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RESEARCH

3N bound state formalism based on 3N free basis states

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Abstract A spin–isospin dependent three-dimensional approach has been applied for the formulation of the three-nucleon bound state, and a new expression for Faddeev equation based on three-nucleon free basis states has been obtained. The advantage of this new expression is that the Faddeev integral equation has been simpler for numerical calculation.

Keywords Faddeev equation · Three-dimensional approach · Three-nucleon bound state

Introduction

During the past years, the three-dimensional (3D) approach has been developed for few-body bound and scattering problems [1–13]. The motivation for developing this approach is introducing a direct solution of the integral equations avoiding the very involved angular momentum algebra occurring for the permutations, transformations and especially for the three-body forces.

In the case of the three-body bound state, the Faddeev equation has been formulated for three identical bosons as a function of vector Jacobi momenta, with the specific stress upon the magnitudes of the momenta and the angles between them [2]. Adding the spin–isospin to the 3D formalism was a major additional task which was carried out in Ref. [5]. In this paper we have attempted to reformulate the three-nucleon (3N) bound state and have obtained a new expression for Faddeev integral equation. To this end, we have used 3N free basis state for representation of 3N wave function.

This manuscript is organized as follows. In Sect. 2, we have derived a new expression for Faddeev equation in a realistic 3D scheme as a function of Jacobi momenta vectors and the spin–isospin quantum numbers. Then, we have chosen suitable coordinate system for describing Faddeev components of total 3N wave function as function of five independent variables for numerical calculations. Finally, in Sect. 3, a summary and an outlook have been presented.

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3N bound state in a 3D momentum representation

Faddeev equation for the 3N bound state with considering pairwise interactions is described by [14]:

$$|\psi^{M_t}\rangle = G_0 t P |\psi^{M_t}\rangle, \tag{1}$$

where $|\psi^{M_t}\rangle$ is Faddeev component of the total 3N wave function, M_t being the projection of total angular momentum along the quantization axis, $P = P_{12}P_{23} + P_{13}P_{23}$ is the sum of cyclic and anti-cyclic permutations of three nucleons, t denotes the two-body transition operator which is determined by a Lippmann–Schwinger equation and G_0 is the free 3N propagator which is given by:

$$G_0 = \frac{1}{E - \frac{p^2}{m} - \frac{3q^2}{4m}}, \tag{2}$$

where E is the binding energy of 3N bound state. To solve Eq. (1) in the momentum space, we introduce the 3N free basis state in a 3D formalism as [6]:

$$\begin{aligned} |\mathbf{pq}\gamma\rangle &\equiv |\mathbf{pq} m_{s_1} m_{s_2} m_{s_3} m_{t_1} m_{t_2} m_{t_3}\rangle \\ &\equiv |\mathbf{q} m_{s_1} m_{t_1}\rangle |\mathbf{p} m_{s_2} m_{s_3} m_{t_2} m_{t_3}\rangle, \end{aligned} \tag{3}$$

This basis state involves two standard Jacobi momenta $\mathbf{p}$ and $\mathbf{q}$ which are the relative momentum vector in the subsystem and the momentum vector of the spectator with respect to the subsystem, respectively [14]. $|\gamma\rangle \equiv |m_{s_1} m_{s_2} m_{s_3} m_{t_1} m_{t_2} m_{t_3}\rangle$ is the spin–isospin parts of the basis state where the quantities $m_{s\_i}(m_{t\_i})$ are the projections of the spin (isospin) of each three nucleons along the quantization axis. The introduced basis states are completed and normalized as:

$$\sum_{\gamma} \int d\mathbf{p} \int d\mathbf{q} |\mathbf{pq}\gamma\rangle\langle\mathbf{pq}\gamma| = 1, \tag{4}$$

$$\langle\mathbf{p}'\mathbf{q}'\gamma'|\mathbf{pq}\gamma\rangle = \delta(\mathbf{p}'-\mathbf{p})\delta(\mathbf{q}'-\mathbf{q})\delta_{\gamma'\gamma}. \tag{5}$$

Now, we start by inserting the completeness relation twice into Eq. (1) as follows:

$$\langle\mathbf{pq}\gamma|\psi^{M_t}\rangle = \frac{1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}} \sum_{\gamma''} \int d\mathbf{p}'' \int d\mathbf{q}'' \sum_{\gamma'} \int d\mathbf{p}' \int d\mathbf{q}' \times \langle\mathbf{pq}\gamma|t|\mathbf{p}''\mathbf{q}''\gamma''\rangle\langle\mathbf{p}''\mathbf{q}''\gamma''|P|\mathbf{p}'\mathbf{q}'\gamma'\rangle\langle\mathbf{p}'\mathbf{q}'\gamma'|\psi^{M_t}\rangle. \tag{6}$$

The matrix elements of the permutation operator P are evaluated as [6]:

$$\begin{aligned}\langle\mathbf{p}''\mathbf{q}''\gamma''|P|\mathbf{p}'\mathbf{q}'\gamma'\rangle &= \delta\left(\mathbf{p}''-\frac{1}{2}\mathbf{q}''-\mathbf{q}'\right)\delta\left(\mathbf{p}'+\mathbf{q}''+\frac{1}{2}\mathbf{q}'\right) \\ &\times \delta_{m''_{s_1} m'_{s_3}} \delta_{m''_{s_2} m'_{s_1}} \delta_{m''_{s_3} m'_{s_2}} \delta_{m''_{t_1} m'_{t_3}} \delta_{m''_{t_2} m'_{t_1}} \delta_{m''_{t_3} m'_{t_2}} \\ &+ \delta\left(\mathbf{p}''+\frac{1}{2}\mathbf{q}''+\mathbf{q}'\right)\delta\left(\mathbf{p}'-\mathbf{q}''-\frac{1}{2}\mathbf{q}'\right) \\ &\times \delta_{m''_{s_1} m'_{s_2}} \delta_{m''_{s_2} m'_{s_3}} \delta_{m''_{s_3} m'_{s_1}} \delta_{m''_{t_1} m'_{t_2}} \delta_{m''_{t_2} m'_{t_3}} \delta_{m''_{t_3} m'_{t_1}},\end{aligned} \tag{7}$$

and for the two-body t-matrix we have:

$$\langle\mathbf{pq}\gamma|t|\mathbf{p}''\mathbf{q}''\gamma''\rangle = \langle\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}|t(\epsilon)|\mathbf{p}''m''_{s_2}m''_{s_3}m''_{t_2}m''_{t_3}\rangle \delta(\mathbf{q}-\mathbf{q}'')\delta_{m_{s_1}m''_{s_1}}\delta_{m_{t_1}m''_{t_1}}, \tag{8}$$

where $\epsilon = E - \frac{3}{4m}q^2$, is the energy carried by a two-body subsystem in a three-nucleon system. Substituting Eqs. (7) and (8) into Eq. (6) yields:

$$\begin{aligned}\langle\mathbf{pq}\gamma|\psi^{M_t}\rangle &= \frac{1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}} \sum_{m'_{s_1} m'_{t_1}} \int d\mathbf{q}' \\ &\times \Big\{ \sum_{m'_{s_2} m'_{t_2}} \langle\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}|t(\epsilon)|\boldsymbol{\pi}m'_{s_1}m'_{s_2}m'_{t_1}m'_{t_2}\rangle \\ &\times \left\langle -\boldsymbol{\pi}'\mathbf{q}'m'_{s_1}m'_{s_2}m_{s_1}m'_{t_1}m'_{t_2}m_{t_1}|\psi^{M_t}\right\rangle \\ &+ \sum_{m'_{s_3} m'_{t_3}} \langle\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}|t(\epsilon)|-\boldsymbol{\pi}m'_{s_3}m'_{s_1}m'_{t_3}m'_{t_1}\rangle \\ &\times \left\langle \boldsymbol{\pi}'\mathbf{q}'m'_{s_1}m_{s_1}m'_{s_3}m'_{t_1}m_{t_1}m'_{t_3}|\psi^{M_t}\right\rangle \Big\} \\ &= \frac{1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}} \sum_{m'_{s_1} m'_{t_1} m'_s m'_t} \int d\mathbf{q}' \\ &\times \Big\{ \langle\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}|t(\epsilon)|\boldsymbol{\pi}m'_{s_1}m'_s m'_{t_1}m'_t\rangle \\ &\times \langle\boldsymbol{\pi}'\mathbf{q}'m'_{s_1}m_{s_1}m'_s m'_{t_1}m'_t m_{t_1}|P_{23}|\psi^{M_t}\rangle \\ &+ \left\langle \mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}|t(\epsilon)P_{23}|\boldsymbol{\pi}m'_{s_1}m'_s m'_{t_1}m'_t\right\rangle \\ &\times \left\langle \boldsymbol{\pi}'\mathbf{q}'m'_{s_1}m_{s_1}m'_s m'_{t_1}m_{t_1}m'_t|\psi^{M_t}\right\rangle \Big\}.\end{aligned} \tag{9}$$

In the last equality, we have used the antisymmetry of Faddeev component of the 3N wave function as:

$$P_{23}|\psi^{M_t}\rangle = -|\psi^{M_t}\rangle, \tag{10}$$

and also we have considered:

$$\boldsymbol{\pi} = \frac{1}{2}\mathbf{q}+\mathbf{q}', \quad \boldsymbol{\pi}' = \mathbf{q}+\frac{1}{2}\mathbf{q}'. \tag{11}$$

The antisymmetrized two-body t-matrix is introduced as [4]:

$$\begin{aligned}&{}_a\langle\mathbf{p}'m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}|t|\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}\rangle_a \\ &= \langle\mathbf{p}'m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}|t(1-P_{23})|\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}\rangle,\end{aligned} \tag{12}$$

where $|\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}\rangle_a$ is the antisymmetrized two-body state which is defined as:

$$|\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}\rangle_a = \frac{1}{\sqrt{2}}(1-P_{23})|\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}\rangle. \tag{13}$$

Hence, the final expression for Faddeev equation is explicitly written:

$$\begin{aligned}\langle\mathbf{pq}\gamma|\psi^{M_t}\rangle &= \frac{-1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}} \sum_{m'_{s_1} m'_{t_1} m'_s m'_t} \int d\mathbf{q}'_a \\ &\times \langle\mathbf{p}m_{s_2}m_{s_3}m_{t_2}m_{t_3}|t(\epsilon)|\boldsymbol{\pi}m'_{s_1}m'_s m'_{t_1}m'_t\rangle_a \\ &\times \langle\boldsymbol{\pi}'\mathbf{q}'m'_{s_1}m_{s_1}m'_s m'_{t_1}m_{t_1}m'_t|\psi^{M_t}\rangle.\end{aligned} \tag{14}$$

As a simplification, we rewrite this equation as:

$$\psi_{\gamma}^{M_t}(\mathbf{p},\mathbf{q}) = \frac{-1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}} \sum_{m''_s m''_t m'_s m'_t} \int d\mathbf{q}' t_{a\, m_{s_2}m_{s_3}m_{t_2}m_{t_3}}^{m''_s m'_s m''_t m'_t}(\mathbf{p},\boldsymbol{\pi};\epsilon)\psi_{\tilde{\gamma}}^{M_t}(\boldsymbol{\pi}',\mathbf{q}'), \tag{15}$$

where we have used index $\tilde{\gamma}$ instead of $m''_s m_{s_1} m'_s m''_t m_{t_1} m'_t$ for simplicity. The previous expression which has been presented in Ref. [5] is:

$$\psi_{\alpha}(\mathbf{p},\mathbf{q}) = \frac{1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}} \sum_{\gamma\gamma'\alpha'} g_{\alpha\gamma} g_{\gamma'\alpha'} \delta_{m_{s_3}m'_{s_1}} \delta_{m_{t_3}m'_{t_1}} \int d\mathbf{q}' t_{a\, m_{s_1}m_{s_2}m_{t_1}m_{t_2}}^{m'_{s_2}m'_{s_3}m'_{t_2}m'_{t_3}}(\mathbf{p},-\boldsymbol{\pi};\epsilon)\psi_{\alpha'}(\boldsymbol{\pi}',\mathbf{q}'), \tag{16}$$

It is clear that in the new Faddeev integral equation, Clebsch–Gordan coefficients $g_{\alpha\gamma}$ and $g_{\gamma'\alpha'}$ and summing up on α' does not exist. Thus, this new expression is simpler for numerical calculations in comparison with the previous one.

For solving Eq. (15), one needs the matrix elements of the antisymmetrized two-body t-matrix. We connect this quantity to its momentum-helicity representation in "Appendix 1". To solve this integral equation numerically, we have to define a suitable coordinate system. It is convenient to choose the spin polarization direction parallel to the z axis and express the momentum vectors in this coordinate system. With this selection, we can write the two-body t-matrix and 3N wave function as (see Appendices 1 and 2):

$$t_{a\,m_{s_2}m_{s_3}m_{t_2}m_{t_3}}^{m''_s m'_s m''_t m'_t}(\mathbf{p},\boldsymbol{\pi};\epsilon) = e^{-i[(m_{s_2}+m_{s_3})\varphi_p-(m''_s+m'_s)\varphi_\pi]}\,{}^{\pm}t_{a\,m_{s_2}m_{s_3}m_{t_2}m_{t_3}}^{m''_s m'_s m''_t m'_t}(p,x_p,\cos\varphi_{p\pi},x_\pi,\pi,y_{p\pi};\epsilon), \qquad (17)$$

$$\psi_\gamma^{M_t}(\mathbf{p},\mathbf{q}) = e^{-i[(m_{s_2}+m_{s_3})\varphi_p+(m_{s_1}-M_t)\varphi_q]}\,{}^{\pm}\psi_\gamma^{M_t}(p,x_p,\cos\varphi_{pq},x_q,q), \qquad (18)$$

$$\psi_{\tilde\gamma}^{M_t}(\boldsymbol{\pi}',\mathbf{q}') = e^{-i[(m_{s_1}+m'_s)\varphi_{\pi'}+(m''_s-M_t)\varphi']}\,{}^{\pm}\psi_{\tilde\gamma}^{M_t}(\pi',x_{\pi'},\cos\varphi_{\pi'q'},x',q'). \qquad (19)$$

where $x'=\hat{\mathbf{q}}'\cdot\hat{\mathbf{z}}$, $\varphi'=\varphi_{q'}$ and the labels $\pm$ are related to the signs of $\sin\varphi_{p\pi}$, $\sin\varphi_{pq}$ and $\sin\varphi_{\pi'q'}$. With considering:

$$\varphi_\pi = \varphi'+\varphi_{\pi q'},\quad \varphi_{\pi'} = \varphi'+\varphi_{\pi'q'}. \qquad (20)$$

Our final Faddeev equation from Eq. (15) is rewritten as:

$$\begin{aligned}{}^{\pm}\psi_\gamma^{M_t}(p,x_p,\cos\varphi_{pq},x_q,q) &= \frac{-1}{E-\frac{p^2}{m}-\frac{3q^2}{4m}}\sum_{m''_s m''_t m'_s m'_t}\int_0^\infty dq'\int_{-1}^{1}dx'\int_0^{2\pi}d\varphi' e^{i(m''_s+m'_s)\varphi_{\pi q'}}e^{-i(m_{s_1}+m'_s)\varphi_{\pi'q'}}\\ &\times e^{i(m_{s_1}-M_t)(\varphi_q-\varphi')}t_{a\,m_{s_2}m_{s_3}m_{t_2}m_{t_3}}^{m''_s m'_s m''_t m'_t}(p,x_p,\cos\varphi_{p\pi},x_\pi,\pi,y_{p\pi};\epsilon)^{\pm}\\ &\times\psi_{\tilde\gamma}^{M_t}(\pi',x_{\pi'},\cos\varphi_{\pi'q'},x',q'),\end{aligned} \qquad (21)$$

where the variables are developed similar to the 3N scattering as [13]:

$$\begin{aligned}
x_q &= \hat{\mathbf{q}}\cdot\hat{\mathbf{z}},\\
x_p &= \hat{\mathbf{p}}\cdot\hat{\mathbf{z}},\\
\pi &= \sqrt{\frac{1}{4}q^2+q'^2+qq'y_{qq'}},\\
\pi' &= \sqrt{q^2+\frac{1}{4}q'^2+qq'y_{qq'}},\\
x_\pi &= \hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{z}} = \frac{\frac{1}{2}qx_q+q'x'}{\pi},\\
x_{\pi'} &= \hat{\boldsymbol{\pi}}'\cdot\hat{\mathbf{z}} = \frac{qx_q+\frac{1}{2}q'x'}{\pi'},\\
y_{p\pi} &= \hat{\mathbf{p}}\cdot\hat{\boldsymbol{\pi}} = \frac{\frac{1}{2}qy_{pq}+q'y_{pq'}}{\pi},\\
y_{\pi q'} &= \hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{q}}' = \frac{\frac{1}{2}qy_{qq'}+q'}{\pi},\\
y_{\pi'q'} &= \hat{\boldsymbol{\pi}}'\cdot\hat{\mathbf{q}}' = \frac{qy_{qq'}+\frac{1}{2}q'}{\pi'},\\
y_{pq} &= \hat{\mathbf{p}}\cdot\hat{\mathbf{q}} = x_px_q+\sqrt{1-x_p^2}\sqrt{1-x_q^2}\cos\varphi_{pq},\\
y_{pq'} &= \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}' = x_px'+\sqrt{1-x_p^2}\sqrt{1-x'^2}\cos(\varphi_p-\varphi'),\\
y_{qq'} &= \hat{\mathbf{q}}\cdot\hat{\mathbf{q}}' = x_qx'+\sqrt{1-x_q^2}\sqrt{1-x'^2}\cos(\varphi_q-\varphi'),\\
\cos\varphi_{p\pi} &= \frac{\hat{\mathbf{p}}\cdot\hat{\boldsymbol{\pi}}-(\hat{\mathbf{p}}\cdot\hat{\mathbf{z}})(\hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{z}})^2}} = \frac{y_{p\pi}-x_px_\pi}{\sqrt{1-x_p^2}\sqrt{1-x_\pi^2}},\\
\cos\varphi_{\pi'q'} &= \frac{\hat{\boldsymbol{\pi}}'\cdot\mathbf{q}'-(\hat{\boldsymbol{\pi}}'\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}'\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\boldsymbol{\pi}}'\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\mathbf{q}}'\cdot\hat{\mathbf{z}})^2}} = \frac{y_{\pi'q'}-x_{\pi'}x'}{\sqrt{1-x_{\pi'}^2}\sqrt{1-x'^2}},\\
\cos\varphi_{\pi q'} &= \frac{\hat{\boldsymbol{\pi}}\cdot\mathbf{q}'-(\hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}'\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\mathbf{q}}'\cdot\hat{\mathbf{z}})^2}} = \frac{y_{\pi q'}-x_\pi x'}{\sqrt{1-x_\pi^2}\sqrt{1-x'^2}}.
\end{aligned} \qquad (22)$$

It is clear that the Faddeev component of the wave function ψ is explicitly calculated as a function of five independent variables. In Appendices 4 and 5, we discuss about the x'- and φ'-integration and also determination of the signs of sine functions without any ambiguity. The Faddeev integral equation (21) represents a set of three-dimensional homogenous integral equations, which after discretization turns into a huge matrix eigenvalue equation. The huge matrix eigenvalue equation requires an iterative solution method. We can use a Lanczos-like scheme that is proved to be very efficient for nuclear few-body problems [15].

In this stage, we discuss about the total number of coupled integral equations. The total number of coupled Faddeev equations for the 3N bound state in a realistic 3D formalism according to the spin–isospin states is given by:

$$N = 2(N_t\times N_s) = 2\left(N_t\times\sum_{i=1}^{3}N_{m_{s_i}}\right), \qquad (23)$$

where N_s and N_t are the total number of spin and isospin states, respectively, and $N_{m\_s\_i}$ is the number of spin states for each nucleon. It is clear that $N_{m\_s\_i}=2$ and $N_t=3$ for our problem. The factor 2 is related to signs of sine functions of azimuthal angles which is explained in "Appendix 5". Consequently, the total number of coupled Faddeev equations for either ^{3}H or ^{3}He is $N=48$. The total 3N wave function $|\Psi^{M_t}\rangle$ is given by [14]:

$$|\Psi^{M_t}\rangle = (1+P)|\psi^{M_t}\rangle. \qquad (24)$$

Now, we derive an expression for the matrix elements of the total 3N wave function by inserting the 3N free basis state as follows:

$$\langle \mathbf{pq}\gamma|\Psi^{M_t}\rangle = \langle \mathbf{pq}\gamma|\psi^{M_t}\rangle + \langle \mathbf{pq}\gamma|P_{12}P_{23}|\psi^{M_t}\rangle + \langle \mathbf{pq}\gamma|P_{13}P_{23}|\psi^{M_t}\rangle. \tag{25}$$

By applying the permutation operator $P_{12}P_{23}$ and $P_{13}P_{23}$ to the 3N free basis state, Eq. (25) can be written as [6]:

$$\langle \mathbf{pq}\gamma|\Psi^{M_t}\rangle = \langle \mathbf{pq}\gamma|\psi^{M_t}\rangle + \langle \mathbf{p}_2\mathbf{q}_2\gamma_2|\psi^{M_t}\rangle + \langle \mathbf{p}_3\mathbf{q}_3\gamma_3|\psi^{M_t}\rangle, \tag{26}$$

with:

$$\mathbf{p}_2 = -\frac{1}{2}\mathbf{p} - \frac{3}{4}\mathbf{q}, \quad \mathbf{q}_2 = \mathbf{p} - \frac{1}{2}\mathbf{q}, \quad \gamma_2 \equiv m_{s_2}m_{s_3}m_{s_1}m_{t_2}m_{t_3}m_{t_1},$$
$$\mathbf{p}_3 = -\frac{1}{2}\mathbf{p} + \frac{3}{4}\mathbf{q}, \quad \mathbf{q}_3 = -\mathbf{p} - \frac{1}{2}\mathbf{q}, \quad \gamma_3 \equiv m_{s_3}m_{s_1}m_{s_2}m_{t_3}m_{t_1}m_{t_2}. \tag{27}$$

As a simplification, Eq. (26) is rewritten as:

$$\Psi^{M_t}_{\gamma}(\mathbf{p},\mathbf{q}) = \psi^{M_t}_{\gamma}(\mathbf{p},\mathbf{q}) + \psi^{M_t}_{\gamma_2}(\mathbf{p}_2,\mathbf{q}_2) + \psi^{M_t}_{\gamma_3}(\mathbf{p}_3,\mathbf{q}_3). \tag{28}$$

Now, we rewrite this equation in the selected coordinate system as:

$$\begin{aligned}\Psi^{M_t}_{\gamma}(\mathbf{p},\mathbf{q}) = {} & e^{-i[(m_{s_2}+m_{s_3})\varphi_p+(m_{s_1}-M_t)\varphi_q]\,\pm}\psi^{M_t}_{\gamma} \\ & \times (p, x_p, \cos\varphi_{pq}, x_q, q) \\ & + e^{-i[(m_{s_3}+m_{s_1})\varphi_{p_2}+(m_{s_2}-M_t)\varphi_{q_2}]\,\pm}\psi^{M_t}_{\gamma_2} \\ & \times (p_2, x_{p_2}, \cos\varphi_{p_2q_2}, x_{q_2}, q_2) \\ & + e^{-i[(m_{s_1}+m_{s_2})\varphi_{p_3}+(m_{s_3}-M_t)\varphi_{q_3}]\,\pm}\psi^{M_t}_{\gamma_3} \\ & \times (p_3, x_{p_3}, \cos\varphi_{p_3q_3}, x_{q_3}, q_3).\end{aligned} \tag{29}$$

By considering:

$$\begin{aligned}\varphi_{p_2} &= \varphi_q + \varphi_{p_2q}, \varphi_{q_2} = \varphi_q + \varphi_{q_2q}, \\ \varphi_{p_3} &= \varphi_q + \varphi_{p_3q}, \varphi_{q_3} = \varphi_q + \varphi_{q_3q},\end{aligned} \tag{30}$$

Eq. (29) can be written as:

$$\begin{aligned}& {}^{\pm}\Psi^{M_t}_{\gamma}(p, x_p, \cos\varphi_{pq}, x_q, q) \\ & = {}^{\pm}\psi^{M_t}_{\gamma}(p, x_p, \cos\varphi_{pq}, x_q, q) + e^{i(m_{s_2}+m_{s_3})\varphi_{pq}} \\ & \quad \times \Big\{ e^{-i[(m_{s_3}+m_{s_1})\varphi_{p_2q}+(m_{s_2}-M_t)\varphi_{q_2q}]\,\pm}\psi^{M_t}_{\gamma_2}(p_2, x_{p_2}, \cos\varphi_{p_2q_2}, x_{q_2}, q_2) \\ & \quad + e^{-i[(m_{s_1}+m_{s_2})\varphi_{p_3q}+(m_{s_3}-M_t)\varphi_{q_3q}]\,\pm}\psi^{M_t}_{\gamma_3}(p_3, x_{p_3}, \cos\varphi_{p_3q_3}, x_{q_3}, q_3) \Big\}.\end{aligned} \tag{31}$$

where:

$$\begin{aligned} p_2 &= |-\frac{1}{2}\mathbf{p} - \frac{3}{4}\mathbf{q}| = \frac{1}{2}\sqrt{p^2 + \frac{9}{4}q^2 + 3pqy_{pq}}, \\ p_3 &= |-\frac{1}{2}\mathbf{p} + \frac{3}{4}\mathbf{q}| = \frac{1}{2}\sqrt{p^2 + \frac{9}{4}q^2 - 3pqy_{pq}}, \\ q_2 &= |\mathbf{p} - \frac{1}{2}\mathbf{q}| = \sqrt{p^2 + \frac{1}{4}q^2 - pqy_{pq}}, \\ q_3 &= |-\mathbf{p} - \frac{1}{2}\mathbf{q}| = \sqrt{p^2 + \frac{1}{4}q^2 + pqy_{pq}}, \\ x_{p_2} &= \hat{\mathbf{p}}_2\cdot\hat{\mathbf{z}} = \frac{-\frac{1}{2}px_p - \frac{3}{4}qx_q}{p_2}, \\ x_{p_3} &= \hat{\mathbf{p}}_3\cdot\hat{\mathbf{z}} = \frac{-\frac{1}{2}px_p + \frac{3}{4}qx_q}{p_3}, \\ x_{q_2} &= \hat{\mathbf{q}}_2\cdot\hat{\mathbf{z}} = \frac{px_p - \frac{1}{2}qx_q}{q_2}, \\ x_{q_3} &= \hat{\mathbf{q}}_3\cdot\hat{\mathbf{z}} = \frac{-px_p - \frac{1}{2}qx_q}{q_3}, \\ \cos\varphi_{p_2q_2} &= \frac{\hat{\mathbf{p}}_2\cdot\hat{\mathbf{q}}_2 - (\hat{\mathbf{p}}_2\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}_2\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\mathbf{q}}_2\cdot\hat{\mathbf{z}})^2}} = \frac{\frac{-\frac{1}{2}p^2+\frac{3}{8}q^2-\frac{1}{2}pqy_{pq}}{p_2q_2} - x_{p_2}x_{q_2}}{\sqrt{1-x^2_{p_2}}\sqrt{1-x^2_{q_2}}}, \\ \cos\varphi_{p_3q_3} &= \frac{\hat{\mathbf{p}}_3\cdot\hat{\mathbf{q}}_3 - (\hat{\mathbf{p}}_3\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}_3\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\mathbf{q}}_3\cdot\hat{\mathbf{z}})^2}} = \frac{\frac{\frac{1}{2}p^2-\frac{3}{8}q^2-\frac{1}{2}pqy_{pq}}{p_2q_2} - x_{p_3}x_{q_3}}{\sqrt{1-x^2_{p_3}}\sqrt{1-x^2_{q_3}}}, \\ \cos\varphi_{p_2q} &= \frac{\hat{\mathbf{p}}_2\cdot\hat{\mathbf{q}} - (\hat{\mathbf{p}}_2\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})^2}} = \frac{\frac{-\frac{1}{2}py_{pq}-\frac{3}{4}q}{p_2} - x_{p_2}x_q}{\sqrt{1-x^2_{p_2}}\sqrt{1-x^2_q}}, \\ \cos\varphi_{p_3q} &= \frac{\hat{\mathbf{p}}_3\cdot\hat{\mathbf{q}} - (\hat{\mathbf{p}}_3\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\mathbf{p}}_3\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})^2}} = \frac{\frac{-\frac{1}{2}py_{pq}+\frac{3}{4}q}{p_3} - x_{p_3}x_q}{\sqrt{1-x^2_{p_3}}\sqrt{1-x^2_q}}, \\ \cos\varphi_{q_2q} &= \frac{\hat{\mathbf{q}}_2\cdot\hat{\mathbf{q}} - (\hat{\mathbf{q}}_2\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\mathbf{q}}_2\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})^2}} = \frac{\frac{py_{pq}-\frac{1}{2}q}{q_2} - x_{q_2}x_q}{\sqrt{1-x^2_{q_2}}\sqrt{1-x^2_q}}, \\ \cos\varphi_{q_3q} &= \frac{\hat{\mathbf{q}}_3\cdot\hat{\mathbf{q}} - (\hat{\mathbf{q}}_3\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\mathbf{q}}_3\cdot\hat{\mathbf{z}})^3}\sqrt{1-(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})^2}} = \frac{\frac{-py_{pq}-\frac{1}{2}q}{q_3} - x_{q_3}x_q}{\sqrt{1-x^2_{q_3}}\sqrt{1-x^2_q}},\end{aligned} \tag{32}$$

The labels $\pm$ are related to the signs of $\sin\varphi_{pq}$, $\sin\varphi_{p_2q_2}$ and $\sin\varphi_{p_3q_3}$ which are determined in "Appendix 5".

Summary and outlook

We extend the recently developed formalism for a new treatment of the Nd scattering in three dimensions for the 3N bound state [13]. We propose a new representation of the 3D Faddeev equation for the 3N bound state including the spin and isospin degrees of freedom in the momentum space. This work provides the necessary formalism for the calculation of the 3N bound state observables which is under preparation.

Appendix 1: Connection between the antisymmetrized NN t-matrix and its helicity representation

In our formulation, we need the matrix elements of the antisymmetrized two-body t-matrix. We connect these matrix elements to the corresponding ones in the momentum-helicity representation. The antisymmetrized momentum-helicity basis state which is parity eigenstate is given by [4]:

$$|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda;t_{23}\rangle^{\pi a}=\frac{1}{\sqrt{2}}(1-P_{23})|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda\rangle_{\pi}|t_{23}\rangle \\ =\frac{1}{\sqrt{2}}(1-\eta_{\pi}(-)^{S_{23}+t_{23}})|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda\rangle_{\pi}|t_{23}\rangle, \quad (33)$$

Here, S_{23} is the total spin, λ is the spin projection along relative momentum of two nucleons, t_{23} is the total isospin and $|t_{23}\rangle \equiv |t_{23}\tau\rangle$ is the total isospin state of the two nucleons. τ is the isospin projection along its quantization axis which reveals the total electric charge of system. For simplicity, τ is suppressed since electric charge is conserved. In Eq. (33), P_{23} is the permutation operator which exchanges the two nucleons labels in all spaces i.e., momentum, spin and isospin spaces, and $|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda\rangle_{\pi}$ is parity eigenstate which is given by:

$$|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda\rangle_{\pi}=\frac{1}{\sqrt{2}}(1+\eta_{\pi}P_{\pi})|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda\rangle, \quad (34)$$

where P_{π} is the parity operator, $\eta_{\pi}=\pm 1$ are the parity eigenvalues and $|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda\rangle$ is momentum-helicity state. The antisymmetrized two-body t-matrix is given by [6]:

$$t_{a m_{s_2} m_{s_3} m_{t_2} m_{t_3}}^{m'_{s_2} m'_{s_3} m'_{t_2} m'_{t_3}}(\mathbf{p},\mathbf{p}';\epsilon)=\frac{1}{4}\delta_{(m_{t_2}+m_{t_3}),(m'_{t_2}+m'_{t_3})}e^{-i(\lambda_0\varphi_p-\lambda'_0\varphi_{p'})} \\ \times\sum_{S_{23}t_{23}\pi}\left(1-\eta_{\pi}(-)^{S_{23}+t_{23}}\right) \\ \times C\left(\frac{1}{2}\frac{1}{2}t_{23};m_{t_2}m_{t_3}\right)C\left(\frac{1}{2}\frac{1}{2}t_{23};m'_{t_2}m'_{t_3}\right) \\ \times C\left(\frac{1}{2}\frac{1}{2}S_{23};m_{s_2}m_{s_3}\lambda_0\right)C\left(\frac{1}{2}\frac{1}{2}S_{23};m'_{s_2}m'_{s_3}\lambda'_0\right) \\ \times\sum_{\lambda\lambda'}d^{S_{23}}_{\lambda_0\lambda}(x_p)d^{S_{23}}_{\lambda'_0\lambda'}(x_{p'})t^{\pi S_{23}t_{23}}_{\lambda\lambda'}(\mathbf{p},\mathbf{p}';\epsilon), \quad (35)$$

where based on momentum-helicity basis states the two-body t-matrix is defined as:

$$t^{\pi S_{23}t_{23}}_{\lambda\lambda'}(\mathbf{p},\mathbf{p}';\epsilon)\equiv{}^{\pi a}\langle\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda;t_{23}|t(\epsilon)|\mathbf{p}';\hat{\mathbf{p}}'S_{23}\lambda';t_{23}\rangle^{\pi a}, \quad (36)$$

These two-body t-matrix elements are connected to the solutions of Lippmann–Schwinger equation as follows:

$$t^{\pi S_{23}t_{23}}_{\lambda\lambda'}(\mathbf{p},\mathbf{p}';\epsilon)=\frac{\sum_{N=-S_{23}}^{S_{23}}e^{iN\varphi_{pp'}}d^{S_{23}}_{N\lambda}(x_p)d^{S_{23}}_{N\lambda'}(x_{p'})}{d^{S_{23}}_{\lambda'\lambda}(y_{pp'})} \\ \times t^{\pi S_{23}t_{23}}_{\lambda\lambda'}(p,p',y_{pp'};\epsilon), \quad (37)$$

where:

$$y_{pp'}=x_p x_{p'}+\sqrt{1-x_p^2}\sqrt{1-x_{p'}^2}\cos\varphi_{pp'}. \quad (38)$$

It should be mentioned that the fully off-shell NN t-matrix $t^{\pi S_{23}t_{23}}_{\lambda\lambda'}(p,p',y_{pp'};\epsilon)$, obeys a set of coupled Lippmann–Schwinger equations which are solved numerically in Ref. [4]. Finally, Eqs. (35) and (17) can be written as:

$$t_{a m_{s_2} m_{s_3} m_{t_2} m_{t_3}}^{m'_{s_2} m'_{s_3} m'_{t_2} m'_{t_3}}(\mathbf{p},\mathbf{p}';\epsilon)=e^{-i[(m_{s_2}+m_{s_3})\varphi_p-(m'_{s_2}+m'_{s_3})\varphi_{p'}]}t^{\pm\, m'_{s_2} m'_{s_3} m'_{t_2} m'_{t_3}}_{a\, m_{s_2} m_{s_3} m_{t_2} m_{t_3}} \\ (p,x_p,\cos\varphi_{pp'},x_{p'},p',y_{pp'};\epsilon), \quad (39)$$

where the labels $\pm$ are related to the sign of $\sin\varphi_{pp'}$ which is determined as:

$$\sin\varphi_{pp'}=\pm\sqrt{1-\cos^2\varphi_{pp'}}. \quad (40)$$

we consider positive sign for $\varphi_{pp'}\in[0,\pi]$ and negative sign for $\varphi_{pp'}\in[\pi,2\pi]$.

Appendix 2: Azimuthal dependency of the 3N wave function

We introduce the 3N momentum-helicity basis state as:

$$|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda,\mathbf{q};\hat{\mathbf{q}}S_1\Lambda\rangle=|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda\rangle|\mathbf{q};\hat{\mathbf{q}}S_1\Lambda\rangle, \quad (41)$$

where:

$$\mathbf{S}_{23}\cdot\hat{\mathbf{p}}|\hat{\mathbf{p}}S_{23}\lambda\rangle=\lambda|\hat{\mathbf{p}}S_{23}\lambda\rangle, \quad (42)$$

$$\mathbf{S}_1\cdot\hat{\mathbf{q}}|\hat{\mathbf{q}}S_1\Lambda\rangle=\Lambda|\hat{\mathbf{q}}S_1\Lambda\rangle. \quad (43)$$

Thus, Faddeev component of the 3N wave function can be written as:

$$\psi^{M_t}_{\gamma}(\mathbf{p},\mathbf{q})=\sum_{S_{23}\lambda S_1\Lambda}\langle\mathbf{pq}\gamma|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda,\mathbf{q};\hat{\mathbf{q}}S_1\Lambda\rangle \\ \langle\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda,\mathbf{q};\hat{\mathbf{q}}S_1\Lambda|\psi^{M_t}\rangle, \quad (44)$$

with considering:

$$|\hat{\mathbf{p}}S_{23}\lambda\rangle=R_S(\hat{\mathbf{p}})|\hat{\mathbf{z}}S_{23}\lambda\rangle=e^{-iS^z_{23}\varphi_p}e^{-iS^y_{23}\theta_p}|\hat{\mathbf{z}}S_{23}\lambda\rangle, \quad (45)$$

$$|\hat{\mathbf{q}}S_1\Lambda\rangle=R_S(\hat{\mathbf{q}})|\hat{\mathbf{z}}S_1\Lambda\rangle=e^{-iS^z_1\varphi_q}e^{-iS^y_1\theta_q}|\hat{\mathbf{z}}S_1\Lambda\rangle, \quad (46)$$

We have written:

$$\langle\mathbf{pq}\gamma|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda,\mathbf{q};\hat{\mathbf{q}}S_1\Lambda\rangle \\ =\langle\mathbf{pq}\gamma|R_S(\hat{\mathbf{p}})R_S(\hat{\mathbf{q}})|\mathbf{p};\hat{\mathbf{z}}S_{23}\lambda,\mathbf{q};\hat{\mathbf{z}}S_1\Lambda\rangle \\ =\langle\mathbf{pq}\gamma|e^{-iS^z_{23}\varphi_p}e^{-iS^y_{23}\theta_p}e^{-iS^z_1\varphi_q}e^{-iS^y_1\theta_q}|\mathbf{p};\hat{\mathbf{z}}S_{23}\lambda,\mathbf{q};\hat{\mathbf{z}}S_1\Lambda\rangle \\ =e^{-im_{s_1}\varphi_q}e^{-i(m_{s_2}+m_{s_3})\varphi_p}\langle\mathbf{pq}\gamma|e^{-iS^y_{23}\theta_p}e^{-iS^y_1\theta_q}|\mathbf{p};\hat{\mathbf{z}}S_{23}\lambda,\mathbf{q};\hat{\mathbf{z}}S_1\Lambda\rangle. \quad (47)$$

Also, with considering:

$$|\mathbf{p};\hat{\mathbf{p}}S_{23}\lambda\rangle=R_{J_p}(\hat{\mathbf{p}})|p\hat{\mathbf{z}};\hat{\mathbf{z}}S_{23}\lambda\rangle \\ =e^{-i(L^z_p+S^z_{23})\varphi_p}e^{-i(L^y_p+S^y_{23})\theta_p}|p\hat{\mathbf{z}};\hat{\mathbf{z}}S_{23}\lambda\rangle, \quad (48)$$

$$|\mathbf{q};\hat{\mathbf{q}}S_1\Lambda\rangle=R_{J_q}(\hat{\mathbf{q}})|q\hat{\mathbf{z}};\hat{\mathbf{z}}S_1\Lambda\rangle \\ =e^{-i(L^z_q+S^z_1)\varphi_q}e^{-i(L^y_q+S^y_1)\theta_q}|q\hat{\mathbf{z}};\hat{\mathbf{z}}S_1\Lambda\rangle, \quad (49)$$

We have written:

$$\begin{aligned}
&\langle \mathbf{p}; \hat{\mathbf{p}}S_{23}\lambda, \mathbf{q}; \hat{\mathbf{q}}S_1\Lambda|\psi^{M_t}\rangle \\
&= \langle p\hat{\mathbf{z}}; \hat{\mathbf{z}}S_{23}\lambda, q\hat{\mathbf{z}}; \hat{\mathbf{z}}S_1\Lambda|R_{J_p}^{-1}(\hat{\mathbf{p}})R_{J_q}^{-1}(\hat{\mathbf{q}})|\psi^{M_t}\rangle \\
&= \langle p\hat{\mathbf{z}}; \hat{\mathbf{z}}S_{23}\lambda, q\hat{\mathbf{z}}; \hat{\mathbf{z}}S_1\Lambda|e^{i(L_p^y+S_{23}^y)\theta_p}e^{i(L_p^z+S_{23}^z)\varphi_p} \\
&\quad e^{i(L_q^y+S_1^y)\theta_q}e^{i(L_q^z+S_1^z)\varphi_q}|\psi^{M_t}\rangle \\
&= \langle p\hat{\mathbf{z}}; \hat{\mathbf{z}}S_{23}\lambda, q\hat{\mathbf{z}}; \hat{\mathbf{z}}S_1\Lambda|e^{i(L_p^y+S_{23}^y)\theta_p}e^{i(L_p^z+S_{23}^z)\varphi_{pq}} \\
&\quad e^{i(L_p^z+S_{23}^z)\varphi_q}e^{i(L_q^y+S_1^y)\theta_q}e^{i(L_q^z+S_1^z)\varphi_q}|\psi^{M_t}\rangle \\
&= e^{iM_t\varphi_q}\langle p\hat{\mathbf{z}}; \hat{\mathbf{z}}S_{23}\lambda, q\hat{\mathbf{z}}; \hat{\mathbf{z}}S_1\Lambda|e^{i(L_p^y+S_{23}^y)\theta_p}e^{i(L_p^z+S_{23}^z)\varphi_{pq}} \\
&\quad e^{i(L_q^y+S_1^y)\theta_q}|\psi^{M_t}\rangle. \qquad (50)
\end{aligned}$$

Consequently, Eq. (44) can be rewritten as:

$$\begin{aligned}
\psi_\gamma^{M_t}(\mathbf{p},\mathbf{q}) &= e^{-i[(m_{s_2}+m_{s_3})\varphi_p+(m_{s_1}-M_t)\varphi_q]} \\
&\times \sum_{S_{23}\lambda S_1\Lambda} \langle \mathbf{pq}\gamma|e^{-iS_{23}^y\theta_p}e^{-iS_1^y\theta_q}|\mathbf{p}; \hat{\mathbf{z}}S_{23}\lambda, \mathbf{q}; \hat{\mathbf{z}}S_1\Lambda\rangle \\
&\times \langle p\hat{\mathbf{z}}; \hat{\mathbf{z}}S_{23}\lambda, q\hat{\mathbf{z}}; \hat{\mathbf{z}}S_1\Lambda|e^{i(L_p^y+S_{23}^y)\theta_p}e^{i(L_p^z+S_{23}^z)\varphi_{pq}} \\
&\quad e^{i(L_q^y+S_1^y)\theta_q}|\psi^{M_t}\rangle. \qquad (51)
\end{aligned}$$

Finally, this equation can be written as:

$$\psi_\gamma^{M_t}(\mathbf{p},\mathbf{q}) \equiv e^{-i[(m_{s_2}+m_{s_3})\varphi_p+(m_{s_1}-M_t)\varphi_q]\pm}\psi_\gamma^{M_t}(p,x_p,\cos\varphi_{pq},x_q,q) \qquad (52)$$

Appendix 3: Parity and time reversal invariance of the total 3N wave function

In this section, we discuss about properties of the total wave function under the parity and time reversal invariance. Parity invariance would mean:

$$\begin{aligned}
\langle \mathbf{pq}\gamma|\Psi^{M_t}\rangle &= \langle \mathbf{pq}\gamma|P_\pi^{-1}P_\pi|\Psi^{M_t}\rangle = \langle -\mathbf{p},-\mathbf{q}\gamma|P_\pi|\Psi^{M_t}\rangle \\
&= \langle -\mathbf{p},-\mathbf{q}\gamma|\Psi^{M_t}\rangle = \langle -\mathbf{p},-\mathbf{q}\gamma|\psi^{M_t}\rangle \\
&\quad + \langle -\mathbf{p}_2,-\mathbf{q}_2\gamma_2|\psi^{M_t}\rangle + \langle -\mathbf{p}_3,-\mathbf{q}_3\gamma_3|\psi^{M_t}\rangle, \qquad (53)
\end{aligned}$$

where we have used $P_\pi|\Psi^{M_t}\rangle = |\Psi^{M_t}\rangle$ for the 3N total wave function. Eq. (53) leads to:

$$\begin{aligned}
&\langle \mathbf{p},\mathbf{q}\gamma|\psi^{M_t}\rangle = \langle -\mathbf{p},-\mathbf{q}\gamma|\psi^{M_t}\rangle, \\
&\langle \mathbf{p}_2,\mathbf{q}_2\gamma_2|\psi^{M_t}\rangle = \langle -\mathbf{p}_2,-\mathbf{q}_2\gamma_2|\psi^{M_t}\rangle, \qquad (54)\\
&\langle \mathbf{p}_3,\mathbf{q}_3\gamma_3|\psi^{M_t}\rangle = \langle -\mathbf{p}_3,-\mathbf{q}_3\gamma_3|\psi^{M_t}\rangle.
\end{aligned}$$

So we have:

$$^{\pm}\psi_\gamma^{M_t}(p,x_p,\cos\varphi_{pq},x_q,q) = e^{-i(M_s-M_t)\pi\pm}\psi_\gamma^{M_t}(p,-x_p,\cos\varphi_{pq},-x_q,q), \qquad (55)$$

$$^{\pm}\Psi_\gamma^{M_t}(p,x_p,\cos\varphi_{pq},x_q,q) = e^{-i(M_s-M_t)\pi\pm}\Psi_\gamma^{M_t}(p,-x_p,\cos\varphi_{pq},-x_q,q), \qquad (56)$$

where $M_s = m_{s\_1} + m_{s\_2} + m_{s\_3}$. The time reversal invariance might be more interesting. The total wave function can be written as:

$$\begin{aligned}
\langle \mathbf{pq}\gamma|\Psi^{M_t}\rangle &= \langle \mathbf{pq}\gamma|T^{-1}T|\Psi^{M_t}\rangle = i^{2M_s}\langle -\mathbf{p},-\mathbf{q},-\gamma|T|\Psi^{M_t}\rangle \\
&= i^{2(M_s+M_t)}\langle -\mathbf{p},-\mathbf{q},-\gamma|\Psi^{-M_t}\rangle. \qquad (57)
\end{aligned}$$

Considering parity and time reversal invariance leads to:

$$\begin{aligned}
\langle \mathbf{pq}\gamma|\Psi^{M_t}\rangle i &=^{2(M_s+M_t)} \langle \mathbf{pq},-\gamma|\Psi^{-M_t}\rangle \\
&= i^{2(M_s+M_t)}\{\langle \mathbf{pq},-\gamma|\psi^{-M_t}\rangle + \langle \mathbf{p}_2\mathbf{q}_2,-\gamma_2|\psi^{-M_t}\rangle \\
&\quad + \langle \mathbf{p}_3\mathbf{q}_3,-\gamma_3|\psi^{-M_t}\rangle\}. \qquad (58)
\end{aligned}$$

So we have:

$$^{\pm}\psi_\gamma^{M_t}(p,x_p,\cos\varphi_{pq},x_q,q) = i^{2(M_s+M_t)\pi\pm}\psi_{-\gamma}^{-M_t}(p,x_p,\cos\varphi_{pq},x_q,q), \qquad (59)$$

$$^{\pm}\Psi_\gamma^{M_t}(p,x_p,\cos\varphi_{pq},x_q,q) = i^{2(M_s+M_t)\pi\pm}\Psi_{-\gamma}^{-M_t}(p,x_p,\cos\varphi_{pq},x_q,q). \qquad (60)$$

Appendix 4: The x'-integration

According to Eq. (21), the x'-integration carried out as:

$$J(p,x_p,\cos\varphi_{pq},x_q,q) = \int_{-1}^{1} dx' C(x')J(\pi',x_{\pi'},\cos\varphi_{\pi'q'},x',q'), \qquad (61)$$

where the C is known function determined by $t_a^{\pm}$ and exponential functions. This equation can be rewritten as:

$$\begin{aligned}
J(p,x_p,\cos\varphi_{pq},x_q,q) &= \int_{-1}^{0} dx' C(x')J(\pi',x_{\pi'},\cos\varphi_{\pi'q'},x',q') \\
&\quad + \int_{0}^{1} dx' C(x')J(\pi',x_{\pi'},\cos\varphi_{\pi'q'},x',q') \\
&= \int_{0}^{1} dx' C(-x')J(\pi'(-x'),x_{\pi'}(-x'), \\
&\quad \cos\varphi_{\pi'q'}(-x'),-x',q') \\
&\quad + \int_{0}^{1} dx' C(x')J(\pi',x_{\pi'},\cos\varphi_{\pi'q'},x',q'), \qquad (62)
\end{aligned}$$

Finally, by considering parity invariance which is described in "Appendix 3", Eq. (62) can be written:

$$J(p,x_p,\cos\varphi_{pq},x_q,q)=\int_0^1 \mathrm{d}x'\{(-)^{M_s+M_t}C(-x')J(\pi'(-x'),\\ -x_{\pi'}(-x'),\cos\varphi_{\pi'q'}(-x'),x',q')\\ +C(x')J(\pi',x_{\pi'},\cos\varphi_{\pi'q'},x',q')\}. \quad (63)$$

Appendix 5: The φ'-integration

According to Eq. (21), the φ'-integration for fixed p, q, x_p, x_q, $\cos\varphi_{pq}$ and q' can be written as:

$$I(\varphi_p,\varphi_q)=\int_0^{2\pi}\mathrm{d}\varphi' e^{im_1(\varphi_q-\varphi')}e^{+im_2\varphi_{\pi q'}}e^{-im_3\varphi_{\pi'q'}}\\ \times A^{\pm}[\cos(\varphi_q-\varphi'),\cos(\varphi_p-\varphi'),\cos\varphi_{pq}]\\ \times sB^{\pm}[\cos(\varphi_q-\varphi')], \quad (64)$$

where the $A^{\pm}$ and $B^{\pm}$ are known functions determined by $t_a^{\pm}$ and $^{\pm}\psi$, respectively. As we know, the exponential functions $e^{+im_2\varphi_{\pi q'}}$ and $e^{-im_3\varphi_{\pi'q'}}$ are functions of $\cos\varphi_{\pi q'}$ and $\cos\varphi_{\pi'q'}$ by considering their sine functions as:

$$\sin\varphi_{\pi q'}=\pm\sqrt{1-\cos\varphi^2_{\pi q'}},\\ \sin\varphi_{\pi'q'}=\pm\sqrt{1-\cos\varphi^2_{\pi'q'}}. \quad (65)$$

Also, the cosine functions $\cos\varphi_{\pi q'}$ and $\cos\varphi_{\pi'q'}$ are function of $\varphi_q-\varphi'$. Substituting $\varphi''=\varphi'-\varphi_q$ leads to:

$$I(\varphi_p,\varphi_q)=\int_0^{2\pi}\mathrm{d}\varphi'' e^{-im_1\varphi''}e^{+im_2\varphi_{\pi q'}}e^{-im_3\varphi_{\pi'q'}}A^{\pm}[\cos\varphi'',\\ \cos(\varphi_{pq}-\varphi''),\cos\varphi_{pq}]B^{\pm}[\cos\varphi'']\\ \equiv I^{\pm}(\varphi_{pq}), \quad (66)$$

where the labels of $I^{\pm}(\cos\varphi_{pq})$ depend on the sign of $\sin\varphi_{pq}$. It is clear that the angles $\varphi_{\pi q'}$ and $\varphi_{\pi'q'}$ belong to the interval $[-\pi, 0]$ when φ'' varies in the interval $[0, \pi]$ and they belong to the interval $[0, \pi]$ when φ'' varies in the interval $[\pi, 2\pi]$. Furthermore, since the labels of $B^{\pm}$ depend on the sign of $\sin\varphi_{\pi'q'}$, thus for $\varphi''\in[0,\pi]$ and $\varphi''\in[\pi,2\pi]$, we can choose negative and positive labels, respectively. Consequently, the integral $I^{\pm}(\cos\varphi_{pq})$ can be decomposed as:

$$I^{\pm}(\varphi_{pq})=\int_0^{\pi}\mathrm{d}\varphi'' e^{-im_1\varphi''}e^{-im_2|\varphi_{\pi q'}|}e^{+im_3|\varphi_{\pi'q'}|}A^{\pm}\\ \times[\cos\varphi'',\cos(\varphi_{pq}-\varphi''),\cos\varphi_{pq}]B^{-}[\cos\varphi'']\\ +\int_{\pi}^{2\pi}\mathrm{d}\varphi'' e^{-im_1\varphi''}e^{+im_2\varphi_{\pi q'}}e^{-im_3\varphi_{\pi'q'}}A^{\pm}\\ \times[\cos\varphi'',\cos(\varphi_{pq}-\varphi''),\cos\varphi_{pq}]B^{+}[\cos\varphi'']. \quad (67)$$

Now, we discuss about the labels of $A^{\pm}$. As we know, the labels of $A^{\pm}$ are related to the sign of $\sin\varphi_{p\pi}$. We can write $\varphi_{p\pi}=\varphi_{pq}-\varphi_{\pi q}$, and then we have:

$$\sin\varphi_{p\pi}=\sin\varphi_{pq}\cos\varphi_{\pi q}-\cos\varphi_{pq}\sin\varphi_{\pi q}, \quad (68)$$

where:

$$\cos\varphi_{\pi q}=\frac{\hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{q}}-(\hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{z}})(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})}{\sqrt{1-(\hat{\boldsymbol{\pi}}\cdot\hat{\mathbf{z}})^2}\sqrt{1-(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})^2}}=\frac{y_{\pi q}-x_\pi x_q}{\sqrt{1-x_\pi}\sqrt{1-x_q^2}},\\ y_{\pi q}=\frac{\frac{1}{2}q+q'y_{qq'}}{\pi},\\ \sin\varphi_{\pi q}=\pm\sqrt{1-\cos^2\varphi_{\pi q}}. \quad (69)$$

It is clear that the angle $\varphi_{\pi q}$ belongs to the interval $[0, \pi]$ when φ'' varies in the interval $[0, \pi]$ and belong to the interval $[\pi, 2\pi]$ when φ'' varies in the interval $[\pi, 2\pi]$. Thus, depending on various intervals of variables φ_{pq} and φ'', we can choose the positive or negative sign for $\sin\varphi_{pq}$ and $\sin\varphi_{\pi q}$, and then we can calculate $\sin\varphi_{p\pi}$ from Eq (68). Consequently, for $\sin\varphi_{p\pi}\in[0,1]$ and $\sin\varphi_{p\pi}\in[-1,0]$, we can consider positive and negative signs of $A^{\pm}$, respectively. Substituting $\varphi'''=2\pi-\varphi''$, in the second integral of Eq. (67) yields:

$$\int_0^{\pi}\mathrm{d}\varphi''' e^{+im_1\varphi'''}e^{+im_2\varphi_{\pi q'}}e^{-im_3\varphi_{\pi'q'}}A^{\pm}\\ \times[\cos\varphi''',\cos(\varphi_{pq}+\varphi'''),\cos\varphi_{pq}]B^{+}[\cos\varphi'''], \quad (70)$$

Therefore, Eq. (67) can be rewritten:

$$I^{\pm}(\varphi_{pq})=\int_0^{\pi}\mathrm{d}\varphi'' e^{-im_1\varphi''}e^{-im_2|\varphi_{\pi q'}|}e^{+im_3|\varphi_{\pi'q'}|}A^{\pm}\\ \times[\cos\varphi'',\cos(\varphi_{pq}-\varphi''),\cos\varphi_{pq}]B^{-}[\cos\varphi'']\\ +\int_0^{\pi}\mathrm{d}\varphi'' e^{+im_1\varphi''}e^{+im_2\varphi_{\pi q'}}e^{-im_3\varphi_{\pi'q'}}A^{\pm}\\ \times[\cos\varphi'',\cos(\varphi_{pq}+\varphi''),\cos\varphi_{pq}]B^{+}[\cos\varphi'']. \quad (71)$$

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