Solving dynamic equations by a semi-analytical method

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Abstract– Dynamic equations can be considered as a mathematical representation of the time-varying behavior of a system, using ordinary differential equations for state dynamics and algebraic equations for output/observation. In this paper, we consider one of the most famous dynamic equations known as the Lienard equation. In mathematics, more specifically in the study of dynamical systems and differential equations, the Lienard equation is a type of second-order ordinary differential equation. One of the applications of the Lenard equation is its use in modeling oscillatory circuits and the development of radios and vacuum tubes. Therefore, due to the wide application of the Lienard equation, it is very important to provide methods that can solve it appropriately. Here, a procedure is presented based on the homotopy perturbation method (HPM) to obtain an approximate solution for the Lienard equation. The homotopy perturbation method is a semi-analytical technique to solve nonlinear ordinary or partial differential equations. This method employs the concept of the homotopy from topology to generate a convergent series solution for nonlinear systems. The HPM is defined as a mathematical technique that assumes the solution of a nonlinear differential equation can be expressed as a power series in a homotopy parameter. In the proposed method, the solution is considered as the sum of an infinite series which converges rapidly to the accurate solutions. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter. Two numerical examples are given to apply the homotopy perturbation method for solving the Lienard equation. Also, the obtained approximate solutions have been compared with their known theoretical solutions and results obtained via the variational iteration method (VIM). The results show the efficiency and capability of HPM in solving dynamic equations.

Keywords: Dynamic equations, Lienard equation, Homotopy perturbation method, Variational iteration method.

1. Introduction

Dynamical equations refer to equations that describe the time evolution of a system in a classical field approach. These equations can be derived entirely classically by expanding in eigenmodes of the waveguide and introducing coupling terms that describe the opto-acoustic interaction.

In this paper, we consider one of the important dynamic equations called the Lienard equation. The general form of this equation is [1]

$$x'' + f(x)x' + g(x) = h(t), \qquad (1)$$

which is not only regarded as a generalization of the damped pendulum equation or a damped spring-mass system (where $f(x)x'$ is the damping force, $g(x)$ is the restoring force and $h(t)$ is the external force), but also used as nonlinearmodels in many physically significant fields when taking different choices for $f(x), g(x)$ and $h(t)$.

For example, the choices $f(x) = \epsilon(x^2 - 1), g(x) = x$ and $h(t) = 0$ lead equation of (1) to the Van der Pol equation served as a nonlinear model of electronic oscillation [2, 3]. Therefore, studying equation of (1) is of physical significance. In the general case, it is commonly believed that it is very difficult to find its exact solution by usual ways [4].

The following special case of equation (1) was studied in [1, 5, 6]:

$$x''(t) + a\,x(t) + b\,x^3(t) + c\,x^5(t) = 0, \qquad (2)$$

where a, b and c are real coefficients. Finding explicit exactand numerical solutions of nonlinear equations efficiently is of major importance and has widespread applications in numerical methods and applied mathematics.

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In this study, we will implement the homotopy perturbation method (in short HPM) to find exact solution and approximate solutions to the Lienard equation for a given nonlinearity (2). The homotopy perturbation method introduced by He [7-10] has been used by many mathematicians and engineers to solve various functional equations. In this method the solution is considered as the sum of an infinite series which converges rapidly to the accurate solutions. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a small parameter. Recently, many research articles have been published regarding the use of the homotopy perturbation method, and we will introduce some of them below.

In 2021, Chun-HuiHe et al. [11] presented a heuristic review on the homotopy perturbation method for non-conservative oscillators. Also, Ji-Huan Heet al. [12], applied homotopy perturbation method for the fractal toda oscillator. In 2022, Ji-Huan He et al. [13] used homotopy perturbation method for fractal duffing oscillator with arbitrary conditions. In 2024, Ali et al. [14] used homotopy perturbation method to analyze heat and mass exchanger for Ree-Eyring hybrid nanofluid through a stretching. Adebisi et al. [15] proposed a mathematical model of COVID-19 by using homotopy perturbation method, focusing on the impact of saturated incidence rates and treatment responses on its dynamical transmission. Also, Roy et al. [16], applied the optimal and modified homotopy perturbation method (OM-HPM) to provide convergent semi-analytical solutions for three different models: duffing oscillator (DO), Rayleigh oscillator (RO) and van der Pol oscillator (vdPO). In 2025, Anakira et al. [17] presented a modified version of homotopy perturbation method to obtain Exact Solutions of nonlinear delay Volterra integro-differential equations. Sharif et al. [18] introduced a novel modification to homotopy perturbation method to analyze the dynamic behavior of pendulums on rotatingframes. Also, Devi et al. [19] used the homotopy perturbation method to investigate population dynamics in oxygen-depleted environments. Recently, Khalili Golmankhaneh et al. [20] presented a novel application of the homotopy perturbation method to a system of coupled fractal Schrödinger–Korteweg–deVries (S-KdV) equations, formulated within the framework of fractal calculus. In the next section, the basic idea of homotopy perturbation method is presented.

2. Homotopy perturbation method

Having successfully overcomes many of the limitations of traditional perturbation methods, the homotopy perturbation method has received the attention of much research in recent decades. The simple implementation of the homotopy perturbation method even when applied to strong nonlinear differential equation is another desired feature that made it one of the most used techniques in recent years. Here, we present a brief introduction to the homotopy perturbation method. To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(t) = 0, \qquad t \in \Omega, \tag{3}$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, \qquad t \in \Gamma, \tag{4}$$

where A is a general differential operator, B is a boundary operator, $f(t)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Eq. (3) can therefore be rewritten as follows

$$L(u) + N(u) - f(t) = 0, \qquad t \in \Omega. \tag{5}$$

By the homotopy technique, we construct a homotopy

$$U(t, p): \Omega \times [0, 1] \to \mathbb{R},$$

which satisfies:

$$H(U, p) = (1 - p)[L(U) - L(u_0)] + p[A(u) - f(t)] = 0, \tag{6}$$

or

$$H(U, p) = L(U) - L(u_0) + pL(u_0) + p[N(u) - f(t)] = 0, \tag{7}$$

where $p \in [0,1]$ is anembedding parameter, u_0 is an initial approximation of Eq.(3), which satisfies the boundary conditions.

Obviously, fromEqs. (6) or (7) we will have

$$H(U, 0) = L(U) - L(u_0) = 0, \tag{8}$$

$$H(U, 1) = A(U) - f(r) = 0. \tag{9}$$

The changing process of p form zero to unity is just that of $U(t, p)$ from $u_0(t)$ to $u(t)$. In topology, this is calledhomotopy. According to the HPM, we can first use theembedding parameter p as a small parameter, and assumethat the solution of Eqs. (7) or (8) can be written as a power series in p:

$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \cdots, \tag{10}$$

and the exact solution is obtained as follows:

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 \ldots = \textstyle\sum_{j=0}^{\infty} U_j. \tag{11}$$

The series (11) is convergent for most cases, and the rate of convergence depends on $L(U)$[21]. For later numerical computation, we let the expression

$$\Phi_n(t) = \textstyle\sum_{j=0}^{\infty} U_j, \tag{12}$$

to denote the n-term approximation to u.

3. HPM for Lienard equation

In this section, we consider Lienard equation (2) with initial conditions

$$x(0) = K_1, \;\; x'(0) = K_2. \tag{13}$$

By using of initial conditions (13), we choose

$$x_0(t) = K_1 t + K_2, \tag{14}$$

that is an initial approximation of the exact solution of Eq.

(2). According to the homotopy perturbation method, we construct the homotopy

$$(1-p)[X''(t)-x_0''(t)] \\ +p[X''(t)+a\,X(t)+b\,X^3(t)+\ c\,X^5(t)]=0, \tag{15}$$

to solve Eq. (2). Therefore

$$(1-p)X''(t)+p[X''(t)+a\,X(t)+b\,X^3(t)+\ c\,X^5(t)]=0, \tag{16}$$

or

$$X''(t)=-p[a\,X(t)+b\,X^3(t)+\ c\,X^5(t)]. \tag{17}$$

By considering the initial conditions (13),we have

$$X(t)=(K_1t+K_2) \\ -p\int_0^t\int_0^s\left(a\,X(r)+b\,X^3(r)+\ c\,X^5(r)\right)dr\,ds. \tag{18}$$

Now suppose that the solution of Eq. (18) has the form

$$X(t)=X_0(t)+pX_1(t)+p^2X_2(t)+p^3X_3(t)+\cdots, \tag{19}$$

where $X_i(t)$ are functions yet to be determined. Substituting (19) into (18), and equating the terms with identical powers of p, we have

$$p^0: X_0(t)=K_1t+K_2,$$

$$p^1: X_1(t)=-p\int_0^t\int_0^s\left(a\,X_0(r)+b\,X_0^3(r)+c\,X_0^5(r)\right)dr\,ds,$$

$$p^2: X_2(t) \\ =-p\int_0^t\int_0^s\left(a\,X_1(r)+b\sum_{k=0}^{1}\sum_{j=0}^{k}X_j(r)X_{k-j}(r)X_{1-k}(r)\right. \\ \left.+c\sum_{w=0}^{1}\sum_{q=0}^{w}\sum_{k=0}^{q}\sum_{j=0}^{k}X_j(r)X_{k-j}(r)X_{q-k}(r)X_{w-q}(r)X_{1-w}(r)\right)dr\,ds,$$

⋮

$$p^i: X_i(t) \\ =-p\int_0^t\int_0^s\left(a\,X_{i-1}(r)+b\sum_{k=0}^{i-1}\sum_{j=0}^{k}X_j(r)X_{k-j}(r)X_{i-k-1}(r)\right. \\ \left.+c\sum_{w=0}^{i-1}\sum_{q=0}^{w}\sum_{k=0}^{q}\sum_{j=0}^{k}X_j(r)X_{k-j}(r)X_{q-k}(r)X_{w-q}(r)X_{i-w-1}(r)\right)dr\,ds. \tag{20}$$

Therefore, the exact solution of (2) can be obtained by setting $p=1$, i.e.

$$x(t)=\lim_{p\to 1}X(t)=\sum_{i=0}^{\infty}X_i(t).$$

For later numerical computation, we let the expression

$$\Phi_j(t)=\sum_{i=0}^{j}X_i(t),$$

to denote the j-term approximation to $x(t)$.

4. Implementation of the method

In this section, we give two numerical examples to apply the HPM for solving the Lienard equation (2).

Example 1: In this example, we follow [1] and consider the following initial conditions

$$x(0)=K_1=\sqrt{-\frac{2a}{b}}, \tag{21}$$

$$x'(0)=K_2=-\frac{a\sqrt{-a}}{\sqrt{\frac{-2a}{b}}b}=-\frac{a}{\sqrt{2b}}, \tag{22}$$

where a and b are arbitrary constants.
By using the formulas obtained via HPM (20) and MATLAB software, we can obtain

$$X_0(t)=K_1+K_2t=\sqrt{-\frac{2a}{b}}-\frac{a}{\sqrt{2b}}t,$$

$$X_1(t)=\frac{\sqrt{2}a}{1680\,b^3\sqrt{b}}\Big[\left(840\,b^3\sqrt{-a}-3360\,acb\sqrt{-a}\right)t^2 \\ +(2800\,a^2bc-700\,ab^3)t^3 \\ +\left(1400\,a^2cb\sqrt{-a}-210\,ab^3\sqrt{-a}\right)t^4 \\ +(21\,a^2b^3-420\,a^3bc)t^5 \\ -70\,a^3cb\sqrt{-a}t^6+5\,a^4bc\,t^7\Big],$$

$$\begin{aligned}X_2(t) = -\frac{\sqrt{2}a^2}{11531520\, b^5\sqrt{b}}\Big[&(19219200\, ab^2c \\ &- 38438400\, a^2c^2 \\ &- 2402400\, b^4)b\sqrt{-a}t^4 \\ &+ (2930928\, ab^5 + 63545280\, a^3bc^2 \\ &- 28060032\, a^2bc)t^4 \\ &+ (1489488\, ab^4 - 19781712\, a^2b^2c \\ &+ 55095040\, a^3c^2)b\sqrt{-a}t^6 \\ &+ (8648640\, a^3b^3c - 29744000\, a^4bc^2 \\ &- 394680\, a^2b^5)t^7 \\ &+ (2471040\, a^3b^2c - 11325600\, a^4c^2 \\ &- 54054\, a^2b^4)b\sqrt{-a}t^8 \\ &+ (3003a^3b^5 + 3146000\, a^5bc^2 \\ &- 44616\, a^4b^3c)t^9 \\ &+ (629200\, a^5c^2 \\ &- 46332\, a^4b^2c)b\sqrt{-a}t^{10} \\ &+ (\,2106\, a^5b^3c - 85800\, a^6bc^2)t^{11} \\ &- 7150\, a^2c^2\, b\sqrt{-a}t^{12} \\ &+ 275\, a^7bc^2t^{13}\Big],\end{aligned}$$

and so on. We approximate exact solution of $x(t)$ with

$$\Phi_2(t) = \sum_{j=0}^{2} X_j(t). \tag{23}$$

The exact solution of $x(t)$ is given by

$$x(t) = \sqrt{\frac{-2a\left(1+\tanh\left(\sqrt{-a}t\right)\right)}{b}}. \tag{24}$$

in [5]. We shall illustrate the accuracy and efficiently of homotopy perturbation method applied to Eq. (2) compared to the variational iteration method applied to same equation [1]. For this purpose, we consider the same initial conditions for Lienard equation (2) as considered specifically in [1], we take $a = -1, b = 4, c = 3$ and $t = 0(0.1)1$. We present in Table 1, absolute errors between the 2-term approximate of HPM and the exact solution. Also, absolute errors between the 2-iteration VIM and the exact solution are presented in Table 1, which is given in [1].

Table 1. The numerical results for $a = -1, b = 4$ and $c = 3$ in Example 1.

	HPM	VIM
t	$\lvert x(t) - \Phi_2(t)\rvert$	$\lvert x(t) - x_2(t)\rvert$
0.1	$2.8154e-12$	$3.3588e-10$
0.2	$9.4426e-10$	$1.7493e-08$
0.3	$2.9896e-08$	$1.4734e-07$
0.4	$3.4870e-07$	$5.3415e-07$
0.5	$2.2890e-06$	$1.0564e-06$
0.6	$1.0400e-05$	$1.0878e-06$
0.7	$3.2905e-05$	$7.6947e-07$
0.8	$7.9427e-05$	$6.6036e-06$
0.9	$1.2890e-04$	$4.5760e-05$
1	$4.0138e-05$	$1.9575e-04$

Example 2: In this example, we follow [1] and consider the following initial conditions

$$x(0) = \sqrt{\frac{M}{2+N}},$$

where

$$M = 4\sqrt{\frac{3a^2}{3b^2-16ac}},$$

$$N = -1 + \frac{\sqrt{3}b}{\sqrt{3b^2-16ac}},$$

and also

$$x'(0) = 0,$$

Where a, b and c are arbitrary constants.
By using the formulas obtained via HPM (20) and MATLAB software, we can obtain

$$X_0(t) = x(0) + x'(0)t = \sqrt{\frac{M}{2+N}},$$

$$\begin{aligned}X_1(t) &= \frac{-6a}{\left(\sqrt{9b^2-48ac}+3b\right)^3}\Big(-3b^2+16ac \\ &\quad - b\sqrt{9b^2-48ac}\Big)\sqrt{-3a\left(\sqrt{9b^2-48ac}+3b\right)}\,t^2,\end{aligned}$$

$$\begin{aligned}X_2(t) &= \frac{6a^2\sqrt{-a\left(\sqrt{9b^2-48ac}+3b\right)}}{\left(\sqrt{9b^2-48ac}+3b\right)^5\left(\sqrt{3b^2-16ac}+\sqrt{3}b\right)} \\ &\quad \times \left((56ac-15b^2)\sqrt{9b^2-48ac} - 45b^3 + 288abc\right) \\ &\quad \times \left(-3b^2+16ac - b\sqrt{9b^2-48ac}\right)t^3,\end{aligned}$$

and so on. We approximate exact solution of $x(t)$ with

$$\Phi_2(t) = \sum_{j=0}^{2} X_j(t). \tag{25}$$

The exact solution of $x(t)$ is given by

$$x(t) = \sqrt{\frac{M\cdot \mathrm{sech}^2\left(\sqrt{-a}t\right)}{2+N\cdot \mathrm{sech}^2\left(\sqrt{-a}t\right)}}. \tag{26}$$

in [5]. This result can be verified of the Taylor series of $x(t)$. We shall illustrate the accuracy and efficiently of homotopy perturbation method applied to Eq. (2) compared to the variational iteration method applied to same equation [1]. For this purpose, we consider the same initial conditions for Lienard equation (2) as considered specifically in [1], we take $a = -1, b = 4, c = 3$ and $t = 0(0.1)1$. We present in Table 2, absolute errors between the

2-term approximate of HPM and the exact solution. Also, absolute errors between the 2-iteration VIM and the exact solution are presented in Table 2, which is given in [1].

Table 2. The numerical results for $a = -1, b = 4$ and $c = 3$ in Example 2.

	HPM	VIM
t	$\lvert x(t) - \Phi_2(t) \rvert$	$\lvert x(t) - x_2(t) \rvert$
0.1	$7.2721e-10$	$4.4279e-08$
0.2	$1.8270e-07$	$2.7277e-06$
0.3	$4.5417e-06$	$2.9172e-05$
0.4	$4.3537e-05$	$1.5024e-04$
0.5	$2.4674e-04$	$5.1335e-04$
0.6	$1.0009e-03$	$1.3428e-03$
0.7	$3.2206e-03$	$2.9032e-03$
0.8	$8.7437e-03$	$5.4296e-03$
0.9	$2.0852e-02$	$9.0425e-03$
1	$4.4907e-02$	$1.3667e-02$

5. Conclusion

In this paper, homotopy perturbation method was employed successfully for solving the Lienard equation. The exact solutions are compared with the approximate solutions obtained by HPM. Also, comparison of the results obtained by HPM with results obtained by VIM is discussed in detail. It is evident that the overall errors can be made smaller by calculating more terms of the sequence of approximate solutions. The results show that the homotopy perturbation method is a powerful mathematical tool for finding the exact and approximate solutions of nonlinear equations.

References

[1] M. Matinfar, H. Hosseinzadeh and M. Ghanbari. A numerical implementation of the variational iteration method for the Lienard equation. World Journal of Modelling and Simulation, 2008, 4(3): 205-210.

[2] J. Guckenheimer, Dynamics of the van der pol equation. IEEE Trans. Circ. Syst., 1980, 27: 938-989.

[3] Z.F. Zhang, T. Ding and H.W. Huang, Qualitative Theory of Differential Equations. Science Press, Peking, 1985.

[4] J.K. Hale, Ordinary Differential Equations. Wiley, New York, 1980.

[5] Z. Feng, On explicit exact solutions for the Lienard equation and its applications. Phys. Lett. A, 2002, 239:50-56.

[6] D. Kong, Explicit exact solutions for the Lienard equation and its applications. Phys. Lett. A, 1995, 196: 301-306.

[7] J.H. He. Application of homotopy perturbation method to nonlinear wave equations. Chaos, Solitons and Fractals. 2005, 26: 695-700.

[8] J.H. He. Homotopy perturbation method for solving boundary value problems. Physics Letters A, 2006, 350: 87-88.

[9] J.H. He. Limit cycle and bifurcation of nonlinear problems. Chaos, Solitons and Fractals. 2005, 26(3): 827-833.

[10] J.H. He. The homotopy perturbation method for nonlinear oscillators with discontinuities. Applied Mathematics and Computation. 2004, 151: 287-292.

[11] C.H. He and Y.O. El-Dib, A heuristic review on the homotopy perturbation method for non-conservative oscillators, Journal of low frequency noise, vibration and active control, 2022, 41(2):572-603.

[12] J.H. He, Y.O. El-Dib and A.A. Mady, Homotopy perturbation method for the fractal toda oscillator, fractal and fractional, 2021, 5(93): 1-8.

[13] J.H. He, M.L. Jiao and C.H. He, Homotopy Perturbation method for fractal duffing oscillator with arbitrary conditions, Fractals, 2022, 30 (9):2250165.

[14] F. Ali, A. Zaib, M. Faizan, S.S. Zafar, S. Alkarni, N.A. Shah, J.D. Chung, Heat and mass exchanger analysis for Ree-Eyring hybrid nanofluid through a stretching sheet utilizing the homotopy perturbation method, Case Studies in Thermal Engineering, 2024, 54: 104014.

[15] A. FolashadeAdebisi, M.O. Olayiwola, I.A. Adediran and A.I. Alaje, A Novel Mathematical Model and Homotopy Perturbation Method Analyzing the Effects of Saturated Incidence and Treatment Rate on COVID-19 Eradication, 2024, 48: 625-636.

[16] T. Roy, B. Rath, J. Asad, D. K Maiti, P. Mallick, and R. Jarrar, Nonlinear oscillators dynamics using optimal and modified homotopy perturbation method, 2024, 43(4): 1469-1480.

[17] N. Anakira, A. Amourah, A. Almalki, A. Alsoboh and T. Sasa, Exact solutions of nonlinear delay Volteraintegro-differential equations using modified homotopy perturbation method, European journal of pure and applied mathematics , 2025, 18 (3): 6462, 1-18.

[18] N. Sharif, M.S. Alam and H.U. Molla, Dynamics of nonlinear pendulum equations: Modified homotopy perturbation method, Journal of Low Frequency Noise, Vibration and Active Control, 2025, 44(3): 1460–1473.

[19] V.S. Devi, S. Thambiayya, G. Senthil, K. Saranya, Homotopy Perturbation Method for Investigating Population Dynamics in Oxygen-Depleted Environments, Discrete Dynamics in Nature and Society, Collections: Discrete Optimization and Decision Making, 2025,https://doi.org/10.1155/ddns/8789320.

[20] A. Khalili Golmankhaneh, D. Pham, I. Stamova, A. Ramazanova and R. Rodríguez-López, Homotopy perturbation method for a system of fractal Schrödinger–Korteweg–de Vries equations, Eur. Phys. J. Spec. Top., 2025,https://doi.org/10.1140/epjs/s11734-025-01807-5.

[21] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems. International Journal of Non-Linear Mechanics. 2000, 35 (1): 37-43.