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# Structural Aspects of Weak Connectedness and Weak Compactness in Bipolar Soft Weak Structures

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**Abstract.** Connectedness and compactness are two essential properties that characterize the structural behavior of topological spaces. Extending these notions to bipolar soft topological spaces is crucial for analyzing systems involving both positive and negative information. In this paper, we define $\widetilde{\widetilde{W}}$-separated $B_PSSs$ and strong $\widetilde{\widetilde{W}}$-separated $B_PSSs$ using bipolar soft weak structures. We also introduce and investigate bipolar soft weak connectedness and bipolar soft weak compactness within bipolar soft weak structures. Furthermore, we define the concepts of bipolar soft weak local connectedness and bipolar soft weak components, supported by illustrative examples to clarify their meanings. Furthermore, we explore the relationships between these new concepts and their classical counterparts, showing that a $B_PS$ $\widetilde{\widetilde{W}}$-connected (resp. disconnected) space is not necessarily $B_PS$ $\widetilde{\widetilde{W}}$-hereditary, and that a $B_PSW$-closed subset of a $B_PS$ $\widetilde{\widetilde{W}}$-compact space may fail to be $B_PS$ $\widetilde{\widetilde{W}}$-compact.

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## 1 Introduction

Soft set theory, introduced by Molodtsov [1] in 1999, provides a powerful mathematical framework for handling uncertainty, vagueness, and imprecision that are often encountered in real-world data. Building on this foundation, Maji et al. [2] established the basic operations and definitions that formalized soft sets, while Ali et al. [3] introduced new operations to enhance their applicability in decision-making problems. The development of soft topology emerged as a natural extension of soft set theory to topological structures, initiated by Shabir and Naz [4], who defined the fundamental concepts of soft open and soft closed sets. Subsequent studies by Aygünoğlu and Aygün [5] deepened the theoretical understanding of soft topological spaces. Ahmad and Hussain [6] explored algebraic structures of soft topology, and Peyghan et al. [7] along with Hussain [8] investigated soft connectedness and related properties. Recent contributions have further enriched the field: Polat et al. [9], and Aydn and Enginolu [10] examined new topological notions, Jafari et al. [11] studied soft topologies induced by soft relations, and Al Ghour [12] introduced soft homogeneous components and soft products, providing novel perspectives on the structural composition of soft spaces. Furthermore, Zakari, Ghareeb, and Omran [13] introduced and investigated the concept of soft weak structures, extending classical soft topological notions by defining and analyzing weaker forms of soft open and soft closed sets within soft topological spaces.

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Recent advances in soft set theory have moved decisively toward bipolar extensions and their topological counterparts, enriching both the theoretical foundations and practical applications of soft topology. Building on classical soft-topological ideas, researchers have introduced bipolar soft structures that capture positivenegative (bipolar) information and enable finer modeling of uncertainty; foundational treatments by [14] and structural properties of bipolar soft topologies by Öztürk [15] and related generalized forms have been developed by Saleh, Asaad and Mohammed [16, 17, 18], while Shabir and collaborators explored bipolar connectedness and compactness notions [19]. These theoretical innovations have been paired with methodological contributions for decision-making and similarity assessment: Demirta and Dalkl applied bipolar fuzzy soft sets to medical diagnosis [20], Demir, Saldaml and Okurer proposed bipolar fuzzy soft filters for multi-criteria group decision-making [21], and Hamad et al. developed similarity measures for bipolar interval-valued fuzzy soft data [22]. Further structural and relational perspectives such as proximity via bipolar fuzzy soft classes and bipolar soft functions were advanced by Saldaml and Demir and by Fadel and Dzul-Kifli [23, 24, 25], while Fujita and Smarandache introduced multi-tier hypergraph frameworks that incorporate bipolar information for modeling complex networks [26, 27]. Recent work on specialized classes, including bipolar soft minimal structures which provides new perspectives for simplifying and analyzing bipolar soft topological systems [28]. Collectively, these contributions show that bipolar soft set theory and bipolar soft topology form a rapidly maturing area that links rigorous topological constructs with concrete decision-making, diagnosis, and network-modeling applications.

Recently, M. Taher and Asaad [29] introduced the concept of bipolar soft weak structures within the framework of bipolar soft topological spaces, establishing a new class of weak topological systems. Their study defines and analyzes key notions such as bipolar soft weak open, closed, closure, interior, and boundary sets, along with corresponding pointwise concepts and neighborhood structures. They defined bipolar soft weak subspaces for these weak structures.

Despite these developments, the concepts of weak connectedness and weak compactness have not yet been examined within bipolar soft topological structures. While soft weak structures and bipolar soft topologies have been studied separately, there is still no unified framework combining both. This gap restricts the analysis of spaces exhibiting weakly connected or weakly compact behaviors under dual (positivenegative) conditions.

Motivated by this limitation, the present paper makes the following contributions:

1. Introduces bipolar soft weak connectedness and bipolar soft weak compactness within bipolar soft weak structures.

2. Defines bipolar soft weak locally connected spaces and bipolar soft weak components, illustrated with examples.

3. Establishes several properties and relationships between the proposed notions and their classical counterparts, showing that a $B_PS\widetilde{\widetilde{\widetilde{W}}}$-connected (resp. disconnected) space is not necessarily $B_PS\,\widetilde{\widetilde{\widetilde{W}}}$-hereditary, and that a $B_PSW$-closed subset of a $B_PS\,\widetilde{\widetilde{\widetilde{W}}}$-compact space may not be $B_PS\,\widetilde{\widetilde{\widetilde{W}}}$-compact.

4. Extends the theoretical foundations of bipolar soft topology and opens new directions for further research in weak topological structures.

## 2 Preliminaries

Within this paper, let $\breve{I}$ be a universe set, and the nonempty set $\breve{\aleph}$ be an entire set of parameters, $P(\breve{I})$ be the family of all subsets of $\breve{I}$. Let $\breve{\pi}$ and $\breve{\sigma}$ be nonempty subsets of $\breve{\aleph}$. This section begins by reviewing essential concepts, including soft weak structures, bipolar soft set logic and provides the foundational background for bipolar soft topological spaces.

**Definition 2.1.** [1] (Soft Set) *A pair $(\check{\zeta}, \breve{\pi})$ is known as a soft set on $\check{\amalg}$, where $\check{\zeta}$ is a mapping from $\breve{\pi}$ into $P(\check{\amalg})$. Meaning that a soft set on $\check{\amalg}$ is a parameterized family of subsets of the universe $\check{\amalg}$. For $\epsilon \in \breve{\pi}$, $(\check{\zeta}, \breve{\pi})$ can be considered as $\epsilon$-elements' set of the soft set $(\check{\zeta}, \breve{\pi})$. As seen, soft set is not a crisp set. Following that, the family of all soft sets on $\check{\amalg}$ is denoted by $SS(\check{\amalg})$. Therefore, a soft set $(\check{\zeta}, \breve{\pi})$ can be dictated as:*

$$(\check{\zeta}, \breve{\pi}) = \{(\epsilon, \check{\zeta}(\epsilon)) : \epsilon \in \breve{\pi},\ \check{\zeta}(\epsilon) \subseteq \check{\amalg}\}.$$

**Definition 2.2.** [13] *A soft subset $(\check{\zeta}, \rho)$ of $(\widetilde{\check{\amalg}}, \rho)$ is called a soft weak compact set, denoted by $S\ \widetilde{W}$-compact set, if each $S\ \widetilde{W}$-open cover of $(\check{\zeta}, \rho)$ has a finite $S$ subcover. A $SWS$ $(\check{\amalg}, \widetilde{W}, \rho)$ is said to be a $S\ \widetilde{W}$-compact space if $(\widetilde{\check{\amalg}}, \rho)$ is a $S\ \widetilde{W}$-compact subset of itself.*

**Definition 2.3.** [2] *Let $\breve{\pi} = \{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$ be a subset of $\check{\aleph}$, the **Not** set of $\breve{\pi}$ is denoted by $\neg\breve{\pi} = \{\neg\epsilon_1, \neg\epsilon_2, ..., \neg\epsilon_n\}$ where, $\neg\epsilon_i = Not\ \epsilon_i$, for all $i$.*

**Definition 2.4.** [14] *A triple $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is known as a bipolar soft set, denoted by $B_PSS$, on $\check{\amalg}$, where $\check{\zeta}$ and $\check{\xi}$ are mappings defined by $\check{\zeta} : \breve{\pi} \longrightarrow P(\check{\amalg})$ and $\check{\xi} : \neg\breve{\pi} \longrightarrow P(\check{\amalg})$ so that $\check{\zeta}(\epsilon) \cap \check{\xi}(\neg\epsilon) = \phi$ for all $\epsilon \in \breve{\pi}$ and $\neg\epsilon \in \neg\breve{\pi}$.*
*So, a $B_PSS$ $(\check{\zeta}, \check{\xi}, \breve{\pi})$ can be dictated as:*

$$(\check{\zeta}, \check{\xi}, \breve{\pi}) = \{(\epsilon, \check{\zeta}(\epsilon), \check{\xi}(\neg\epsilon)) : \epsilon \in \breve{\pi} \text{ and } \check{\zeta}(\epsilon) \ \cap \ \check{\xi}(\neg\epsilon) \ = \ \phi\}.$$

*We denote $B_PSS(\check{\amalg})$ by the set of all $B_PSSs$ on $\check{\amalg}$.*

**Definition 2.5.** [14] *For any two $B_PSSs$ $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$, it is stated that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ is a bipolar soft $(B_PS)$ subset of $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$ if:*

*(i)* $\breve{\pi} \subseteq \breve{\sigma}$ *and,*

*(ii)* $\check{\zeta}_1(\epsilon) \subseteq \check{\zeta}_2(\epsilon)$ *and* $\check{\xi}_2(\neg\epsilon) \subseteq \check{\xi}_1(\neg\epsilon)$ *for all* $\epsilon \in \breve{\pi}$ *and* $\neg\epsilon \in \neg\breve{\pi}$.

*This relationship is denoted by $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$. Likewise, it is stated that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ is a $B_PS$ superset of $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$, denoted by $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\supseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$, if $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$ is a $B_PS$ subset of $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$.*

**Definition 2.6.** [14] *A $B_PSS$ $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is considered a null $B_PSS$ denoted by $(\Phi, \widetilde{\widetilde{\check{\amalg}}}, \breve{\pi})$, if $\check{\zeta}(\epsilon) = \phi$ for all $\epsilon \in \breve{\pi}$ and $\check{\xi}(\neg\epsilon) = \check{\amalg}$ for all $\neg\epsilon \in \neg\breve{\pi}$.*

**Definition 2.7.** [14] *A $B_PSS$ $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is considered an absolute $B_PSS$ denoted by $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$, if $\check{\zeta}(\epsilon) = \check{\amalg}$ for all $\epsilon \in \breve{\pi}$ and $\check{\xi}(\neg\epsilon) = \phi$ for all $\neg\epsilon \in \neg\breve{\pi}$.*

**Definition 2.8.** [14] *Let $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$ be two $B_PSSs$, then the $B_PS$ union of these $B_PSSs$ is the $B_PSS(\check{\delta}, \check{\gamma}, \dot{\kappa})$, where $\dot{\kappa} = \breve{\pi} \cap \breve{\sigma}$ is a nonempty set and for all $\epsilon \in \dot{\kappa}$, there is*
*$\check{\delta}(\epsilon) = \check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon), \epsilon \in \breve{\pi} \cap \breve{\sigma} \neq \phi$ and $\check{\gamma}(\neg\epsilon) = \check{\xi}_1(\neg\epsilon) \cap \check{\xi}_2(\neg\epsilon), \neg\epsilon \in \neg\breve{\pi} \cap \neg\breve{\sigma} \neq \phi$. This operation is denoted as $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\sigma}) = (\check{\delta}, \check{\gamma}, \dot{\kappa})$.*

**Definition 2.9.** *Let $\{(\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) : i \in I\}$ be any family of $B_PSSs$, then $\widetilde{\widetilde{\bigcup}}_{i\in I} (\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) = (\check{\delta}, \check{\gamma}, \dot{\kappa})$, where $\check{\delta}(\epsilon) = \check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) \cup \ ...$ and $\check{\gamma}(\neg\epsilon) = \check{\xi}_1(\neg\epsilon) \cap \check{\xi}_2(\neg\epsilon) \cap \ ...$ .*

**Definition 2.10.** [14] *Let* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$ *be two* $B_PSSs$*, then the* $B_PS$ *intersection of these* $B_PSSs$ *is the* $B_PSS(\check{\delta}, \check{\gamma}, \dot{\kappa})$*, where* $\dot{\kappa} = \breve{\pi} \cap \breve{\sigma}$ *is a nonempty set and for all* $\epsilon \in \dot{\kappa}$*, there is* $\check{\delta}(\epsilon) = \check{\zeta}_1(\epsilon) \cap \check{\zeta}_2(\epsilon), \epsilon \in \breve{\pi} \cap \breve{\sigma} \neq \phi$ *and* $\check{\gamma}(\neg\epsilon) = \check{\xi}_1(\neg\epsilon) \cup \check{\xi}_2(\neg\epsilon), \neg\epsilon \in \neg\breve{\pi} \cap \neg\breve{\sigma} \neq \phi$*. This operation is denoted as* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\sigma}) = (\check{\delta}, \check{\gamma}, \dot{\kappa})$*.*

**Definition 2.11.** *Let* $\{(\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) : i \in I\}$ *be any family of* $B_PSSs$*, then* $\widetilde{\widetilde{\bigcap}}_{i\in I}(\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) = (\check{\delta}, \check{\gamma}, \dot{\kappa})$*, where* $\check{\delta}(\epsilon) = \check{\zeta}_1(\epsilon) \cap \check{\zeta}_2(\epsilon) \cap \ldots$ *and* $\check{\gamma}(\neg\epsilon) = \check{\xi}_1(\neg\epsilon) \cup \check{\xi}_2(\neg\epsilon) \cup \ldots$ .

**Proposition 2.12.** [19] *If* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\amalg})$*, then*

*(i)* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, (\check{\zeta}, \check{\xi}, \breve{\pi})^c = (\check{\delta}, \Phi, \breve{\pi})$*, where* $\check{\delta}(\epsilon) = \check{\zeta}(\epsilon) \cup \check{\zeta}^c(\epsilon) \subseteq \check{\amalg}$ *for each* $\epsilon \in \breve{\pi}$ *and* $\Phi(\neg\epsilon) = \check{\xi}(\neg\epsilon) \cap \check{\xi}^c(\neg\epsilon) = \phi$ *for each* $\neg\epsilon \in \neg\breve{\pi}$*.*

*(ii)* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}, \check{\xi}, \breve{\pi})^c = (\Phi, \check{\gamma}, \breve{\pi})$*, where* $\Phi(\epsilon) = \check{\zeta}(\epsilon) \cap \check{\zeta}^c(\epsilon) = \phi$ *for each* $\epsilon \in \breve{\pi}$ *and* $\check{\gamma}(\neg\epsilon) = \check{\xi}(\neg\epsilon) \cup \check{\xi}^c(\neg\epsilon) \subseteq \check{\amalg}$ *for each* $\neg\epsilon \in \neg\breve{\pi}$*.*
*Further,* $(\check{\zeta}, \check{\xi}, \breve{\pi})$*,* $(\check{\zeta}, \check{\xi}, \breve{\pi})^c$ *will always satisfy* $\check{\zeta}(\epsilon) \cup \check{\zeta}^c(\epsilon) = \check{\xi}(\neg\epsilon) \cup \check{\xi}^c(\neg\epsilon)$ *for all* $\epsilon \in \breve{\pi}$*.*

**Definition 2.13.** [15] *If* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\amalg})$*, and* $\check{Y}$ *is a nonempty subset of* $\check{\amalg}$*, then the sub* $B_PS$ *set of* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *over* $\check{Y}$ *denoted by* $({}^{\check{Y}}\check{\zeta}, {}^{\check{Y}}\check{\xi}, \breve{\pi})$*, and is defined as follows:*

$$^{\check{Y}}\check{\zeta}(\epsilon) = \check{Y} \cap \check{\zeta}(\epsilon) \text{ and } {}^{\check{Y}}\check{\xi}(\neg\epsilon) = \check{Y} \cap \check{\xi}(\neg\epsilon), \text{ for each } \epsilon \in \breve{\pi} \text{ and } \neg\epsilon \in \neg\breve{\pi}.$$

**Definition 2.14.** [19] *Let* $\widetilde{\widetilde{\tau}}$ *be the family of* $B_PSSs$ *on* $\check{\amalg}$ *with* $\breve{\pi}$ *as the set of parameters, then,* $\widetilde{\widetilde{\tau}}$ *be considered a bipolar soft topology* $(B_PST)$ *on* $\check{\amalg}$ *if:*

*(i)* $(\Phi, \widetilde{\widetilde{\check{\amalg}}}, \breve{\pi})$ *and* $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ *belong to* $\widetilde{\widetilde{\tau}}$*.*

*(ii) The* $B_PS$ *union of any number of* $B_PSSs$ *in* $\widetilde{\widetilde{\tau}}$ *belongs to* $\widetilde{\widetilde{\tau}}$*.*

*(iii) The* $B_PS$ *intersection of finite number of* $B_PSSs$ *in* $\widetilde{\widetilde{\tau}}$ *belongs to* $\widetilde{\widetilde{\tau}}$*.*

*Then* $(\check{\amalg}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ *has the name of a bipolar soft topological space* $(B_PSTS)$ *on* $\check{\amalg}$*.*
*Every member of* $\widetilde{\widetilde{\tau}}$ *is known as a bipolar soft open set, denoted by* $B_PS$*-open. The complement of a* $B_PS$*-open set is* $B_PS$*-closed.*

**Proposition 2.15.** [15] *Let* $(\check{\amalg}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ *be a* $B_PSTS$ *on* $\check{\amalg}$ *and* $\check{Y}$ *be a nonempty subset of* $\check{\amalg}$*, then,* $\widetilde{\widetilde{\tau}}_{\check{Y}} = \{({}^{\check{Y}}\check{\zeta}, {}^{\check{Y}}\check{\xi}, \breve{\pi}) : (\check{\zeta}, \check{\xi}, \breve{\pi}) \in \widetilde{\widetilde{\tau}}\}$ *is* $B_PST$ *on* $\check{Y}$*. The family* $\widetilde{\widetilde{\tau}}_{\check{Y}}$ *is known as a* $B_PS$ *subspace topology.*

**Definition 2.16.** [15] *Let* $(\check{\amalg}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ *be a* $B_PSTS$ *on* $\check{\amalg}$ *and* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$*, then the family* $\widetilde{\widetilde{\tau}}_{(\check{\zeta}, \check{\xi}, \breve{\pi})} = \left\{(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) : (\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) \widetilde{\widetilde{\in}} \widetilde{\widetilde{\tau}} \text{ and } i \in I\right\}$ *is a* $B_PS$ *subspace topology on* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *and* $\left(\check{\amalg}_{(\check{\zeta}, \check{\xi}, \breve{\pi})}, \widetilde{\widetilde{\tau}}_{(\check{\zeta}, \check{\xi}, \breve{\pi})}, \breve{\pi}, \neg\breve{\pi}\right)$ *has the name of a* $B_PS$ *topological subspace of* $(\check{\amalg}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$*.*

**Definition 2.17.** [19] *Two* $B_PSSs$ $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are said to be disjoint* $B_PSSs$ *if* $\check{\zeta}_1(\epsilon) \cap \check{\zeta}_2(\epsilon) = \phi$ *for all* $\epsilon \in \breve{\pi}$*.*

**Definition 2.18.** [19] *Let* $(\check{\amalg}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ *be a* $B_PSTS$ *on* $\check{\amalg}$*. A* $B_PS$ *separation of* $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ *is a pair* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$*,* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *of non-null disjoint* $B_PS$ *open sets on* $\check{\amalg}$ *such that* $\check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) = \check{\amalg}$ *for all* $\epsilon \in \breve{\pi}$*.*

**Definition 2.19.** [19] *A $B_PSTS$ $(\check{\mathrm{I}}, \widetilde{\widetilde{\tau}}, \check{\pi}, \neg\check{\pi})$ is said to be a $B_PS$ disconnected space if there exists a $B_PS$ separation of $(\widetilde{\widetilde{\check{\mathrm{I}}}}, \Phi, \check{\pi})$. Further, $(\check{\mathrm{I}}, \widetilde{\widetilde{\tau}}, \check{\pi}, \neg\check{\pi})$ is said to be a $B_PS$ connected space if it is not a $B_PS$ disconnected space.*

**Definition 2.20.** [19] *A property $\mathcal{P}$ of a $B_PSTS$ $(\check{\mathrm{I}}, \widetilde{\widetilde{\tau}}, \check{\pi}, \neg\check{\pi})$ is said to be $B_PS$ hereditary if every $B_PS$ subspace $(Y, \widetilde{\widetilde{\tau}}_{\check{Y}}, \check{\pi}, \neg\check{\pi})$ of $(\check{\mathrm{I}}, \widetilde{\widetilde{\tau}}, \check{\pi}, \neg\check{\pi})$ also possesses the property $\mathcal{P}$.*

**Definition 2.21.** [19] *A family $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) : (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\tau}}\}_{\delta\in\Delta}$ of $B_PSSs$ is said to be a $B_PS$ cover of a $B_PSS$ $(\check{\zeta}, \check{\xi}, \check{\pi})$ if:*

$$(\check{\zeta}, \check{\xi}, \check{\pi}) \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}).$$

*Furthermore, it is called the $B_PS$ open cover of a $B_PSS$ $(\check{\zeta}, \check{\xi}, \check{\pi})$ if each member of $\check{\Delta}$ is a $B_PS$ open set. A $B_PS$ subcover of $\check{\Delta}$ is a subfamily of $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})\}_{\delta\in\Delta}$ which is also a $B_PS$ open cover.*

**Definition 2.22.** [19] *A bipolar soft subset $(\check{\zeta}, \check{\xi}, \check{\pi})$ of $(\widetilde{\widetilde{\check{\mathrm{I}}}}, \Phi, \check{\pi})$ is called a bipolar soft compact set, denoted by, $B_PS$ compact set, if each $B_PS$ open cover of $(\check{\zeta}, \check{\xi}, \check{\pi})$ has a finite $B_PS$ subcover. A $B_PSTS$ $(\check{\mathrm{I}}, \widetilde{\widetilde{\tau}}, \check{\pi}, \neg\check{\pi})$ is said to be $B_PS$ compact space if $(\widetilde{\widetilde{\check{\mathrm{I}}}}, \Phi, \check{\pi})$ is a $B_PS$ compact subset of itself.*

**Definition 2.23.** [29] *Let $\widetilde{\widetilde{\widetilde{W}}}$ be a family of $B_PS$ subsets on $\check{\mathrm{I}}$, then $\widetilde{\widetilde{\widetilde{W}}}$ is considered a $(B_PSWS)$ on $\check{\mathrm{I}}$ if $(\Phi, \widetilde{\widetilde{\check{\mathrm{I}}}}, \check{\pi}) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\widetilde{W}}}$.*

*Then, $(\check{\mathrm{I}}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ is known as a $B_PSWS$ on $\check{\mathrm{I}}$. The members of $\widetilde{\widetilde{\widetilde{W}}}$ are considered bipolar soft $\widetilde{\widetilde{\widetilde{W}}}$-open sets, denoted by $B_PSW$-open in $\check{\mathrm{I}}$.*

*And $(\check{\zeta}, \check{\xi}, \check{\pi})$ is considered bipolar soft $\widetilde{\widetilde{\widetilde{W}}}$-closed, denoted by $B_PSW$-closed if its $B_PSW$ complement $(\check{\zeta}, \check{\xi}, \check{\pi})^c$ is $B_PSW$-open.*

**Theorem 2.24.** [29] *Let $(\check{\mathrm{I}}, \widetilde{W}, \check{\pi})$ be a $SWS$. Then the family $\widetilde{\widetilde{\widetilde{W}}}$ consisting of $B_PSS$ $(\check{\zeta}, \check{\xi}, \check{\pi})$ such that $(\check{\zeta}, \check{\pi}) \,\widetilde{\in}\, \widetilde{W}$ and $\check{\xi}(\neg\epsilon) = \check{\mathrm{I}} \setminus \check{\zeta}(\epsilon)$ for all $\neg\epsilon \in \neg\check{\pi}$ defines a $B_PSWS$ on $\check{\mathrm{I}}$.*

**Definition 2.25.** [29] *Let $(\check{\mathrm{I}}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ be a $B_PSWS$ and $(\check{\zeta}, \check{\xi}, \check{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\mathrm{I}})$. Then the $B_PSW$-closure of $(\check{\zeta}, \check{\xi}, \check{\pi})$, denoted by $B_PSW$-$cl(\check{\zeta}, \check{\xi}, \check{\pi})$, is the $B_PS$ intersection of all $B_PSW$-closed sets containing $(\check{\zeta}, \check{\xi}, \check{\pi})$. So, the $B_PSW$-closure can be dictated as:*

$$B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \check{\pi}) = \widetilde{\widetilde{\bigcap}}\{(\check{\delta}, \check{\gamma}, \check{\pi}) : (\check{\delta}, \check{\gamma}, \check{\pi}) \text{ is } B_PSW\text{-closed and } (\check{\delta}, \check{\gamma}, \check{\pi}) \,\widetilde{\widetilde{\supseteq}}\, (\check{\zeta}, \check{\xi}, \check{\pi})\}.$$

**Definition 2.26.** [29] *Let $(\check{\mathrm{I}}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ be a $B_PSWS$ and $(\check{\zeta}, \check{\xi}, \check{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\mathrm{I}})$. Then the $B_PSW$-interior of $(\check{\zeta}, \check{\xi}, \check{\pi})$, denoted by $B_PSW$-$int(\check{\zeta}, \check{\xi}, \check{\pi})$, is the $B_PS$ union of all $B_PSW$-open subsets of $(\check{\zeta}, \check{\xi}, \check{\pi})$.*
*So, the set can be dictated as:*

$$B_PSW\text{-}int(\check{\zeta}, \check{\xi}, \check{\pi}) = \widetilde{\widetilde{\bigcup}}\{(\check{\delta}, \check{\gamma}, \check{\pi}) : (\check{\delta}, \check{\gamma}, \check{\pi}) \widetilde{\widetilde{\in}} \widetilde{\widetilde{\widetilde{W}}} \text{and} \, (\check{\delta}, \check{\gamma}, \check{\pi}) \widetilde{\widetilde{\subseteq}} (\check{\zeta}, \check{\xi}, \check{\pi})\}.$$

**Definition 2.27.** [29] *Let $(\check{\mathrm{I}}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ be a $B_PSWS$ and $(\check{\zeta}, \check{\xi}, \check{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\mathrm{I}})$, then the bipolar soft $\widetilde{\widetilde{\widetilde{W}}}$-boundary of $(\check{\zeta}, \check{\xi}, \check{\pi})$, denoted by $B_PSW$-$b(\check{\zeta}, \check{\xi}, \check{\pi})$, is defined as*

$$B_PSW\text{-}b(\check{\zeta}, \check{\xi}, \check{\pi}) = B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \check{\pi}) \,\widetilde{\widetilde{\cap}}\, B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \check{\pi})^c.$$

**Proposition 2.28.** [29] *Let $(\check{\mathrm{I}}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ be a $B_PSWS$ on $\check{\mathrm{I}}$ and $\check{Y}$ be a nonempty subset of $\check{\mathrm{I}}$, then, $\widetilde{\widetilde{\widetilde{W}}}_{\check{Y}} = \{({}^{\check{Y}}\check{\zeta}, {}^{\check{Y}}\check{\xi}, \check{\pi}) : (\check{\zeta}, \check{\xi}, \check{\pi}) \in \widetilde{\widetilde{\widetilde{W}}}\}$ is $B_PSW$ on $\check{Y}$. The family $\widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$ is known as a $B_PS$ subspace topology.*

**Proposition 2.29.** *Let* $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ *be a* $B_PSWSS$ *of* $B_PSWS$ $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *over* $\check{\text{II}}$ *and* $({}^{\check{Y}}\check{\zeta}, {}^{\check{Y}}\check{\xi}, \breve{\pi})$ *be a* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$*-closed set in* $\check{Y}$*. Then* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is a* $B_PSW$*-closed set in* $\check{\text{II}}$*.*

**Definition 2.30.** [29] *Let* $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *be a* $B_PSWS$ *and* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\text{II}})$*, then the family*

$$\widetilde{\widetilde{\widetilde{W}}}_{(\check{\zeta},\check{\xi},\breve{\pi})} = \{(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) : (\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\widetilde{W}}},\ i \in \mathcal{I}\}$$

*is said to be a bipolar soft weak subspace* $B_PSWSS$ *on* $(\check{\zeta}, \check{\xi}, \breve{\pi})$*.*

## 3 $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-Connected Sets

This section shows $\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$ ($S\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$) using $B_PSWS$ providing some of their properties. In addition, $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected sets in terms of $B_PSWS$ are presented, obtaining properties and their relations.

**Definition 3.1.** *Let* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *be two* $B_PSSs$ *in* $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *which are not null. Then*

- *(i)* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are known as* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$ *if* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, B_PSW\text{-}cl\,(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$ *and* $B_PSW\text{-}cl\,(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$*.*
- *(ii)* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are known as strong* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$ *(*$S\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$*) if* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, B_PSW\text{-}cl\,(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \widetilde{\widetilde{\check{\text{II}}}}, \breve{\pi})$ *and* $B_PSW\text{-}cl\,(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \widetilde{\widetilde{\check{\text{II}}}}, \breve{\pi})$*.*

**Proposition 3.2.** *Every* $S\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$ *on* $\check{\text{II}}$ *is a* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$ *on* $\check{\text{II}}$*.*

**Proof.** Follows directly from definitions. □

**Proposition 3.3.** *Any two* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$ *(*$S\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$*) are disjoint* $B_PSSs$*.*

**Proof.** Obvious. □

**Remark 3.4.** *Note that disjoint* $B_PSSs$ *may not be* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$ *(*$S\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$*); meaning that the converse of Proposition 3.3 is not true as shown in the following example.*

**Example 3.5.** Let $\check{\text{II}} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and

$$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \widetilde{\widetilde{\check{\text{II}}}}, \breve{\pi}), (\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi})\}$$

be a $B_PSWS$ over $\check{\text{II}}$ where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$,$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$,$(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\text{II}})$, defined as follows

$$\begin{aligned}
(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_3, \hbar_4\}, \{\hbar_2\})\},\\
(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_1, \hbar_3\}, \{\hbar_2, \hbar_4\}), (\epsilon_2, \{\hbar_1, \hbar_3\}, \{\hbar_2\})\},\\
(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_1, \hbar_2, \hbar_3\}, \phi), (\epsilon_2, \{\hbar_1, \hbar_3, \hbar_4\}, \{\hbar_2\})\}.
\end{aligned}$$

Now, assume that $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$ are disjoint $B_PSSs$ over $\check{\text{II}}$ defined by

$$\begin{aligned}
(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_1, \hbar_2, \hbar_4\}, \{\hbar_3\}), (\epsilon_2, \{\hbar_1, \hbar_2, \hbar_4\}, \{\hbar_3\})\},\\
(\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_3\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_2\})\}.
\end{aligned}$$

Then $B_PSW-cl\ (\check{\delta}_1,\check{\gamma}_1,\breve{\pi}) = B_PSW-cl\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\tilde{\tilde{\check{\Pi}}},\Phi,\breve{\pi})$ and $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\tilde{\tilde{\cap}}\ B_PSW-cl\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$, $B_PSW-cl\ (\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$. But $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\Phi,\check{\gamma},\breve{\pi})$. Thus, $B_PSSs$ $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$, $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ are disjoint $B_PSSs$ but not $\widetilde{\widetilde{W}}$-separated $B_PSSs$ ($S\widetilde{\widetilde{W}}$-separated $B_PSSs$).

**Proposition 3.6.** *Let be any two $B_PSSs$ on $\check{\Pi}$. Then the following are correct:*

*(i) If $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ are two $\widetilde{\widetilde{W}}$-separated $B_PSSs$ over $\check{\Pi}$ with $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$. Then, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ also are $\widetilde{\widetilde{W}}$-separated $B_PSSs$ over $\check{\Pi}$.*

*(ii) If $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ are two $S\widetilde{\widetilde{W}}$-separated $B_PSSs$ over $\check{\Pi}$ with $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$. Then, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ also are $S\widetilde{\widetilde{W}}$-separated $B_PSSs$ over $\check{\Pi}$.*

**Proof.**

(*i*) Given $\widetilde{\widetilde{W}}$-separated $B_PSSs$ $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$. Then
$(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\Phi,\check{\xi},\breve{\pi})$. Since $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$, then
$B_PSW\text{-}cl\,(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $B_PSW\text{-}cl\,(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$. Therefore, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW\text{-}cl\,(\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = B_PSW\text{-}cl\,(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\Phi,\check{\gamma},\breve{\pi})$. Hence, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ are $\widetilde{\widetilde{W}}$-separated $B_PSSs$ over $\check{\Pi}$.

(*ii*) Let given $S\widetilde{\widetilde{W}}$-separated $B_PSSs$ $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$. Then
$(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\Phi,\tilde{\tilde{\check{\Pi}}},\breve{\pi})$. Since $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$, then
$B_PSW\text{-}cl\,(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $B_PSW\text{-}cl\,(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$. Therefore, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW\text{-}cl\,(\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = B_PSW\text{-}cl\,(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\Phi,\tilde{\tilde{\check{\Pi}}},\breve{\pi})$. Hence, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ are $S\widetilde{\widetilde{W}}$-separated $B_PSSs$ over $\check{\Pi}$.

□

**Theorem 3.7.** *Two $\widetilde{\widetilde{W}}$-closed subsets $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ of $B_PSWS$ $(\check{\Pi},\widetilde{\widetilde{W}},\breve{\pi},\neg\breve{\pi})$ over $\check{\Pi}$ are $\widetilde{\widetilde{W}}$-separated $B_PSSs$ if and only if they are disjoint $B_PSSs$.*

**Proof.** The first condition is obvious. Conversely, assume that $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ are disjoint $\widetilde{\widetilde{W}}$-closed. So, $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\Phi,\check{\xi},\breve{\pi})$ and $B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi}) = (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$, $B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ and hence

$$(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW-cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = B_PSW-cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\Phi,\check{\xi},\breve{\pi})$$

showing that $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ are $\widetilde{\widetilde{W}}$-separated $B_PSSs$ over $\check{\Pi}$. □

**Remark 3.8.** *Two disjoint $\widetilde{\widetilde{W}}$-open sets $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ are not necessarily $\widetilde{\widetilde{W}}$-separated.*

**Example 3.9.** Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}, \breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and

$$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi})\}$$

be a $B_PSWS$ over $\check{\amalg}$ where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, $(\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\amalg})$, defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1, \hbar_4\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1, \hbar_4\})\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_3\}, \{\hbar_1, \hbar_4\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_1, \hbar_4\})\},$$
$$(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \{\hbar_1, \hbar_4\}), (\epsilon_2, \{\hbar_2, \hbar_3\}, \{\hbar_1, \hbar_4\})\}.$$

Obviously, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are disjoint $\widetilde{\widetilde{\widetilde{W}}}$-open but not $\widetilde{\widetilde{\widetilde{W}}}$-separated as $B_PSW-cl$ $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) =$ $B_PSW-cl$ $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$, which implies that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ $\tilde{\tilde{\cap}}$ $B_PSW-cl$ $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $B_PSW$-$cl(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ $\tilde{\tilde{\cap}}$ $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. Hence the conclusion.

**Corollary 3.10.** *If two $\widetilde{\widetilde{\widetilde{W}}}$-closed subsets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ of $B_PSWS$ $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ over $\check{\amalg}$ are $S\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$, then they are disjoint $B_PSSs$.*

**Proof.** Follows directly from Proposition 3.2 and Theorem 3.7. □

**Remark 3.11.** *The example below shows that the converse of Corollary 3.10 is not true in general. Hence two disjoint $\widetilde{\widetilde{\widetilde{W}}}$-closed sets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are not necessarily $S\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$ .*

**Example 3.12.** Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}, \breve{\pi} = \{\epsilon_1\}$ and

$$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$$

be a $B_PSWS$ over $\check{\amalg}$ where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\amalg})$, defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_2\})\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_3\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_1\})\}.$$

Obviously, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c$ are disjoint $\widetilde{\widetilde{\widetilde{W}}}$-closed but not $S\widetilde{\widetilde{\widetilde{W}}}$-strong separated as $B_PSW$-$cl(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$ $= B_PSW$-$cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c = (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$, which implies that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c \tilde{\tilde{\cap}}$ $B_PSW$-$cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c = (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$, $B_PSW$-$cl(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$ $\tilde{\tilde{\cap}}$ $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c = (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c$. Hence the conclusion.

**Definition 3.13.** *A $B_PS$ subset $(\check{\zeta}, \check{\xi}, \breve{\pi})$ of $B_PSWS$ $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ over $\check{\amalg}$ is called $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected over $\check{\amalg}$ if there exist $\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$ of $(\check{\zeta}, \check{\xi}, \breve{\pi})$. Otherwise, a $B_PSS$ $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is called $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected over $\check{\amalg}$.*

**Remark 3.14.** *The null $B_PSS$ $(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi})$ is always $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected.*

**Definition 3.15.** *Let $\hbar_\upsilon^\epsilon$, $\hbar'^{\epsilon'}_{\upsilon'}$ $\tilde{\tilde{\in}}$ $B_PSP(\check{\amalg})_{(\breve{\pi},\neg\breve{\pi})}$ of a $B_PSWS$ $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Then, $\hbar_\upsilon^\epsilon$ and $\hbar'^{\epsilon'}_{\upsilon'}$ are called $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected points if they are contained in $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected set over $\check{\amalg}$.*

**Proposition 3.16.** *Let* $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *be a* $B_PSWS$ *over* $\check{\text{II}}$ *and* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *be a* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-connected set such that* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$*, where* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$*. Then* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *or* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$*.*

**Proof.** From $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$, then $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, B_PSW\text{-}cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$ and $B_PSW\text{-}cl\,(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$. Since $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, then $(\check{\zeta}, \check{\xi}, \breve{\pi}) = (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, \big((\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\widetilde{\widetilde{\cup}}(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big) = \big((\check{\zeta}, \check{\xi}, \breve{\pi})\widetilde{\widetilde{\cap}}(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\big) \,\widetilde{\widetilde{\cup}}\, ((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}))$. We state that at least one of the $B_PSSs$ $\big((\check{\zeta}, \check{\xi}, \breve{\pi})\widetilde{\widetilde{\cap}}(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\big)$ and $\big((\check{\zeta}, \check{\xi}, \breve{\pi})\widetilde{\widetilde{\cap}}(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big)$ is null $B_PSS$. Now, suppose that if possible non of these $B_PSSs$ is null, hence,

$$(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \neq (\Phi, \check{\xi}, \breve{\pi}) \text{ and } (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \neq (\Phi, \check{\xi}, \breve{\pi}).$$

Thus,

$$\begin{aligned}
&((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})) \,\widetilde{\widetilde{\cap}}\, B_PSW - cl\big((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big) \\
&\qquad \widetilde{\widetilde{\subseteq}} \big((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\big) \,\widetilde{\widetilde{\cap}}\, \big(B_PSW - cl(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, B_PSW - cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big) \\
&\qquad = \big((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, B_PSW - cl(\check{\zeta}, \check{\xi}, \breve{\pi})\big) \,\widetilde{\widetilde{\cap}}\, \big((\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, B_PSW - cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big) \\
&\qquad = (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\Phi, \check{\xi}, \breve{\pi}) \\
&\qquad = (\Phi, \check{\xi}, \breve{\pi}).
\end{aligned}$$

Similarly,

$$B_PSW\text{-}cl\big((\check{\zeta}, \check{\xi}, \breve{\pi})\widetilde{\widetilde{\cap}}(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\big) \,\widetilde{\widetilde{\cap}}\, \big((\check{\zeta}, \check{\xi}, \breve{\pi})\widetilde{\widetilde{\cap}}(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big) = (\Phi, \check{\xi}, \breve{\pi}).$$

Therefore, $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$. Thus, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ can be expressed as $B_PS$ union of a pair of $\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$. So, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected. Which is a contradiction. Hence, at least one of the $B_PSSs$ $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ is null $B_PSS$. Now, if $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$, then $(\check{\zeta}, \check{\xi}, \breve{\pi}) = (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ which implies that $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. If $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$, then $(\check{\zeta}, \check{\xi}, \breve{\pi}) = (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ implying that $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$. Therefore, either $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ or $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. □

**Corollary 3.17.** *If* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is a* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-connected subset of a* $B_PSWS$ $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *such that* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *where* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are both* $B_PSW$*-closed and nonnull disjoint* $B_PSSs$*. Then,* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* $B_PSSs$*.*

**Proof.** Follows directly from Proposition 3.16 and Theorem 3.7. □

**Proposition 3.18.** *Let* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *be* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-connected and* $(\check{\delta}, \check{\gamma}, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\text{II}})$ *such that* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\delta}, \check{\gamma}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})$*. Then* $(\check{\delta}, \check{\gamma}, \breve{\pi})$ *is* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-connected. Specifically,* $B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is also* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-connected.*

**Proof.** Suppose that $(\check{\delta}, \check{\gamma}, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected. Then, there exist nonnull $B_PSSs$ $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ in which

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, B_PSW\text{-}cl\,(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = B_PSW\text{-}cl\,(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$$

and $(\check{\delta}, \check{\gamma}, \breve{\pi}) = (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$.

From $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}, \check{\gamma}, \breve{\pi}) = (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, it follows from Proposition 3.16 that $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ or $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. Let $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ thus, $B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}cl(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ then,

$$B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}cl(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi}),$$

but $(\Phi, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, therefore,

$$B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi}).$$

So, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})$ then, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})$ implies that $B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. Hence, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$. This is a contradiction because $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ is nonnull $B_PSS$. Therefore, $(\check{\delta}, \check{\gamma}, \breve{\pi})$ is $B_PS\ \tilde{\tilde{\tilde{W}}}$-connected. Also, from $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})$, implies that $B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})$ is $B_PS\ \tilde{\tilde{\tilde{W}}}$-connected. □

**Proposition 3.19.** *Let* $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) : \delta \in \Delta\}$ *be the family of* $B_PS\ \tilde{\tilde{\tilde{W}}}$*-connected sets such that* $\tilde{\tilde{\bigcap}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \neq (\Phi, \check{\xi}, \breve{\pi})$. *Then* $\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ *is* $B_PS\ \tilde{\tilde{\tilde{W}}}$*-connected.*

**Proof.** Assume $(\check{\delta}, \check{\gamma}, \breve{\pi}) = \tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is not $B_PS\ \tilde{\tilde{\tilde{W}}}$-connected. Thus, there exist two nonnull disjoint $B_PSW$-open sets $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$ such that $(\check{\delta}, \check{\gamma}, \breve{\pi}) = (\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$. For each $\delta \in \Delta$, $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ are disjoint $B_PSW$-open sets in $(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ in which

$$\begin{aligned}&\big((\check{\delta}_1, \check{\gamma}_1, \breve{\pi})\tilde{\tilde{\cap}}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big) \,\tilde{\tilde{\cup}}\, \big((\check{\delta}_2, \check{\gamma}_2, \breve{\pi})\tilde{\tilde{\cap}}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big)\\ &= \big((\check{\delta}_1, \check{\gamma}_1, \breve{\pi})\tilde{\tilde{\cup}}(\check{\delta}_2, \check{\gamma}_2, \breve{\pi})\big) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) = (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}).\end{aligned}$$

Now, from $(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is a $B_PS\ \tilde{\tilde{\tilde{W}}}$-connected set, one of the $B_PSSs$ $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is a null $B_PSSs$, say, $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) = (\Phi, \check{\gamma}, \breve{\pi})$. Then, $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) = (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ which implies that $(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$ for all $\delta \in \Delta$ and hence $\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$, that is, $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$. This given, $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) = (\Phi, \check{\gamma}, \breve{\pi})$. This is a contradiction because $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi})$ is nonnull $B_PSS$. Hence, $(\check{\delta}, \check{\gamma}, \breve{\pi})$ is a $B_PS\ \tilde{\tilde{\tilde{W}}}$-connected. □

**Proposition 3.20.** *For any two* $B_PSPs$ $\hbar^{\epsilon}_{\upsilon}$, $\hbar'^{\epsilon'}_{\upsilon'}$ $\tilde{\tilde{\in}}$ $(\check{\zeta}, \check{\xi}, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\text{II}})$ *in a* $B_PSWS$ $(\check{\text{II}}, \tilde{\tilde{\tilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *are contained in some* $B_PS\ \tilde{\tilde{\tilde{W}}}$*-connected set* $(\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}, \check{\xi}, \breve{\pi})$. *Then* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is* $B_PS\ \tilde{\tilde{\tilde{W}}}$*-connected.*

**Proof.** Let $(\check{\zeta}, \check{\xi}, \breve{\pi})$ be a $B_PS\ \tilde{\tilde{\tilde{W}}}$-disconnected set. Thus, there is a $\tilde{\tilde{\tilde{W}}}$-separated $B_PSSs$ $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ of $(\check{\zeta}, \check{\xi}, \breve{\pi})$. Then, there are two $B_PSPs$ $\hbar^{\epsilon}_{\upsilon}$, $\hbar'^{\epsilon'}_{\upsilon'}$ in which $\hbar^{\epsilon}_{\upsilon} \,\tilde{\tilde{\in}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $\hbar'^{\epsilon'}_{\upsilon'} \,\tilde{\tilde{\in}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. Through the assumption, there is a $B_PS\ \tilde{\tilde{\tilde{W}}}$-connected set $(\check{\delta}, \check{\gamma}, \breve{\pi})$ containing $\hbar^{\epsilon}_{\upsilon}$, $\hbar'^{\epsilon'}_{\upsilon'}$ such that

$$(\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}, \check{\xi}, \breve{\pi}) = (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}).$$

Thus, by Proposition 3.16, we have $(\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ or $(\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. This implies that

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \neq (\Phi, \check{\zeta}, \breve{\pi}).$$

This is contradiction since $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $\widetilde{\widetilde{W}}$-separated $B_PSSs$. So, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-connected. □

**Proposition 3.21.** *Let $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) : \delta \in \Delta\}$ be the family of $B_PS$ $\widetilde{\widetilde{W}}$-connected sets such that one of the members of this family intersects every other member. Then, $\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-connected.*

**Proof.** Let $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})$ be a fixed member of the given family such that $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \neq (\Phi, \check{\zeta}, \breve{\pi})$ for every $\delta \in \Delta$. Then, $(\check{\delta}_\delta, \check{\gamma}_\delta, \breve{\pi}) = (\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-connected for each $\delta \in \Delta$, hence by Proposition 3.20. Now,

$$\begin{aligned}\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) &= \widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} \big((\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})\widetilde{\widetilde{\cup}}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big)\\ &= (\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, \big(\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big).\end{aligned}$$

Since $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})$ is one of the family $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) : \delta \in \Delta\}$ and

$$\begin{aligned}\widetilde{\widetilde{\bigcap}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) &= \widetilde{\widetilde{\bigcap}}_{\delta\in\Delta} \big((\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})\widetilde{\widetilde{\cup}}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big)\\ &= (\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, \big(\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big) \neq (\Phi, \check{\xi}, \breve{\pi}).\end{aligned}$$

From $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})$ intersects every $(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$. Therefore, $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \neq (\Phi, \check{\xi}, \breve{\pi})$. Hence, by Proposition 3.19, $\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-connected. □

**Proposition 3.22.** *For each two $\hbar^{\epsilon}_{\upsilon}$, $\hbar'^{\epsilon'}_{\upsilon'}$ $\widetilde{\widetilde{\in}}$ $B_PSP(\check{\text{II}})_{(\breve{\pi},\neg\breve{\pi})}$ of a $B_PSWS$ $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ are $B_PS$ $\widetilde{\widetilde{W}}$-connected, then $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-connected.*

**Proof.** Let $\hbar^{\epsilon}_{\upsilon}$ be a fixed $B_PSP$ in a $B_PSWS$ $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$. Then, for each $\hbar^{\epsilon}_{\upsilon}$ $B_PS$ different than $\hbar'^{\epsilon'}_{\upsilon'}$, we have $B_PS$ $\widetilde{\widetilde{W}}$-connected, say, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ containing $\hbar^{\epsilon}_{\upsilon}$ and $\hbar'^{\epsilon'}_{\upsilon'}$. Since $\hbar^{\epsilon}_{\upsilon} \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\bigcap}}_{\hbar^{\epsilon}_{\upsilon}\widetilde{\widetilde{\in}}(\widetilde{\widetilde{\check{\text{II}}}},\Phi,\breve{\pi})} (\check{\zeta}, \check{\xi}, \breve{\pi})$, it follows from Proposition 3.19 that $\widetilde{\widetilde{\bigcup}}_{\hbar^{\epsilon}_{\upsilon}\widetilde{\widetilde{\in}}(\widetilde{\widetilde{\check{\text{II}}}},\Phi,\breve{\pi})} (\check{\zeta}, \check{\xi}, \breve{\pi}) = (\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-connected. □

## 4 $B_PS$ $\widetilde{\widetilde{W}}$-Connected Spaces

In this section, the concept of bipolar soft weak connected ($B_PS$ $\widetilde{\widetilde{W}}$-connected) structure is presented. Also, some properties and results of this new concept of $B_PSWS$ are discussed.

**Definition 4.1.** *Let $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ be a $B_PSWS$. A $B_PS$ $\widetilde{\widetilde{W}}$-separation of $(\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ is defined to be the nonnull disjoint $B_PSW$-open sets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ such that $\check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) = \check{\text{II}}$ for each $\epsilon \in \breve{\pi}$.*

**Definition 4.2.** *A $B_PSWS$ $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is called $B_PS$ $\widetilde{\widetilde{W}}$-disconnected if $(\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ has $B_PS$ $\widetilde{\widetilde{W}}$-separation. That is, there exist nonnull disjoint $B_PSW$-open sets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ such that $\check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) = \check{\text{II}}$ for all $\epsilon \in \breve{\pi}$. Otherwise, $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is said to be $B_PS$ $\widetilde{\widetilde{W}}$-connected.*

**Remark 4.3.** *Suppose that* $\mid \check{\amalg} \mid = 1$, *there are only three* $B_PSWS$ *in* $\check{\amalg}$ *(that is,* $(\Phi, \tilde{\tilde{\check{\amalg}}}, \check{\pi})$, $(\tilde{\tilde{\check{\amalg}}}, \Phi, \check{\pi})$ *and* $(\Phi, \Phi, \check{\pi})$*) are* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-connected spaces, then we will have four* $\widetilde{\widetilde{\widetilde{W}}}$*-structures* $B_PSWS$*:-*

*(i)* $\widetilde{\widetilde{\widetilde{W}}}_1 = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \check{\pi}), (\tilde{\tilde{\check{\amalg}}}, \Phi, \check{\pi}), (\Phi, \Phi, \check{\pi})\}$.

*(ii)* $\widetilde{\widetilde{\widetilde{W}}}_2 = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \check{\pi}), (\tilde{\tilde{\check{\amalg}}}, \Phi, \check{\pi})\}$.

*(iii)* $\widetilde{\widetilde{\widetilde{W}}}_3 = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \check{\pi}), (\Phi, \Phi, \check{\pi})\}$.

*(iv)* $\widetilde{\widetilde{\widetilde{W}}}_4 = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \check{\pi})\}$.

Now, we suppose that $\mid \check{\amalg} \mid > 1$, for the rest of our work.

**Example 4.4.** Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$, $\check{\pi} = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $\neg\check{\pi} = \{\neg\epsilon_1, \neg\epsilon_2, \neg\epsilon_3\}$. Suppose that $\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \check{\pi}), (\check{\zeta}_1, \check{\xi}_1, \check{\pi}), (\check{\zeta}_2, \check{\xi}_2, \check{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) \tilde{\tilde{\in}} B_PSS(\check{\amalg})$ defined as follows:

$$(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_3\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_2, \hbar_3\}, \{\hbar_1, \hbar_4\}), (\epsilon_3, \{\hbar_1, \hbar_2\}, \{\hbar_3\})\} and$$
$$(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) = \{(\epsilon_1, \{\hbar_3, \hbar_4\}, \{\hbar_1, \hbar_2\}), (\epsilon_2, \{\hbar_1, \hbar_2, \hbar_3\}, \{\hbar_4\}), (\epsilon_3, \{\hbar_1, \hbar_4\}, \phi)\}.$$

Thus, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space since there does not exist $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \check{\pi})$.

**Example 4.5.** Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}$ and $\check{\pi} = \{\epsilon_1, \epsilon_2\}$. So, the $B_PSWS$ $\widetilde{\widetilde{\widetilde{W}}}$ over $\check{\amalg}$ is given by $\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \check{\pi}), (\check{\zeta}_1, \check{\xi}_1, \check{\pi}), (\check{\zeta}_2, \check{\xi}_2, \check{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) \tilde{\tilde{\in}} B_PSS(\check{\amalg})$ defined as follows:

$$(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_1\}, \{\hbar_2\})\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) = \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_2, \hbar_3\}, \{\hbar_1\})\}.$$

Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected since $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \check{\pi})$ form a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \check{\pi})$.

**Proposition 4.6.** *Let* $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) : \delta \in \Delta\}$ *be a family of* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-connected sets such that* $\tilde{\tilde{\bigcap}}_{\delta \in \Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) \neq (\Phi, \check{\xi}, \check{\pi})$. *Then* $\tilde{\tilde{\bigcup}}_{\delta \in \Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ *is* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-connected.*

**Proof.** Suppose $(\check{\delta}, \check{\gamma}, \check{\pi}) = \tilde{\tilde{\bigcup}}_{\delta \in \Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ is not $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected. Then, there exist two nonnull disjoint $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-open sets $(\check{\delta}_1, \check{\gamma}_1, \check{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \check{\pi})$ such that $(\check{\delta}, \check{\gamma}, \check{\pi}) = (\check{\delta}_1, \check{\gamma}_1, \check{\pi}) \tilde{\tilde{\cup}} (\check{\delta}_2, \check{\gamma}_2, \check{\pi})$. For each $\delta \in \Delta$, $(\check{\delta}_1, \check{\gamma}_1, \check{\pi}) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \check{\pi}) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ are disjoint $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-open sets in $(\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ such that

$$\big((\check{\delta}_1, \check{\gamma}_1, \check{\pi}) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})\big) \tilde{\tilde{\cup}} \big((\check{\delta}_2, \check{\gamma}_2, \check{\pi}) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})\big)$$
$$= \big((\check{\delta}_1, \check{\gamma}_1, \check{\pi}) \tilde{\tilde{\cup}} (\check{\delta}_2, \check{\gamma}_2, \check{\pi})\big) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) = (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}).$$

Now, $(\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected sets, one of the $B_PSSs$ $(\check{\delta}_1, \check{\gamma}_1, \check{\pi}) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \check{\pi}) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ is a null $B_PSSs$, say, $(\check{\delta}_1, \check{\gamma}_1, \check{\pi}) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) = (\Phi, \check{\gamma}, \check{\pi})$. Then, $(\check{\delta}_2, \check{\gamma}_2, \check{\pi}) \tilde{\tilde{\cap}} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) = (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi})$ which implies that $(\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) \tilde{\tilde{\subseteq}} (\check{\delta}_2, \check{\gamma}_2, \check{\pi})$ for all $\delta \in \Delta$ and hence $\tilde{\tilde{\bigcup}}_{\delta \in \Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \check{\pi}) \tilde{\tilde{\subseteq}} (\check{\delta}_2, \check{\gamma}_2, \check{\pi})$, that is, $(\check{\delta}_1, \check{\gamma}_1, \check{\pi}) \tilde{\tilde{\cup}} (\check{\delta}_2, \check{\gamma}_2, \check{\pi}) \tilde{\tilde{\subseteq}} (\check{\delta}_2, \check{\gamma}_2, \check{\pi})$. This gives, $(\check{\delta}_1, \check{\gamma}_1, \check{\pi}) = (\Phi, \check{\gamma}, \check{\pi})$. This is a contradiction, because $(\check{\delta}_1, \check{\gamma}_1, \check{\pi})$ is nonnull $B_PSS$. Hence, $(\check{\delta}, \check{\gamma}, \check{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected. □

**Theorem 4.7.** *A $B_PSWS$ $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ over $\breve{\Pi}$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space if and only if there are two $B_PSW$-closed sets $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ and $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ such that $\breve{\xi}_1(\neg\epsilon) \neq \phi$, $\breve{\xi}_2(\neg\epsilon) \neq \phi$ for some $\neg\epsilon \in \neg\breve{\pi}$, and $\breve{\xi}_1(\neg\epsilon)$ $\cup$ $\breve{\xi}_2(\neg\epsilon) = \breve{\Pi}$ for each $\neg\epsilon \in \neg\breve{\pi}$ and $\breve{\xi}_1(\neg\epsilon) \cap \breve{\xi}_2(\neg\epsilon) = \phi$ for each $\neg\epsilon \in \neg\breve{\pi}$.*

**Proof.** Suppose that $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected. Then, there exist $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\breve{\Pi}}}, \Phi, \breve{\pi})$, say, $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ and $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$. Then,

$$\begin{aligned}
&\breve{\zeta}_1(\epsilon) \cup \breve{\zeta}_2(\epsilon) = \breve{\Pi} \ for\ all\ \epsilon \in \breve{\pi},\\
&\breve{\zeta}_1(\epsilon) \cap \breve{\zeta}_2(\epsilon) = \phi \ for\ all\ \epsilon \in \breve{\pi}\ and\\
&\breve{\zeta}_1(\epsilon) \neq \phi, \breve{\zeta}_2(\epsilon) \neq \phi \ for\ some\ \epsilon \in \breve{\pi}.
\end{aligned}$$

Since $\breve{\zeta}_1(\epsilon) = \breve{\xi}_1^c(\neg\epsilon)$ and $\breve{\zeta}_2(\epsilon) = \breve{\xi}_2^c(\neg\epsilon)$. Now, we get

$$\begin{aligned}
&\breve{\xi}_1^c(\neg\epsilon) \cup \breve{\xi}_2^c(\neg\epsilon) = \breve{\Pi} \ for\ all\ \epsilon \in \breve{\pi},\\
&\breve{\xi}_1^c(\neg\epsilon) \cap \breve{\xi}_2^c(\neg\epsilon) = \phi \ for\ all\ \epsilon \in \breve{\pi}\ and\\
&\breve{\xi}_1^c(\epsilon) \neq \phi, \breve{\xi}_2(\epsilon) \neq \phi \ for\ some\ \epsilon \in \breve{\pi}.
\end{aligned}$$

From, $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$, $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}$, then $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})^c$ and $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})^c$ are $B_PSW$-closed sets.
Conversely, assuming that there are $B_PSW$-closed sets $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$, $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ such that

$$\begin{aligned}
&\breve{\xi}_1(\neg\epsilon) \cup \breve{\xi}_2(\neg\epsilon) = \breve{\Pi} \ for\ all\ \neg\epsilon \in \neg\breve{\pi},\\
&\breve{\xi}_1(\neg\epsilon) \cap \breve{\xi}_2(\neg\epsilon) = \phi \ for\ all\ \neg\epsilon \in \neg\breve{\pi}\ and\\
&\breve{\xi}_1(\neg\epsilon) \neq \phi, \breve{\xi}_2(\neg\epsilon) \neq \phi \ for\ some\ \neg\epsilon \in \neg\breve{\pi}.
\end{aligned}$$

Then $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})^c$, $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})^c$ are $B_PSW$-open sets such that

$$\begin{aligned}
&\breve{\zeta}_1^c(\epsilon) = \breve{\xi}_1(\neg\epsilon) \neq \phi \ and\ \breve{\zeta}_2^c(\epsilon) = \breve{\xi}_2(\neg\epsilon) \neq \phi \ for\ some\ \epsilon \in \breve{\pi},\\
&\breve{\zeta}_1^c(\epsilon) \cup \breve{\zeta}_2^c(\epsilon) = \breve{\xi}_1(\neg\epsilon) \cup \breve{\xi}_2(\neg\epsilon) = \breve{\Pi} \ for\ all\ \epsilon \in \breve{\pi}\ and\\
&\breve{\zeta}_1^c(\epsilon) \cap \breve{\zeta}_2^c(\epsilon) = \breve{\xi}_1(\neg\epsilon) \cap \breve{\xi}_2(\neg\epsilon) = \phi \ for\ all\ \epsilon \in \breve{\pi}.
\end{aligned}$$

Thus, $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})^c$ and $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})^c$ form $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\breve{\Pi}}}, \Phi, \breve{\pi})$. Thus, $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space. □

**Theorem 4.8.** *The $B_PS$ intersection of a pair of $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected spaces over a common universal set is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected.*

**Proof.** Let $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ and $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ be two $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_i$-connected spaces over $\breve{\Pi}$, $i = 1, 2$ and $\widetilde{\widetilde{\widetilde{W}}}$ $= \widetilde{\widetilde{\widetilde{W}}}_1 \,\tilde{\tilde{\cap}}\, \widetilde{\widetilde{\widetilde{W}}}_2$. We need to show that the space $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected. If we say that $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is not $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected. Then there exist two $B_PSSs$ $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$,$(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}$, which forms a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\breve{\Pi}}}, \Phi, \breve{\pi})$ in $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. From $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$, $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}$, then $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$, $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}_1$ and $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$, $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}_2$. This lead to $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ and $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ form a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$-separation of $(\tilde{\tilde{\breve{\Pi}}}, \Phi, \breve{\pi})$ in $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ and also $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ and $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ form a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$-separation of $(\tilde{\tilde{\breve{\Pi}}}, \Phi, \breve{\pi})$ in $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ which is the contradiction to given hypothesis. Therefore, $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space over $\breve{\Pi}$. □

**Remark 4.9.** *The $B_PS$ union of a pair of $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected spaces over the common universal set may not be $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected. As shown in the following example.*

**Example 4.10.** Let $\check{\amalg} = \{\hbar_1, \hbar_2\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$, $\widetilde{\widetilde{\widetilde{W}}}_1 = \{(\Phi, \widetilde{\widetilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\}$ and $\widetilde{\widetilde{\widetilde{W}}}_2 = \{(\Phi, \widetilde{\widetilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$, where

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \phi, \check{\amalg}), (\epsilon_2, \check{\amalg}, \phi)\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \check{\amalg}, \phi), (\epsilon_2, \phi, \check{\amalg})\}.$$

Clearly $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ and $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ are $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected spaces over $\check{\amalg}$ where $\widetilde{\widetilde{\widetilde{W}}} = \widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\cup}\, \widetilde{\widetilde{\widetilde{W}}}_2$. We note that $\widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\cup}\, \widetilde{\widetilde{\widetilde{W}}}_2 = \{(\Phi, \widetilde{\widetilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$ is not a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space over $\check{\amalg}$ since $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ form a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ in $\widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\cup}\, \widetilde{\widetilde{\widetilde{W}}}_2$.

**Proposition 4.11.** *The $B_PS$ union of a pair of $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected spaces over the common universal set is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected.*

**Proof.** Obvious. □

**Remark 4.12.** *The $B_PS$ intersection of a pair of $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected spaces over the common universal set is not necessarily a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space, as shown in the following example.*

**Example 4.13.** Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$,
$\widetilde{\widetilde{\widetilde{W}}}_1 = \{(\Phi, \widetilde{\widetilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$ and
$\widetilde{\widetilde{\widetilde{W}}}_2 = \{(\Phi, \widetilde{\widetilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi}), (\check{\zeta}_4, \check{\xi}_4, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, $(\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$, $(\check{\zeta}_4, \check{\xi}_4, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\amalg})$ defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_1, \hbar_2\}, \{\hbar_3\})\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \phi), (\epsilon_2, \{\hbar_3\}, \{\hbar_1\})\},$$
$$(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_3\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_1, \hbar_3\}, \{\hbar_2\})\} \textit{ and}$$
$$(\check{\zeta}_4, \check{\xi}_4, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1\})\}.$$

Clearly $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ and $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ are $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected spaces over $\check{\amalg}$ where $\widetilde{\widetilde{\widetilde{W}}} = \widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\widetilde{\cap}}\, \widetilde{\widetilde{\widetilde{W}}}_2$. We note that $\widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\widetilde{\cap}}\, \widetilde{\widetilde{\widetilde{W}}}_2 = \{(\Phi, \widetilde{\widetilde{\check{\amalg}}}, \breve{\pi})\}$ is not a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space over $\check{\amalg}$ since there is no two $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ in $\widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\widetilde{\cap}}\, \widetilde{\widetilde{\widetilde{W}}}_2$.

**Proposition 4.14.** *Let $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ be a $B_PSWS$ over $\check{\amalg}$. If there exist a nonnull, nonabsolute $B_PSW$-clopen set $(\check{\zeta}, \check{\xi}, \breve{\pi})$ over $\check{\amalg}$ with $\check{\zeta}(\epsilon) \cup \check{\zeta}^c(\epsilon) = \check{\amalg}$ for each $\epsilon \in \breve{\pi}$, then $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected.*

**Proof.** Since $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is a nonnull, nonabsolute $B_PSW$-clopen set, then $(\check{\zeta}, \check{\xi}, \breve{\pi})^c$ is a nonnull nonabsolute $B_PSW$-clopen set. By Proposition 2.12 and the assumption, we get

$$\check{\zeta}(\epsilon) \cup \check{\zeta}^c(\epsilon) = \check{\amalg}, \text{ for each } \epsilon \in \breve{\pi}, \text{ and } \check{\xi}(\neg\epsilon) \cap \check{\xi}^c(\neg\epsilon) = \phi, \text{ for each } \neg\epsilon \in \neg\breve{\pi},$$

and

$$\check{\zeta}(\epsilon) \cap \check{\zeta}^c(\epsilon) = \phi, \text{ for each } \epsilon \in \breve{\pi}, \text{ and } \check{\xi}(\neg\epsilon) \cup \check{\xi}^c(\neg\epsilon) = \check{\amalg}, \text{ for each } \neg\epsilon \in \neg\breve{\pi},$$

Therefore, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ and $(\check{\zeta}, \check{\xi}, \breve{\pi})^c$ form a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$. Hence, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space. □

**Remark 4.15.** *If there exist a nonnull, nonabsolute $B_PSW$-open set, $B_PSW$-closed set, then $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ may not be a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space. As shown in the following example.*

**Example 4.16.** Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and
$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \tilde{\tilde{\in}} B_PSS(\check{\amalg})$ defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_2\}, \{\hbar_3\}), (\epsilon_2, \{\hbar_1\}, \{\hbar_3\})\} and$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_3\}, \{\hbar_1, \hbar_2\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_1\})\}.$$

Obviously, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ is nonnull, nonabsolute $B_PSW$-clopen but $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is not a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space since there does not exist $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$.

**Proposition 4.17.** *Let $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ and $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ be two $B_PSWSs$ over $\check{\amalg}$. Then,*

*(i) If $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$-connected such that $\widetilde{\widetilde{\widetilde{W}}}_2 \tilde{\tilde{\subseteq}} \widetilde{\widetilde{\widetilde{W}}}_1$, then $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$-connected.*

*(ii) If $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$-disconnected such that $\widetilde{\widetilde{\widetilde{W}}}_1 \tilde{\tilde{\subseteq}} \widetilde{\widetilde{\widetilde{W}}}_2$, then $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$-disconnected.*

**Proof.**

(i) Assume that $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$-connected such that $\widetilde{\widetilde{\widetilde{W}}}_2 \tilde{\tilde{\subseteq}} \widetilde{\widetilde{\widetilde{W}}}_1$. Assume the contrary that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ in $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$. Since $\widetilde{\widetilde{\widetilde{W}}}_2 \tilde{\tilde{\subseteq}} \widetilde{\widetilde{\widetilde{W}}}_1$, then $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ in $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$. This is contradiction. Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$-connected.

(ii) Let $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ be a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$-disconnected such that $\widetilde{\widetilde{\widetilde{W}}}_1 \tilde{\tilde{\subseteq}} \widetilde{\widetilde{\widetilde{W}}}_2$. Assume the contrary that $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$-connected space. Since $\widetilde{\widetilde{\widetilde{W}}}_1 \tilde{\tilde{\subseteq}} \widetilde{\widetilde{\widetilde{W}}}_2$, then by (i), we get $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$-connected. This is contradiction. Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$-disconnected.

□

**Proposition 4.18.** *Let $\big((\check{\zeta}, \check{\xi}, \breve{\pi}), \widetilde{\widetilde{\widetilde{W}}}_{(\check{\zeta}, \check{\xi}, \breve{\pi})}, \breve{\pi}, \neg\breve{\pi}\big)$ be $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected, then $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected.*

**Proof.** Let $\big((\check{\zeta}, \check{\xi}, \breve{\pi}), \widetilde{\widetilde{\widetilde{W}}}_{(\check{\zeta}, \check{\xi}, \breve{\pi})}, \breve{\pi}, \neg\breve{\pi}\big)$ be a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space. Assume $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected, then there exist $\widetilde{\widetilde{\widetilde{W}}}$-separated $B_PSSs$, say, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ of $(\check{\zeta}, \check{\xi}, \breve{\pi})$, so by Theorem 3.7 that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\check{\zeta}, \check{\xi}, \breve{\pi})$. This is a contradiction. Thus, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space. □

**Definition 4.19.** A property $\mathcal{P}$ of a $B_PSWS$ $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ is said to be a $B_PS$ weak hereditary property ($B_PS\widetilde{\widetilde{\widetilde{W}}}$- hereditary property) if every $B_PSWSS$ $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \check{\pi}, \neg\check{\pi})$ of $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ also has the property $\mathcal{P}$.

**Remark 4.20.** *The $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space (resp. $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space) is not necessarily a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-hereditary property. As shown in the following example.*

**Example 4.21.** Let $\check{\Pi} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\check{\pi} = \{\epsilon_1, \epsilon_2\}$ and
$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\tilde{\Pi}}}, \check{\pi}), (\check{\zeta}_1, \check{\xi}_1, \check{\pi}), (\check{\zeta}_2, \check{\xi}_2, \check{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) \,\tilde{\tilde{\in}}\, B_PSS(\check{\Pi})$, defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2, \hbar_3\}), (\epsilon_2, \{\hbar_1\}, \{\hbar_2, \hbar_3\})\} \textit{ and}$$
$$(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1, \hbar_3\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1, \hbar_3\})\}.$$

Therefore, $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space.
Now let $\check{Y} = \{\hbar_1, \hbar_2\}$, then $\widetilde{\widetilde{\widetilde{W}}}_{\check{Y}} = \{(\Phi, \tilde{\tilde{\check{Y}}}, \check{\pi}), ({}^{\check{Y}}\check{\zeta}_1, {}^{\check{Y}}\check{\xi}_1, \check{\pi}), ({}^{\check{Y}}\check{\zeta}_2, {}^{\check{Y}}\check{\xi}_2, \check{\pi})\}$, such that

$$({}^{\check{Y}}\check{\zeta}_1, {}^{\check{Y}}\check{\xi}_1, \check{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_1\}, \{\hbar_2,\})\} \textit{ and}$$
$$({}^{\check{Y}}\check{\zeta}_2, {}^{\check{Y}}\check{\xi}_2, \check{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1\})\}.$$

Clearly, $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \check{\pi}, \neg\check{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected subspace of $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$. While $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space.

**Example 4.22.** Let $\check{\Pi} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\check{\pi} = \{\epsilon_1, \epsilon_2\}$ and
$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\tilde{\Pi}}}, \check{\pi}), (\check{\zeta}_1, \check{\xi}_1, \check{\pi}), (\check{\zeta}_2, \check{\xi}_2, \check{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) \,\tilde{\tilde{\in}}\, B_PSS(\check{\Pi})$, defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1, \hbar_3\})\} \textit{ and}$$
$$(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) = \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \phi), (\epsilon_2, \{\hbar_1, \hbar_3\}, \{\hbar_2\})\}.$$

Therefore, $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space.
Let $\check{Y} = \{\hbar_3\}$, then $\widetilde{\widetilde{\widetilde{W}}}_{\check{Y}} = \{(\Phi, \tilde{\tilde{\check{Y}}}, \check{\pi}), ({}^{\check{Y}}\check{\zeta}_1, {}^{\check{Y}}\check{\xi}_1, \check{\pi}), ({}^{\check{Y}}\check{\zeta}_2, {}^{\check{Y}}\check{\xi}_2, \check{\pi})\}$, such that

$$({}^{\check{Y}}\check{\zeta}_1, {}^{\check{Y}}\check{\xi}_1, \check{\pi}) = \{(\epsilon_1, \phi, \phi), (\epsilon_2, \phi, \check{Y})\},$$
$$({}^{\check{Y}}\check{\zeta}_2, {}^{\check{Y}}\check{\xi}_2, \check{\pi}) = \{(\epsilon_1, \check{Y}, \phi), (\epsilon_2, \check{Y}, \phi)\}.$$

Clearly, $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \check{\pi}, \neg\check{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected subspace of $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$. While $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected space.

## 5 $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-Locally Connected Spaces and $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-Components

In this section, a new type of connected set is studied, known as $\widetilde{\widetilde{\widetilde{W}}}$-locally connected in $B_PSWS$. Furthermore, $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-component with some properties.

**Definition 5.1.** *A $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-component of $B_PSWS$ $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ corresponding to $\hbar_\upsilon^\epsilon$ is the $B_PS$ union of all $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected $(\check{\zeta}, \check{\xi}, \check{\pi}) \,\tilde{\tilde{\subseteq}}\, (\tilde{\tilde{\tilde{\Pi}}}, \Phi, \check{\pi})$ which contains $\hbar_\upsilon^\epsilon$. It is denoted by $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$ that is*

$$B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) = \widetilde{\widetilde{\bigcup}}\{(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi}) : \hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, (\check{\zeta}, \check{\xi}, \breve{\pi}) \text{ and } (\check{\zeta}, \check{\xi}, \breve{\pi}) \text{ is } B_PS\ \widetilde{\widetilde{W}}\text{-connected}\}.$$

**Definition 5.2.** *A $B_PSWS$ $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is called $B_PS$ $\widetilde{\widetilde{W}}$-locally connected at $\hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ if for every $B_PS$ $\widetilde{\widetilde{W}}$-open set $(\check{\zeta}, \check{\xi}, \breve{\pi})$ containing $\hbar_\upsilon^\epsilon$, there is a $B_PS$ $\widetilde{\widetilde{W}}$-connected open $(\check{\delta}, \check{\gamma}, \breve{\pi})$ containing $\hbar_\upsilon^\epsilon$ such that $\hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, (\check{\delta}, \check{\gamma}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta}, \check{\xi}, \breve{\pi})$. A $B_PSWS$ $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is said to be $B_PS$ $\widetilde{\widetilde{W}}$-locally connected if it is $B_PS$ $\widetilde{\widetilde{W}}$-locally connected at every $B_PSP$ $\hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$. Otherwise, it is said to be $B_PS$ $\widetilde{\widetilde{W}}$-locally disconnected.*

**Remark 5.3.** *$B_PS$ $\widetilde{\widetilde{W}}$-locally connectedness and $B_PS$ $\widetilde{\widetilde{W}}$-connectedness are independent as shown below.*

**Example 5.4.** Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and $\widetilde{\widetilde{W}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi}), (\check{\zeta}_4, \check{\xi}_4, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, $(\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$, $(\check{\zeta}_4, \check{\xi}_4, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\amalg})$, defined as follows

$$\begin{aligned}
&(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_2\}, \phi)\},\\
&(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \phi)\},\\
&(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon_1, \check{\amalg}, \phi), (\epsilon_2, \phi, \check{\amalg})\} \text{ and}\\
&(\check{\zeta}_4, \check{\xi}_4, \breve{\pi}) = \{(\epsilon_1, \phi, \check{\amalg}), (\epsilon_2, \check{\amalg}, \phi)\}.
\end{aligned}$$

Then $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-locally connected space but not $B_PS$ $\widetilde{\widetilde{W}}$-connected.

**Example 5.5.** Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and $\widetilde{\widetilde{W}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi}), (\check{\zeta}_4, \check{\xi}_4, \breve{\pi})\}$ where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, $(\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$, $(\check{\zeta}_4, \check{\xi}_4, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\amalg})$, defined as follows

$$\begin{aligned}
&(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_3\})\},\\
&(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_2\}, \phi)\},\\
&(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_3\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1, \hbar_3\})\},\\
&(\check{\zeta}_4, \check{\xi}_4, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_2\}, \phi), (\epsilon_2, \{\hbar_2\}, \{\hbar_1, \hbar_3\})\}.
\end{aligned}$$

Then, $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-connected space but not $B_PS$ $\widetilde{\widetilde{W}}$-locally connected because $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ is the $B_PS$ $\widetilde{\widetilde{W}}$-open set containing $\hbar_{1_{\hbar_2}}^{\epsilon_1}$, but there is no $B_PS$ $\widetilde{\widetilde{W}}$-connected open subset of $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ containing $\hbar_{1_{\hbar_2}}^{\epsilon_1}$.

**Remark 5.6.** *For a $B_PSWS$ $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$, we have*

(i) *According to Proposition 4.6, every $B_PS$ $\widetilde{\widetilde{W}}$-component of a $B_PSP$ is the largest $B_PS$ $\widetilde{\widetilde{W}}$-connected set containing this $B_PSP$.*

(ii) *If $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{W}}$-connected space, then $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ is only the $B_PS$ $\widetilde{\widetilde{W}}$-component of each $B_PSP$.*

**Example 5.7.** Consider the $B_PSWS$ in Example 5.4, we have the following:

$B_PSW\text{-}Co(\hbar_{1_{\hbar_2}}^{\epsilon_1}) = B_PSW\text{-}Co(\hbar_{1_{\hbar_3}}^{\epsilon_1}) = B_PSW\text{-}Co(\hbar_{2_{\hbar_1}}^{\epsilon_1}) = B_PSW\text{-}Co(\hbar_{2_{\hbar_3}}^{\epsilon_1}) = B_PSW\text{-}Co(\hbar_{3_{\hbar_1}}^{\epsilon_1}) = B_PSW\text{-}Co(\hbar_{3_{\hbar_2}}^{\epsilon_1}) = (\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$ and
$B_PSW\text{-}Co(\hbar_{1_{\hbar_2}}^{\epsilon_2}) = B_PSW\text{-}Co(\hbar_{1_{\hbar_3}}^{\epsilon_2}) = B_PSW\text{-}Co(\hbar_{2_{\hbar_1}}^{\epsilon_2}) = B_PSW\text{-}Co(\hbar_{2_{\hbar_3}}^{\epsilon_2}) = B_PSW\text{-}Co(\hbar_{3_{\hbar_1}}^{\epsilon_2}) = B_PSW\text{-}Co(\hbar_{3_{\hbar_2}}^{\epsilon_2}) = (\check{\zeta}_4, \check{\xi}_4, \breve{\pi})$.

**Theorem 5.8.** *A $B_PSWS$ $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-locally connected if and only if the $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-components of $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-open sets are $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-open sets.*

**Proof.** Assume that the space $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-locally connected. Let $(\check{\zeta}, \check{\xi}, \breve{\pi})$ be $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-open and $B_PSW\text{-}Co$ be a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-component of $(\check{\zeta}, \check{\xi}, \breve{\pi})$. If $\hbar^{\epsilon}_{\upsilon} \tilde{\tilde{\in}} B_PSW\text{-}Co$ and since $\hbar^{\epsilon}_{\upsilon} \in (\check{\zeta}, \check{\xi}, \breve{\pi})$, there is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected open set $(\check{\delta}, \check{\gamma}, \breve{\pi})$ such that $\hbar^{\epsilon}_{\upsilon} \tilde{\tilde{\in}} (\check{\delta}, \check{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\check{\zeta}, \check{\xi}, \breve{\pi})$. Now, as $B_PSW\text{-}Co$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-component of $\hbar^{\epsilon}_{\upsilon}$ and $(\check{\delta}, \check{\gamma}, \breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected, we have $\hbar^{\epsilon}_{\upsilon} \tilde{\tilde{\in}} (\check{\delta}, \check{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} B_PSW\text{-}Co$. This shows that $B_PSW\text{-}Co$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-open.

Conversely, let $\hbar^{\epsilon}_{\upsilon} \tilde{\tilde{\in}} (\tilde{\tilde{\check{\Pi}}}, \Phi, \breve{\pi})$ be arbitrary and $(\check{\zeta}, \check{\xi}, \breve{\pi})$ be a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-open set containing $\hbar^{\epsilon}_{\upsilon}$. Suppose $B_PSW\text{-}Co$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-component of $(\check{\zeta}, \check{\xi}, \breve{\pi})$ such that $\hbar^{\epsilon}_{\upsilon} \tilde{\tilde{\in}} B_PSW\text{-}Co$. Now, $B_PSW\text{-}Co$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected open set with $\hbar^{\epsilon}_{\upsilon} \tilde{\tilde{\in}} B_PSW\text{-}Co \tilde{\tilde{\subseteq}} (\check{\zeta}, \check{\xi}, \breve{\pi})$. This proves the theorem. □

**Theorem 5.9.** *Let $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ be a $B_PSWS$, then*

*(i) The family of all distinct $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-components of a $B_PSPs$ of $(\tilde{\tilde{\check{\Pi}}}, \Phi, \breve{\pi})$ forms a partition of $(\tilde{\tilde{\check{\Pi}}}, \Phi, \breve{\pi})$.*

*(ii) For every $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-component $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon})$, we have $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon}) = B_PSW\text{-}clB_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon})$.*

**Proof.**

(*i*) Let $\{B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon}):\ \hbar^{\epsilon}_{\upsilon} \tilde{\tilde{\in}} (\tilde{\tilde{\check{\Pi}}}, \Phi, \breve{\pi})\}$ be a family of all distinct $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-components of $(\tilde{\tilde{\check{\Pi}}}, \Phi, \breve{\pi})$. Clearly, $(\tilde{\tilde{\check{\Pi}}}, \Phi, \breve{\pi}) = \tilde{\tilde{\bigcup}} \{B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon}) : \hbar^{\epsilon}_{\upsilon} \tilde{\tilde{\in}} (\tilde{\tilde{\check{\Pi}}}, \Phi, \breve{\pi})\}$. Suppose that there are two distinct $B_PSPs$ $\hbar^{\epsilon}_{\upsilon}$ and $\hbar'^{\epsilon'}_{\upsilon'}$ such that $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon}) \tilde{\tilde{\cap}} B_PSW\text{-}Co(\hbar'^{\epsilon'}_{\upsilon'}) \neq (\Phi, \check{\xi}, \breve{\pi})$. By Proposition 4.6, $(\check{\zeta}, \check{\xi}, \breve{\pi}) = B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon}) \tilde{\tilde{\cup}} B_PSW\text{-}Co(\hbar'^{\epsilon'}_{\upsilon'})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected set. This contradicts that $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon})$ and $B_PSW\text{-}Co(\hbar'^{\epsilon'}_{\upsilon'})$ are the largest $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected sets containing $\hbar^{\epsilon}_{\upsilon}$ and $\hbar'^{\epsilon'}_{\upsilon'}$ respectively. Hence $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon}) \tilde{\tilde{\cap}} B_PSW\text{-}Co(\hbar'^{\epsilon'}_{\upsilon'}) = (\Phi, \check{\xi}, \breve{\pi})$.

(*ii*) Since $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected and $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon}) \tilde{\tilde{\subseteq}} B_PSW\text{-}clB_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon})$, it follows since Proposition 3.18 that $B_PSW\text{-}clB_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected set also. Since $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon})$ is the largest $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-connected set containing $\hbar^{\epsilon}_{\upsilon}$ . Hence, $B_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon}) = B_PSW\text{-}clB_PSW\text{-}Co(\hbar^{\epsilon}_{\upsilon})$.

□

## 6 $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-Compact Spaces

Because of the compactness property importance, this section researches it in $B_PSWSs$ with some essential theorems.

**Definition 6.1.** *A family $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}):\ (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \tilde{\tilde{\in}} \widetilde{\widetilde{\widetilde{W}}}\}_{\delta\in\Delta}$ of $B_PSW$-open sets on $\check{\Pi}$ is said to be a $B_PSW$-open cover of a $B_PSS$ $(\check{\zeta}, \check{\xi}, \breve{\pi})$ if, $(\check{\zeta}, \check{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} \tilde{\tilde{\cup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$. Furthermore, a $B_PS$ subcover is a subfamily of $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ which is also a $B_PSW$-open cover.*

**Definition 6.2.** *A $B_PS$ subset $(\check{\zeta}, \check{\xi}, \breve{\pi})$ of $(\tilde{\tilde{\check{\text{Ц}}}}, \Phi, \breve{\pi})$ is known as a bipolar soft weak compact set, denoted as a $B_PS$ $\widetilde{\widetilde{W}}$-compact set, if each $B_PSW$-open cover of $(\check{\zeta}, \check{\xi}, \breve{\pi})$ has a finite $B_PS$ subcover.*

**Definition 6.3.** *A $B_PSWS$ $(\check{\text{Ц}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is said to be $B_PS$ $\widetilde{\widetilde{W}}$-compact space if $(\tilde{\tilde{\check{\text{Ц}}}}, \Phi, \breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{W}}$-compact subset of itself.*

**Example 6.4.** Let $\check{\text{Ц}} = \mathbb{N}$ be the set of all natural numbers, $\breve{\pi} = \{\epsilon\}$ and $\widetilde{\widetilde{W}} = \{(\Phi, \tilde{\tilde{\check{\text{Ц}}}}, \breve{\pi}), (\check{\zeta}_n, \check{\xi}_n, \breve{\pi}) : n \in \mathbb{N}\}$ where,

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon, \{1, 2\}, \mathbb{N} \setminus \{1, 2\})\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon, \{1, 3\}, \mathbb{N} \setminus \{1, 3\})\},$$
$$(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon, \{1, 4\}, \mathbb{N} \setminus \{1, 4\})\},$$
$$\vdots$$

Then, a $B_PSWS$ $(\check{\text{Ц}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is not $B_PS$ $\widetilde{\widetilde{W}}$-compact, since $(\tilde{\tilde{\check{\text{Ц}}}}, \Phi, \breve{\pi}) = \tilde{\tilde{\cup}}_{n \in \mathbb{N}} (\check{\zeta}_n, \check{\xi}_n, \breve{\pi})$. Whereas $(\tilde{\tilde{\check{\text{Ц}}}}, \Phi, \breve{\pi}) \neq \tilde{\tilde{\cup}}_{i=1}^{n} (\check{\zeta}_i, \check{\xi}_i, \breve{\pi})$.

**Remark 6.5.** *Let $\check{\text{Ц}}$ be a finite universe set and let $(\tilde{\tilde{\check{\text{Ц}}}}, \Phi, \breve{\pi}) \tilde{\tilde{\notin}} \widetilde{\widetilde{W}}$. If $B_PS$ union of some $B_PSW$-open sets is $(\tilde{\tilde{\check{\text{Ц}}}}, \Phi, \breve{\pi})$, then $(\tilde{\tilde{\check{\text{Ц}}}}, \Phi, \breve{\pi})$ is $B_PSW$-compact space.*

**Example 6.6.** Let $\check{\text{Ц}} = \{\hbar_1, \hbar_2, \hbar_3\}, \breve{\pi} = \{\epsilon\}$ and

$$\widetilde{\widetilde{W}} = \{(\Phi, \tilde{\tilde{\check{\text{Ц}}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi})\}$$

be a $B_PSWS$ over $\check{\text{Ц}}$ where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, $(\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\text{Ц}})$, defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon, \{\hbar_1\}, \{\hbar_2\})\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon, \{\hbar_2\}, \{\hbar_3\})\} \, and$$
$$(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon, \{\hbar_3\}, \{\hbar_1\})\}.$$

Then, a $B_PSWS$ $(\check{\text{Ц}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{W}}$-compact.

**Remark 6.7.** *Every $B_PSW$-closed subset of a $B_PS$ $\widetilde{\widetilde{W}}$-compact space is not necessarily $B_PS$ $\widetilde{\widetilde{W}}$-compact.*

**Example 6.8.** Let $\check{\text{Ц}} = \mathbb{N}$, $\breve{\pi} = \{\epsilon\}$ and

$$\widetilde{\widetilde{W}} = \{(\Phi, \tilde{\tilde{\check{\text{Ц}}}}, \breve{\pi}), (\check{\zeta}_0, \check{\xi}_0, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_n, \check{\xi}_n, \breve{\pi}) : n \in \mathbb{N} \setminus \{1\}\}$$

be a $B_PSWS$ over $\check{\text{Ц}}$ where $(\check{\zeta}_0, \check{\xi}_0, \breve{\pi})$, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_n, \check{\xi}_n, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\text{Ц}})$, defined as follows

$$(\check{\zeta}_0, \check{\xi}_0, \breve{\pi}) = \{(\epsilon, \{1, 4, 5, ...\}, \{2, 3\})\},$$
$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon, \{1\}, \{2, 3, 4, 5, ...\})\} \, and$$
$$(\check{\zeta}_n, \check{\xi}_n, \breve{\pi}) = \{(\epsilon, \{2, 3, 4, 5, ..., n\}, \{1\}) : n \in \mathbb{N} \setminus \{1\}\}.$$

Then, a $B_PSWS$ $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-compact. But the $B_PSW$-closed set $\{(\epsilon, \{2,3,4,5,...\}, \{1\})\}$ is not a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-compact set, since the family $\{(\check{\zeta}_n, \check{\xi}_n, \breve{\pi}) : n \in \mathbb{N} \setminus \{1\}\}$ is a $B_PSW$-open cover of $\{(\epsilon, \{2,3,4,5,...\}, \{1\})\}$. That is

$$\{(\epsilon, \{2,3,4,5,...\}, \{1\})\} \stackrel{\approx}{\subseteq} \widetilde{\widetilde{\cup}}_{n\in\mathbb{N}\setminus\{1\}} (\check{\zeta}_n, \check{\xi}_n, \breve{\pi}).$$

Then this $B_PSW$-open cover has no finite $B_PS$ subcover. That is

$$\{(\epsilon, \{2,3,4,5,...\}, \{1\})\} \stackrel{\approx}{\nsubseteq} \widetilde{\widetilde{\cup}}_{n=2}^{k} (\check{\zeta}_n, \check{\xi}_n, \breve{\pi}),\ for\ k \in \mathbb{N} \setminus \{1\}.$$

**Proposition 6.9.** *If* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *is a* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-compact space, then* $(\check{\amalg}, \widetilde{W}, \breve{\pi})$ *is an* $S$ $\widetilde{W}$*-compact space.*

**Proof.** Straightforward. □

**Proposition 6.10.** *If* $(\check{\amalg}, \widetilde{W}, \breve{\pi})$ *is an* $S$ $\widetilde{W}$*-compact space and* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *is a* $B_PSWS$ *constructed since Theorem 2.24, then* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *is a* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$*-compact space.*

**Proof.** Let $(\check{\amalg}, \widetilde{W}, \breve{\pi})$ be an $S$ $\widetilde{W}$-compact space and $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ be a $B_PSW$-open cover of $(\tilde{\tilde{\tilde{\amalg}}}, \Phi, \breve{\pi})$. That is

$$(\tilde{\tilde{\tilde{\amalg}}}, \Phi, \breve{\pi}) \stackrel{\approx}{\subseteq} \widetilde{\widetilde{\cup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}).$$

Then, $\check{\amalg} = \cup\{\check{\zeta}_\delta(\epsilon)\}_{\delta\in\Delta}$ for all $\epsilon \in \breve{\pi}$. Since $(\check{\amalg}, \widetilde{W}, \breve{\pi})$ is an $S$ $\widetilde{W}$-compact space, then $\check{\amalg} = \cup\{\check{\zeta}_{\delta_i}(\epsilon) : i = 1,2,..,n\}_{\delta_i\in\Delta}$. Since $\check{\xi}(\neg\epsilon) = \check{\amalg} \setminus \check{\zeta}(\epsilon)$ for all $\epsilon \in \breve{\pi}$, then $\Phi = \cap\{\check{\xi}_{\delta_i}(\neg\epsilon) : i = 1,2,...,n\}_{\delta_i\in\Delta}$. Hence, $(\tilde{\tilde{\tilde{\amalg}}}, \Phi, \breve{\pi}) \stackrel{\approx}{\subseteq} \widetilde{\widetilde{\cup}}_{i=1}^{n} (\check{\zeta}_{\delta_i}, \check{\xi}_{\delta_i}, \breve{\pi})$. Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}$-compact. □

**Theorem 6.11.** *Let* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ *and* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ *be* $B_PSWSs$*. Then*

- (*i*) *If* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ *is a* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$*-compact space on* $\check{\amalg}$ *and* $\widetilde{\widetilde{\widetilde{W}}}_1 \stackrel{\approx}{\subseteq} \widetilde{\widetilde{\widetilde{W}}}_2$*. Then* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ *is a* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$*-compact space on* $\check{\amalg}$*.*
- (*ii*) *If* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ *is not* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$*-compact space on* $\check{\amalg}$ *and* $\widetilde{\widetilde{\widetilde{W}}}_1 \stackrel{\approx}{\subseteq} \widetilde{\widetilde{\widetilde{W}}}_2$*. Then* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ *is also not* $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$*-compact space on* $\check{\amalg}$*.*

**Proof.**

(*i*) Let $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ be a $B_PSW_1$-open cover of $(\tilde{\tilde{\tilde{\amalg}}}, \Phi, \breve{\pi})$ in $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$. Since $\widetilde{\widetilde{\widetilde{W}}}_1 \stackrel{\approx}{\subseteq} \widetilde{\widetilde{\widetilde{W}}}_2$, then $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ is the $B_PSW_2$-open cover of $(\tilde{\tilde{\tilde{\amalg}}}, \Phi, \breve{\pi})$ by the $B_PSW_2$-open sets of $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$. Since $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_2$-compact space. Thus,

$$(\tilde{\tilde{\tilde{\amalg}}}, \Phi, \breve{\pi}) \stackrel{\approx}{\subseteq} \widetilde{\widetilde{\bigcup}}_{\delta=1}^{n} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}), \text{ for some } \delta_1, \delta_2, ..., \delta_n \in \Delta.$$

Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS$ $\widetilde{\widetilde{\widetilde{W}}}_1$-compact space.

(*ii*) Let $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ be not $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_1$-compact space on $\check{\text{Ï}}$ and $\widetilde{\widetilde{\widetilde{W}}}_1 \mathrel{\widetilde{\widetilde{\subseteq}}} \widetilde{\widetilde{\widetilde{W}}}_2$. Assume if possible $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_2$-compact space on $\check{\text{Ï}}$. By (*i*), $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ is also a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_1$-compact. This is a contradiction. Hence, $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is not $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_2$-compact space on $\check{\text{Ï}}$.

□

**Theorem 6.12.** *Let* $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ *be a* $B_PSWSS$ *of* $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. *Then* $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ *is a* $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$*-compact space if and only if every* $B_PSW$*-open cover of* $(\widetilde{\widetilde{\check{Y}}}, \Phi, \breve{\pi})$ *by* $B_PSW$*-open set in* $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *contains a finite* $B_PS$ *subcover.*

**Proof.** Let $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ be a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-compact space and $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ be a $B_PSW$-open cover of $(\widetilde{\widetilde{\check{Y}}}, \Phi, \breve{\pi})$ by $B_PSW$-open set in $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Now, $\check{Y} \subseteq \bigcup_{\delta\in\Delta} \big(\check{Y} \cap \check{\zeta}_\delta(\epsilon)\big)$ for each $\epsilon \in \breve{\pi}$ and $\phi \supseteq \bigcap_{\delta\in\Delta} \big(\check{Y} \cap \check{\xi}_\delta(\neg\epsilon)\big)$ for each $\neg\epsilon \in \neg\breve{\pi}$. Thus, $\check{\Delta}_Y = \{(^{\check{Y}}\check{\zeta}_\delta, ^{\check{Y}}\check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-open cover of $(\widetilde{\widetilde{\check{Y}}}, \Phi, \breve{\pi})$. Since $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-compact space, then there is a finite $B_PS$ subcover, say, $\{(^{\check{Y}}\check{\zeta}_{\delta_i}, ^{\check{Y}}\check{\xi}_\delta, \breve{\pi})\}_{i=1}^n$ such that,

$$(\widetilde{\widetilde{\check{Y}}}, \Phi, \breve{\pi}) \mathrel{\widetilde{\widetilde{\subseteq}}} \widetilde{\widetilde{\bigcup}}_{i=1}^n (^{\check{Y}}\check{\zeta}_{\delta_i}, ^{\check{Y}}\check{\xi}_{\delta_i}, \breve{\pi}), \text{ for some } \delta_1, \delta_2, ..., \delta_n \in \Delta.$$

Thus, implies that $\big\{(\check{\zeta}_{\delta_i}, \check{\xi}_{\delta_i}, \breve{\pi})\big\}_{i=1}^n$ is a finite $B_PS$ subcover of $(\widetilde{\widetilde{\check{Y}}}, \Phi, \breve{\pi})$ by $B_PSW$-open set in $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Conversely, suppose $\check{\Delta}_{\check{Y}} = \{(^{\check{Y}}\check{\zeta}_\delta, ^{\check{Y}}\check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ which is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-open cover of $(\widetilde{\widetilde{\check{Y}}}, \Phi, \breve{\pi})$. Then, clearly $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ is a $B_PSW$-open cover of $(\widetilde{\widetilde{\check{Y}}}, \Phi, \breve{\pi})$ by $B_PSW$-open set in $(\check{\text{Ï}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Thus, by given hypothesis we have $\{(\check{\zeta}_{\delta_i}, \check{\xi}_\delta, \breve{\pi})\}_{i=1}^n$ is a finite $B_PS$ subcover of $(\widetilde{\widetilde{\check{Y}}}, \Phi, \breve{\pi})$. Therefore, $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-compact space. □

# 7 Conclusions And Future Research

The aim of this paper was to define a new bipolar soft weak structure named bipolar soft weak connectedness and bipolar soft weak compactness, and to introduce the principles of bipolar soft weak locally connected and bipolar soft weak component. The fundamental concepts of $B_PSWS$, which are related to bipolar soft sets, are continuously presented and explored, as well as the definitions and examples needed to explain the concepts. Additionally, the paper has invalidated some $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-locally connected space and $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-component features in $BSWSs$. We provided a definition, demonstrated the way the ideas of $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-locally connected spaces and $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected are distinct, and explored the ways in which the $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected subsets are $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-components. Therefore, the main definitions and results of compactness in $B_PSWSs$ were demonstrated.

Future research on bipolar soft weak structures may focus on several key areas, including continuous mappings and separation axioms.

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