRT also wishes to acknowledge A. Akhvan and F. Omidi for some their comments. This work has been supported in parts by IAUCTB.

Appendix: some useful mathematical relations

Here in this appendix, we present some useful relations that we have used in this paper. Let us choose a five-dimensional metric with coordinate t, r, x, y, z as follows

$$\begin{pmatrix} -\frac{f(r)}{r^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r^2 f(r)} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r^2} \end{pmatrix}$$

The determinant of the induced metric reads as

$$\frac{1}{r^6 f(r)} + \frac{x'(r)^2}{r^6}$$

Therefore, two normal vectors are obtained as

$$\left\{\frac{1}{\sqrt{\frac{r^2}{f(r)}}}, 0, 0, 0, 0\right\},$$
$$\left\{0, -\frac{x'(r)}{\sqrt{r^2 f(r) x'(r)^2 + r^2}}, \frac{1}{\sqrt{r^2 f(r) x'(r)^2 + r^2}}, 0, 0\right\}$$

The nonzero component of the extrinsic curvature ten reads as

$$\mathcal{K}_{11} = \frac{-rf'(r)x'(r) + 2f(r)^2 x'(r)^3 + 2f(r)\big(x'(r) - rx''(r)\big)}{2r^2 f(r)\big(f(r)x'(r)^2 + 1\big)^{5/2}}$$
$$\mathcal{K}_{12} = \frac{x'(r)\big(-rf'(r)x'(r) + 2f(r)^2 x'(r)^3 + 2f(r)\big(x'(r) - rx''(r)\big)\big)}{2r^2\big(f(r)x'(r)^2 + 1\big)^{5/2}}$$
$$\mathcal{K}_{21} = \frac{x'(r)\big(-rf'(r)x'(r) + 2f(r)^2 x'(r)^3 + 2f(r)\big(x'(r) - rx''(r)\big)\big)}{2r^2\big(f(r)x'(r)^2 + 1\big)^{5/2}}$$
$$\mathcal{K}_{22} = \frac{f(r)x'(r)^2\big(-rf'(r)x'(r) + 2f(r)^2 x'(r)^3 + 2f(r)\big(x'(r) - rx''(r)\big)\big)}{2r^2\big(f(r)x'(r)^2 + 1\big)^{5/2}}$$
$$\mathcal{K}_{33} = \frac{f(r)x'(r)}{r^2\sqrt{f(r)x'(r)^2 + 1}}$$
$$\mathcal{K}_{44} = \frac{f(r)x'(r)}{r^2\sqrt{f(r)x'(r)^2 + 1}}$$

and also one finds

$$R_{\mu\nu} n_i^\mu n_i^\nu = \frac{rf'(r) - 4f(r)}{f(r)x'(r)^2 + 1} - \frac{f(r)x'(r)^2\big(r^2 f''(r) - 5rf'(r) + 8f(r)\big)}{2\big(f(r)x'(r)^2 + 1\big)} + \frac{1}{2}\big(-r\big(rf''(r) - 5f'(r)\big) - 8f(r)\big)$$
$$R_{\mu\nu\alpha\beta} n_i^\mu n_i^\alpha n_j^\nu n_j^\beta = -2\left(\frac{f(r)x'(r)^2\big(r^2 f''(r) - 2rf'(r) + 2f(r)\big)}{2\big(f(r)x'(r)^2 + 1\big)} - \frac{rf'(r) - 2f(r)}{2\big(f(r)x'(r)^2 + 1\big)}\right)$$

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