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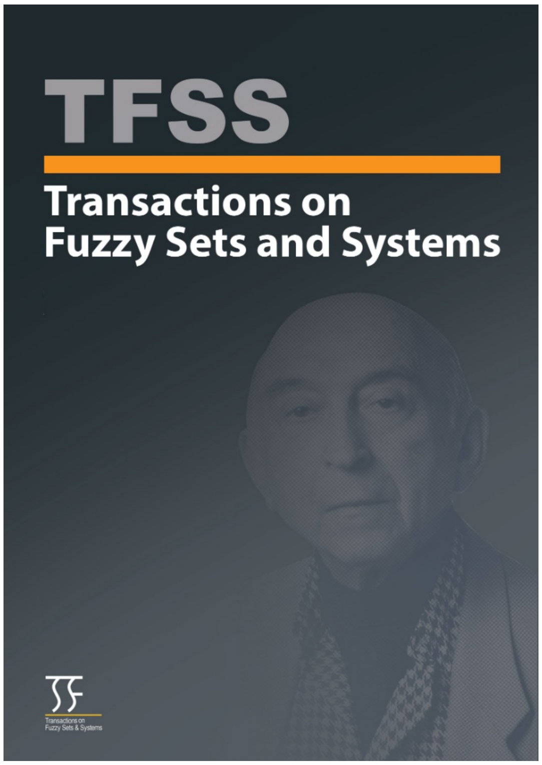

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Author(s):

Bekas S. M.Taher, Department of Mathematics, University of Zakho, Zakho, Iraq.
E-mail: bekes81math@gmail.com
Baravan A. Asaad, Department of Mathematics, University of Zakho, Zakho, Iraq. Department of Computer Science, Cihan University, Duhok, Iraq. E-mail: baravan.asaad@uoz.edu.krd

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Structural Aspects of Weak Connectedness and Weak Compactness in Bipolar Soft Weak Structures

Bekas S. M. Taher , **Baravan A. Asaad***

Abstract. Connectedness and compactness are two essential properties that characterize the structural behavior of topological spaces. Extending these notions to bipolar soft topological spaces is crucial for analyzing systems involving both positive and negative information. In this paper, we define $\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs and strong $\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs using bipolar soft weak structures. We also introduce and investigate bipolar soft weak connectedness and bipolar soft weak compactness within bipolar soft weak structures. Furthermore, we define the concepts of bipolar soft weak local connectedness and bipolar soft weak components, supported by illustrative examples to clarify their meanings. Furthermore, we explore the relationships between these new concepts and their classical counterparts, showing that a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected (resp. disconnected) space is not necessarily B_PS $\widetilde{\widetilde{\widetilde{W}}}$-hereditary, and that a B_PSW-closed subset of a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact space may fail to be B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact.

AMS Subject Classification 2020: 54A05; 54B05; 54D05; 54D30
Keywords and Phrases: Bipolar soft set, Bipolar soft weak structure, Bipolar soft weak connectedness (locally connectedness), Bipolar soft weak components, Bipolar soft weak compactness.

1 Introduction

Soft set theory, introduced by Molodtsov [1] in 1999, provides a powerful mathematical framework for handling uncertainty, vagueness, and imprecision that are often encountered in real-world data. Building on this foundation, Maji et al. [2] established the basic operations and definitions that formalized soft sets, while Ali et al. [3] introduced new operations to enhance their applicability in decision-making problems. The development of soft topology emerged as a natural extension of soft set theory to topological structures, initiated by Shabir and Naz [4], who defined the fundamental concepts of soft open and soft closed sets. Subsequent studies by Aygünoğlu and Aygün [5] deepened the theoretical understanding of soft topological spaces. Ahmad and Hussain [6] explored algebraic structures of soft topology, and Peyghan et al. [7] along with Hussain [8] investigated soft connectedness and related properties. Recent contributions have further enriched the field: Polat et al. [9], and Aydn and Enginolu [10] examined new topological notions, Jafari et al. [11] studied soft topologies induced by soft relations, and Al Ghour [12] introduced soft homogeneous components and soft products, providing novel perspectives on the structural composition of soft spaces. Furthermore, Zakari, Ghareeb, and Omran [13] introduced and investigated the concept of soft weak structures, extending classical soft topological notions by defining and analyzing weaker forms of soft open and soft closed sets within soft topological spaces.

***Corresponding Author:** Baravan A. Asaad, Email: baravan.asaad@uoz.edu.krd, ORCID: 0000-0002-2049-2044
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Recent advances in soft set theory have moved decisively toward bipolar extensions and their topological counterparts, enriching both the theoretical foundations and practical applications of soft topology. Building on classical soft-topological ideas, researchers have introduced bipolar soft structures that capture positivenegative (bipolar) information and enable finer modeling of uncertainty; foundational treatments by [14] and structural properties of bipolar soft topologies by Öztürk [15] and related generalized forms have been developed by Saleh, Asaad and Mohammed [16, 17, 18], while Shabir and collaborators explored bipolar connectedness and compactness notions [19]. These theoretical innovations have been paired with methodological contributions for decision-making and similarity assessment: Demirta and Dalkl applied bipolar fuzzy soft sets to medical diagnosis [20], Demir, Saldaml and Okurer proposed bipolar fuzzy soft filters for multi-criteria group decision-making [21], and Hamad et al. developed similarity measures for bipolar interval-valued fuzzy soft data [22]. Further structural and relational perspectives such as proximity via bipolar fuzzy soft classes and bipolar soft functions were advanced by Saldaml and Demir and by Fadel and Dzul-Kifli [23, 24, 25], while Fujita and Smarandache introduced multi-tier hypergraph frameworks that incorporate bipolar information for modeling complex networks [26, 27]. Recent work on specialized classes, including bipolar soft minimal structures which provides new perspectives for simplifying and analyzing bipolar soft topological systems [28]. Collectively, these contributions show that bipolar soft set theory and bipolar soft topology form a rapidly maturing area that links rigorous topological constructs with concrete decision-making, diagnosis, and network-modeling applications.

Recently, M. Taher and Asaad [29] introduced the concept of bipolar soft weak structures within the framework of bipolar soft topological spaces, establishing a new class of weak topological systems. Their study defines and analyzes key notions such as bipolar soft weak open, closed, closure, interior, and boundary sets, along with corresponding pointwise concepts and neighborhood structures. They defined bipolar soft weak subspaces for these weak structures.

Despite these developments, the concepts of weak connectedness and weak compactness have not yet been examined within bipolar soft topological structures. While soft weak structures and bipolar soft topologies have been studied separately, there is still no unified framework combining both. This gap restricts the analysis of spaces exhibiting weakly connected or weakly compact behaviors under dual (positivenegative) conditions.

Motivated by this limitation, the present paper makes the following contributions:

1. Introduces bipolar soft weak connectedness and bipolar soft weak compactness within bipolar soft weak structures.

2. Defines bipolar soft weak locally connected spaces and bipolar soft weak components, illustrated with examples.

3. Establishes several properties and relationships between the proposed notions and their classical counterparts, showing that a $B_PS\widetilde{\widetilde{\widetilde{W}}}$-connected (resp. disconnected) space is not necessarily $B_PS\,\widetilde{\widetilde{\widetilde{W}}}$-hereditary, and that a B_PSW-closed subset of a $B_PS\,\widetilde{\widetilde{\widetilde{W}}}$-compact space may not be $B_PS\,\widetilde{\widetilde{\widetilde{W}}}$-compact.

4. Extends the theoretical foundations of bipolar soft topology and opens new directions for further research in weak topological structures.

2 Preliminaries

Within this paper, let $\check{\amalg}$ be a universe set, and the nonempty set $\check{\aleph}$ be an entire set of parameters, $\mathrm{P}(\check{\amalg})$ be the family of all subsets of $\check{\amalg}$. Let $\check{\pi}$ and $\check{\sigma}$ be nonempty subsets of $\check{\aleph}$. This section begins by reviewing essential concepts, including soft weak structures, bipolar soft set logic and provides the foundational background for bipolar soft topological spaces.

Definition 2.1. [1] (Soft Set) *A pair* $(\check{\zeta}, \breve{\pi})$ *is known as a soft set on* $\check{\amalg}$*, where* $\check{\zeta}$ *is a mapping from* $\breve{\pi}$ *into* $P(\check{\amalg})$*. Meaning that a soft set on* $\check{\amalg}$ *is a parameterized family of subsets of the universe* $\check{\amalg}$*. For* $\epsilon \in \breve{\pi}$*,* $(\check{\zeta}, \breve{\pi})$ *can be considered as* ϵ*-elements' set of the soft set* $(\check{\zeta}, \breve{\pi})$*. As seen, soft set is not a crisp set. Following that, the family of all soft sets on* $\check{\amalg}$ *is denoted by* $SS(\check{\amalg})$*. Therefore, a soft set* $(\check{\zeta}, \breve{\pi})$ *can be dictated as:*

$$(\check{\zeta}, \breve{\pi}) = \{(\epsilon, \check{\zeta}(\epsilon)) : \epsilon \in \breve{\pi},\ \check{\zeta}(\epsilon) \subseteq \check{\amalg}\}.$$

Definition 2.2. [13] *A soft subset* $(\check{\zeta}, \rho)$ *of* $(\tilde{\check{\amalg}}, \rho)$ *is called a soft weak compact set, denoted by* $S\ \widetilde{W}$*-compact set, if each* $S\ \widetilde{W}$*-open cover of* $(\check{\zeta}, \rho)$ *has a finite* S *subcover. A* SWS $(\check{\amalg}, \widetilde{W}, \rho)$ *is said to be a* $S\ \widetilde{W}$*-compact space if* $(\tilde{\check{\amalg}}, \rho)$ *is a* $S\ \widetilde{W}$*-compact subset of itself.*

Definition 2.3. [2] *Let* $\breve{\pi} = \{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$ *be a subset of* $\check{\aleph}$*, the **Not** set of* $\breve{\pi}$ *is denoted by* $\neg\breve{\pi} = \{\neg\epsilon_1, \neg\epsilon_2, ..., \neg\epsilon_n\}$ *where,* $\neg\epsilon_i = Not\ \epsilon_i$*, for all* i*.*

Definition 2.4. [14] *A triple* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is known as a bipolar soft set, denoted by* B_PSS*, on* $\check{\amalg}$*, where* $\check{\zeta}$ *and* $\check{\xi}$ *are mappings defined by* $\check{\zeta} : \breve{\pi} \longrightarrow P(\check{\amalg})$ *and* $\check{\xi} : \neg\breve{\pi} \longrightarrow P(\check{\amalg})$ *so that* $\check{\zeta}(\epsilon) \cap \check{\xi}(\neg\epsilon) = \phi$ *for all* $\epsilon \in \breve{\pi}$ *and* $\neg\epsilon \in \neg\breve{\pi}$*.*
So, a B_PSS $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *can be dictated as:*

$$(\check{\zeta}, \check{\xi}, \breve{\pi}) = \{(\epsilon, \check{\zeta}(\epsilon), \check{\xi}(\neg\epsilon)) : \epsilon \in \breve{\pi} \text{ and } \check{\zeta}(\epsilon)\ \cap\ \check{\xi}(\neg\epsilon)\ =\ \phi\}.$$

We denote $B_PSS(\check{\amalg})$ *by the set of all* B_PSSs *on* $\check{\amalg}$*.*

Definition 2.5. [14] *For any two* B_PSSs $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$*, it is stated that* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *is a bipolar soft* (B_PS) *subset of* $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$ *if:*

(i) $\breve{\pi} \subseteq \breve{\sigma}$ *and,*

(ii) $\check{\zeta}_1(\epsilon) \subseteq \check{\zeta}_2(\epsilon)$ *and* $\check{\xi}_2(\neg\epsilon) \subseteq \check{\xi}_1(\neg\epsilon)$ *for all* $\epsilon \in \breve{\pi}$ *and* $\neg\epsilon \in \neg\breve{\pi}$*.*

This relationship is denoted by $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$*. Likewise, it is stated that* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *is a* B_PS *superset of* $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$*, denoted by* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\ \tilde{\tilde{\supseteq}}\ (\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$*, if* $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$ *is a* B_PS *subset of* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$*.*

Definition 2.6. [14] *A* B_PSS $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is considered a null* B_PSS *denoted by* $(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi})$*, if* $\check{\zeta}(\epsilon) = \phi$ *for all* $\epsilon \in \breve{\pi}$ *and* $\check{\xi}(\neg\epsilon) = \check{\amalg}$ *for all* $\neg\epsilon \in \neg\breve{\pi}$*.*

Definition 2.7. [14] *A* B_PSS $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is considered an absolute* B_PSS *denoted by* $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$*, if* $\check{\zeta}(\epsilon) = \check{\amalg}$ *for all* $\epsilon \in \breve{\pi}$ *and* $\check{\xi}(\neg\epsilon) = \phi$ *for all* $\neg\epsilon \in \neg\breve{\pi}$*.*

Definition 2.8. [14] *Let* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$ *be two* B_PSSs*, then the* B_PS *union of these* B_PSSs *is the* $B_PSS(\check{\delta}, \check{\gamma}, \dot{\kappa})$*, where* $\dot{\kappa} = \breve{\pi} \cap \breve{\sigma}$ *is a nonempty set and for all* $\epsilon \in \dot{\kappa}$*, there is*
$\check{\delta}(\epsilon) = \check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon), \epsilon \in \breve{\pi} \cap \breve{\sigma} \neq \phi$ *and* $\check{\gamma}(\neg\epsilon) = \check{\xi}_1(\neg\epsilon) \cap \check{\xi}_2(\neg\epsilon), \neg\epsilon \in \neg\breve{\pi} \cap \neg\breve{\sigma} \neq \phi$*. This operation is denoted as* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\ \tilde{\tilde{\cup}}\ (\check{\zeta}_2, \check{\xi}_2, \breve{\sigma}) = (\check{\delta}, \check{\gamma}, \dot{\kappa})$*.*

Definition 2.9. *Let* $\{(\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) : i \in I\}$ *be any family of* B_PSSs*, then* $\tilde{\tilde{\bigcup}}_{i\in I}(\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) = (\check{\delta}, \check{\gamma}, \dot{\kappa})$*, where* $\check{\delta}(\epsilon) = \check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) \cup\ ...$ *and* $\check{\gamma}(\neg\epsilon) = \check{\xi}_1(\neg\epsilon) \cap \check{\xi}_2(\neg\epsilon) \cap\ ...$.

Definition 2.10. [14] *Let* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\sigma})$ *be two* B_PSSs*, then the* B_PS *intersection of these* B_PSSs *is the* $B_PSS(\check{\delta}, \check{\gamma}, \dot{\kappa})$*, where* $\dot{\kappa} = \breve{\pi} \cap \breve{\sigma}$ *is a nonempty set and for all* $\epsilon \in \dot{\kappa}$*, there is* $\check{\delta}(\epsilon) = \check{\zeta}_1(\epsilon) \cap \check{\zeta}_2(\epsilon), \epsilon \in \breve{\pi} \cap \breve{\sigma} \neq \phi$ *and* $\check{\gamma}(\neg\epsilon) = \check{\xi}_1(\neg\epsilon) \cup \check{\xi}_2(\neg\epsilon), \neg\epsilon \in \neg\breve{\pi} \cap \neg\breve{\sigma} \neq \phi$*. This operation is denoted as* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\sigma}) = (\check{\delta}, \check{\gamma}, \dot{\kappa})$.

Definition 2.11. *Let* $\{(\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) : i \in I\}$ *be any family of* B_PSSs*, then* $\widetilde{\widetilde{\cap}}_{i\in I}\,(\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) = (\check{\delta}, \check{\gamma}, \dot{\kappa})$*, where* $\check{\delta}(\epsilon) = \check{\zeta}_1(\epsilon) \cap \check{\zeta}_2(\epsilon) \cap \ldots$ *and* $\check{\gamma}(\neg\epsilon) = \check{\xi}_1(\neg\epsilon) \cup \check{\xi}_2(\neg\epsilon) \cup \ldots$.

Proposition 2.12. [19] *If* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\text{II}})$*, then*

(*i*) $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, (\check{\zeta}, \check{\xi}, \breve{\pi})^c = (\check{\delta}, \Phi, \breve{\pi})$*, where* $\check{\delta}(\epsilon) = \check{\zeta}(\epsilon) \cup \check{\zeta}^c(\epsilon) \subseteq \check{\text{II}}$ *for each* $\epsilon \in \breve{\pi}$ *and* $\Phi(\neg\epsilon) = \check{\xi}(\neg\epsilon) \cap \check{\xi}^c(\neg\epsilon) = \phi$ *for each* $\neg\epsilon \in \neg\breve{\pi}$.

(*ii*) $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}, \check{\xi}, \breve{\pi})^c = (\Phi, \check{\gamma}, \breve{\pi})$*, where* $\Phi(\epsilon) = \check{\zeta}(\epsilon) \cap \check{\zeta}^c(\epsilon) = \phi$ *for each* $\epsilon \in \breve{\pi}$ *and* $\check{\gamma}(\neg\epsilon) = \check{\xi}(\neg\epsilon) \cup \check{\xi}^c(\neg\epsilon) \subseteq \check{\text{II}}$ *for each* $\neg\epsilon \in \neg\breve{\pi}$.
Further, $(\check{\zeta}, \check{\xi}, \breve{\pi})$*,* $(\check{\zeta}, \check{\xi}, \breve{\pi})^c$ *will always satisfy* $\check{\zeta}(\epsilon) \cup \check{\zeta}^c(\epsilon) = \check{\xi}(\neg\epsilon) \cup \check{\xi}^c(\neg\epsilon)$ *for all* $\epsilon \in \breve{\pi}$.

Definition 2.13. [15] *If* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\text{II}})$*, and* $\check{Y}$ *is a nonempty subset of* $\check{\text{II}}$*, then the sub* B_PS *set of* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *over* $\check{Y}$ *denoted by* $({}^{\check{Y}}\check{\zeta}, {}^{\check{Y}}\check{\xi}, \breve{\pi})$*, and is defined as follows:*

$${}^{\check{Y}}\check{\zeta}(\epsilon) = \check{Y} \cap \check{\zeta}(\epsilon) \text{ and } {}^{\check{Y}}\check{\xi}(\neg\epsilon) = \check{Y} \cap \check{\xi}(\neg\epsilon), \text{ for each } \epsilon \in \breve{\pi} \text{ and } \neg\epsilon \in \neg\breve{\pi}.$$

Definition 2.14. [19] *Let* $\widetilde{\widetilde{\tau}}$ *be the family of* B_PSSs *on* $\check{\text{II}}$ *with* $\breve{\pi}$ *as the set of parameters, then,* $\widetilde{\widetilde{\tau}}$ *be considered a bipolar soft topology* (B_PST) *on* $\check{\text{II}}$ *if:*

(*i*) $(\Phi, \widetilde{\widetilde{\check{\text{II}}}}, \breve{\pi})$ *and* $(\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ *belong to* $\widetilde{\widetilde{\tau}}$.

(*ii*) *The* B_PS *union of any number of* B_PSSs *in* $\widetilde{\widetilde{\tau}}$ *belongs to* $\widetilde{\widetilde{\tau}}$.

(*iii*) *The* B_PS *intersection of finite number of* B_PSSs *in* $\widetilde{\widetilde{\tau}}$ *belongs to* $\widetilde{\widetilde{\tau}}$.

Then $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ *has the name of a bipolar soft topological space* (B_PSTS) *on* $\check{\text{II}}$.
Every member of $\widetilde{\widetilde{\tau}}$ *is known as a bipolar soft open set, denoted by* B_PS*-open. The complement of a* B_PS*-open set is* B_PS*-closed.*

Proposition 2.15. [15] *Let* $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ *be a* B_PSTS *on* $\check{\text{II}}$ *and* $\check{Y}$ *be a nonempty subset of* $\check{\text{II}}$*, then,* $\widetilde{\widetilde{\tau}}_{\check{Y}} = \{({}^{\check{Y}}\check{\zeta}, {}^{\check{Y}}\check{\xi}, \breve{\pi}) : (\check{\zeta}, \check{\xi}, \breve{\pi}) \in \widetilde{\widetilde{\tau}}\}$ *is* B_PST *on* $\check{Y}$*. The family* $\widetilde{\widetilde{\tau}}_{\check{Y}}$ *is known as a* B_PS *subspace topology.*

Definition 2.16. [15] *Let* $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ *be a* B_PSTS *on* $\check{\text{II}}$ *and* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$*, then the family* $\widetilde{\widetilde{\tau}}_{(\check{\zeta},\check{\xi},\breve{\pi})} = \left\{(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) : (\check{\zeta}_i, \check{\xi}_i, \breve{\pi}) \widetilde{\widetilde{\in}} \widetilde{\widetilde{\tau}} \text{ and } i \in I\right\}$ *is a* B_PS *subspace topology on* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *and* $\left(\check{\text{II}}_{(\check{\zeta},\check{\xi},\breve{\pi})}, \widetilde{\widetilde{\tau}}_{(\check{\zeta},\check{\xi},\breve{\pi})}, \breve{\pi}, \neg\breve{\pi}\right)$ *has the name of a* B_PS *topological subspace of* $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$.

Definition 2.17. [19] *Two* B_PSSs $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are said to be disjoint* B_PSSs *if* $\check{\zeta}_1(\epsilon) \cap \check{\zeta}_2(\epsilon) = \phi$ *for all* $\epsilon \in \breve{\pi}$.

Definition 2.18. [19] *Let* $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ *be a* B_PSTS *on* $\check{\text{II}}$*. A* B_PS *separation of* $(\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ *is a pair* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$*,* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *of non-null disjoint* B_PS *open sets on* $\check{\text{II}}$ *such that* $\check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) = \check{\text{II}}$ *for all* $\epsilon \in \breve{\pi}$.

Definition 2.19. [19] *A B_PSTS $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ is said to be a B_PS disconnected space if there exists a B_PS separation of $(\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$. Further, $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ is said to be a B_PS connected space if it is not a B_PS disconnected space.*

Definition 2.20. [19] *A property $\mathcal{P}$ of a B_PSTS $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ is said to be B_PS hereditary if every B_PS subspace $(Y, \widetilde{\widetilde{\tau}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ of $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ also possesses the property $\mathcal{P}$.*

Definition 2.21. [19] *A family $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) : (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\tau}}\}_{\delta\in\Delta}$ of B_PSSs is said to be a B_PS cover of a B_PSS $(\check{\zeta}, \check{\xi}, \breve{\pi})$ if:*

$$(\check{\zeta}, \check{\xi}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\subseteq}}} \widetilde{\widetilde{\cup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}).$$

Furthermore, it is called the B_PS open cover of a B_PSS $(\check{\zeta}, \check{\xi}, \breve{\pi})$ if each member of $\check{\Delta}$ is a B_PS open set. A B_PS subcover of $\check{\Delta}$ is a subfamily of $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ which is also a B_PS open cover.

Definition 2.22. [19] *A bipolar soft subset $(\check{\zeta}, \check{\xi}, \breve{\pi})$ of $(\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ is called a bipolar soft compact set, denoted by, B_PS compact set, if each B_PS open cover of $(\check{\zeta}, \check{\xi}, \breve{\pi})$ has a finite B_PS subcover. A B_PSTS $(\check{\text{II}}, \widetilde{\widetilde{\tau}}, \breve{\pi}, \neg\breve{\pi})$ is said to be B_PS compact space if $(\widetilde{\widetilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ is a B_PS compact subset of itself.*

Definition 2.23. [29] *Let $\widetilde{\widetilde{W}}$ be a family of B_PS subsets on $\check{\text{II}}$, then $\widetilde{\widetilde{W}}$ is considered a (B_PSWS) on $\check{\text{II}}$ if $(\Phi, \widetilde{\widetilde{\check{\text{II}}}}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{W}}$.*

Then, $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is known as a B_PSWS on $\check{\text{II}}$. The members of $\widetilde{\widetilde{W}}$ are considered bipolar soft $\widetilde{\widetilde{W}}$-open sets, denoted by B_PSW-open in $\check{\text{II}}$.

And $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is considered bipolar soft $\widetilde{\widetilde{W}}$-closed, denoted by B_PSW-closed if its B_PSW complement $(\check{\zeta}, \check{\xi}, \breve{\pi})^c$ is B_PSW-open.

Theorem 2.24. [29] *Let $(\check{\text{II}}, \widetilde{W}, \breve{\pi})$ be a SWS. Then the family $\widetilde{\widetilde{W}}$ consisting of B_PSS $(\check{\zeta}, \check{\xi}, \breve{\pi})$ such that $(\check{\zeta}, \breve{\pi}) \mathrel{\widetilde{\in}} \widetilde{W}$ and $\check{\xi}(\neg\epsilon) = \check{\text{II}} \setminus \check{\zeta}(\epsilon)$ for all $\neg\epsilon \in \neg\breve{\pi}$ defines a B_PSWS on $\check{\text{II}}$.*

Definition 2.25. [29] *Let $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ be a B_PSWS and $(\check{\zeta}, \check{\xi}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\in}}} B_PSS(\check{\text{II}})$. Then the B_PSW-closure of $(\check{\zeta}, \check{\xi}, \breve{\pi})$, denoted by B_PSW-$cl(\check{\zeta}, \check{\xi}, \breve{\pi})$, is the B_PS intersection of all B_PSW-closed sets containing $(\check{\zeta}, \check{\xi}, \breve{\pi})$. So, the B_PSW-closure can be dictated as:*

$$B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi}) = \widetilde{\widetilde{\bigcap}}\{(\check{\delta}, \check{\gamma}, \breve{\pi}) : (\check{\delta}, \check{\gamma}, \breve{\pi}) \text{ is } B_PSW\text{-closed and } (\check{\delta}, \check{\gamma}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\supseteq}}} (\check{\zeta}, \check{\xi}, \breve{\pi})\}.$$

Definition 2.26. [29] *Let $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ be a B_PSWS and $(\check{\zeta}, \check{\xi}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\in}}} B_PSS(\check{\text{II}})$. Then the B_PSW-interior of $(\check{\zeta}, \check{\xi}, \breve{\pi})$, denoted by B_PSW-$int(\check{\zeta}, \check{\xi}, \breve{\pi})$, is the B_PS union of all B_PSW-open subsets of $(\check{\zeta}, \check{\xi}, \breve{\pi})$.*
So, the set can be dictated as:

$$B_PSW\text{-}int(\check{\zeta}, \check{\xi}, \breve{\pi}) = \widetilde{\widetilde{\bigcup}}\{(\check{\delta}, \check{\gamma}, \breve{\pi}) : (\check{\delta}, \check{\gamma}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{W}} \text{ and } (\check{\delta}, \check{\gamma}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\subseteq}}} (\check{\zeta}, \check{\xi}, \breve{\pi})\}.$$

Definition 2.27. [29] *Let $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ be a B_PSWS and $(\check{\zeta}, \check{\xi}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\in}}} B_PSS(\check{\text{II}})$, then the bipolar soft $\widetilde{\widetilde{W}}$-boundary of $(\check{\zeta}, \check{\xi}, \breve{\pi})$, denoted by B_PSW-$b(\check{\zeta}, \check{\xi}, \breve{\pi})$, is defined as*

$$B_PSW\text{-}b(\check{\zeta}, \check{\xi}, \breve{\pi}) = B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi}) \mathrel{\widetilde{\widetilde{\cap}}} B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})^c.$$

Proposition 2.28. [29] *Let $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ be a B_PSWS on $\check{\text{II}}$ and $\check{Y}$ be a nonempty subset of $\check{\text{II}}$, then, $\widetilde{\widetilde{W}}_{\check{Y}} = \{(^{\check{Y}}\check{\zeta}, ^{\check{Y}}\check{\xi}, \breve{\pi}) : (\check{\zeta}, \check{\xi}, \breve{\pi}) \in \widetilde{\widetilde{W}}\}$ is B_PSW on $\check{Y}$. The family $\widetilde{\widetilde{W}}_{\check{Y}}$ is known as a B_PS subspace topology.*

Proposition 2.29. *Let* $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \check{\pi}, \neg\check{\pi})$ *be a* B_PSWSS *of* B_PSWS $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ *over* $\check{\Pi}$ *and* $(^{\check{Y}}\check{\zeta}, ^{\check{Y}}\check{\xi}, \check{\pi})$ *be a* B_PS $\widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$*-closed set in* $\check{Y}$*. Then* $(\check{\zeta}, \check{\xi}, \check{\pi})$ *is a* B_PSW*-closed set in* $\check{\Pi}$.

Definition 2.30. [29] *Let* $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ *be a* B_PSWS *and* $(\check{\zeta}, \check{\xi}, \check{\pi}) \;\tilde{\tilde{\in}}\; B_PSS(\check{\Pi})$*, then the family*

$$\widetilde{\widetilde{\widetilde{W}}}_{(\check{\zeta},\check{\xi},\check{\pi})} = \{(\check{\zeta}, \check{\xi}, \check{\pi}) \;\tilde{\tilde{\cap}}\; (\check{\zeta}_i, \check{\xi}_i, \check{\pi}) : (\check{\zeta}_i, \check{\xi}_i, \check{\pi}) \;\tilde{\tilde{\in}}\; \widetilde{\widetilde{\widetilde{W}}},\ i \in \mathcal{I}\}$$

is said to be a bipolar soft weak subspace B_PSWSS *on* $(\check{\zeta}, \check{\xi}, \check{\pi})$.

3 B_PS $\widetilde{\widetilde{\widetilde{W}}}$-Connected Sets

This section shows $\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs ($S\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs) using B_PSWS providing some of their properties. In addition, B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected sets in terms of B_PSWS are presented, obtaining properties and their relations.

Definition 3.1. *Let* $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \check{\pi})$ *be two* B_PSSs *in* $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \check{\pi}, \neg\check{\pi})$ *which are not null. Then*

(*i*) $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \check{\pi})$ *are known as* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs *if* $(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) \;\tilde{\tilde{\cap}}\; B_PSW\text{-}cl\,(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) = (\Phi, \check{\xi}, \check{\pi})$ *and* $B_PSW\text{-}cl\,(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) \;\tilde{\tilde{\cap}}\; (\check{\zeta}_2, \check{\xi}_2, \check{\pi}) = (\Phi, \check{\xi}, \check{\pi})$.

(*ii*) $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \check{\pi})$ *are known as strong* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs ($S\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs) *if* $(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) \;\tilde{\tilde{\cap}}\; B_PSW\text{-}cl\,(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) = (\Phi, \tilde{\tilde{\check{\Pi}}}, \check{\pi})$ *and* $B_PSW\text{-}cl\,(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) \;\tilde{\tilde{\cap}}\; (\check{\zeta}_2, \check{\xi}_2, \check{\pi}) = (\Phi, \tilde{\tilde{\check{\Pi}}}, \check{\pi})$.

Proposition 3.2. *Every* $S\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs *on* $\check{\Pi}$ *is a* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs *on* $\check{\Pi}$.

Proof. Follows directly from definitions. □

Proposition 3.3. *Any two* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs ($S\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs) *are disjoint* B_PSSs.

Proof. Obvious. □

Remark 3.4. *Note that disjoint* B_PSSs *may not be* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs ($S\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs)*; meaning that the converse of Proposition 3.3 is not true as shown in the following example.*

Example 3.5. Let $\check{\Pi} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$, $\check{\pi} = \{\epsilon_1, \epsilon_2\}$ and

$$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\Pi}}}, \check{\pi}), (\tilde{\tilde{\check{\Pi}}}, \Phi, \check{\pi}), (\check{\zeta}_1, \check{\xi}_1, \check{\pi}), (\check{\zeta}_2, \check{\xi}_2, \check{\pi}), (\check{\zeta}_3, \check{\xi}_3, \check{\pi})\}$$

be a B_PSWS over $\check{\Pi}$ where $(\check{\zeta}_1, \check{\xi}_1, \check{\pi})$,$(\check{\zeta}_2, \check{\xi}_2, \check{\pi})$,$(\check{\zeta}_3, \check{\xi}_3, \check{\pi}) \;\tilde{\tilde{\in}}\; B_PSS(\check{\Pi})$, defined as follows

$$\begin{aligned}
(\check{\zeta}_1, \check{\xi}_1, \check{\pi}) &= \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_3, \hbar_4\}, \{\hbar_2\})\},\\
(\check{\zeta}_2, \check{\xi}_2, \check{\pi}) &= \{(\epsilon_1, \{\hbar_1, \hbar_3\}, \{\hbar_2, \hbar_4\}), (\epsilon_2, \{\hbar_1, \hbar_3\}, \{\hbar_2\})\},\\
(\check{\zeta}_3, \check{\xi}_3, \check{\pi}) &= \{(\epsilon_1, \{\hbar_1, \hbar_2, \hbar_3\}, \phi), (\epsilon_2, \{\hbar_1, \hbar_3, \hbar_4\}, \{\hbar_2\})\}.
\end{aligned}$$

Now, assume that $(\check{\delta}_1, \check{\gamma}_1, \check{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \check{\pi})$ are disjoint B_PSSs over $\check{\Pi}$ defined by

$$\begin{aligned}
(\check{\delta}_1, \check{\gamma}_1, \check{\pi}) &= \{(\epsilon_1, \{\hbar_1, \hbar_2, \hbar_4\}, \{\hbar_3\}), (\epsilon_2, \{\hbar_1, \hbar_2, \hbar_4\}, \{\hbar_3\})\},\\
(\check{\delta}_2, \check{\gamma}_2, \check{\pi}) &= \{(\epsilon_1, \{\hbar_3\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_2\})\}.
\end{aligned}$$

Then $B_PSW-cl\ (\check{\delta}_1,\check{\gamma}_1,\breve{\pi}) = B_PSW-cl\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\tilde{\tilde{\check{\amalg}}},\Phi,\breve{\pi})$ and $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\tilde{\tilde{\cap}}\ B_PSW-cl\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$, $B_PSW-cl\ (\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$. But $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\Phi,\check{\gamma},\breve{\pi})$. Thus, B_PSSs $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$, $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ are disjoint B_PSSs but not $\tilde{\tilde{\tilde{W}}}$-separated B_PSSs ($S\tilde{\tilde{\tilde{W}}}$-separated B_PSSs).

Proposition 3.6. *Let be any two B_PSSs on $\check{\amalg}$. Then the following are correct:*

(i) If $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ are two $\tilde{\tilde{\tilde{W}}}$-separated B_PSSs over $\check{\amalg}$ with $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$. Then, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ also are $\tilde{\tilde{\tilde{W}}}$-separated B_PSSs over $\check{\amalg}$.

(ii) If $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ are two $S\tilde{\tilde{\tilde{W}}}$-separated B_PSSs over $\check{\amalg}$ with $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$. Then, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ also are $S\tilde{\tilde{\tilde{W}}}$-separated B_PSSs over $\check{\amalg}$.

Proof.

(*i*) Given $\tilde{\tilde{\tilde{W}}}$-separated B_PSSs $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$. Then $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\Phi,\check{\xi},\breve{\pi})$. Since $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$, then $B_PSW\text{-}cl\,(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $B_PSW\text{-}cl\,(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$. Therefore, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW\text{-}cl\,(\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = B_PSW\text{-}cl\,(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\Phi,\check{\gamma},\breve{\pi})$. Hence, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ are $\tilde{\tilde{\tilde{W}}}$-separated B_PSSs over $\check{\amalg}$.

(*ii*) Let given $S\tilde{\tilde{\tilde{W}}}$-separated B_PSSs $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$. Then $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\Phi,\tilde{\tilde{\check{\amalg}}},\breve{\pi})$. Since $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$, then $B_PSW\text{-}cl\,(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $B_PSW\text{-}cl\,(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})\ \tilde{\tilde{\subseteq}}\ B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$. Therefore, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW\text{-}cl\,(\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = B_PSW\text{-}cl\,(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\delta}_2,\check{\gamma}_2,\breve{\pi}) = (\Phi,\tilde{\tilde{\check{\amalg}}},\breve{\pi})$. Hence, $(\check{\delta}_1,\check{\gamma}_1,\breve{\pi})$ and $(\check{\delta}_2,\check{\gamma}_2,\breve{\pi})$ are $S\tilde{\tilde{\tilde{W}}}$-separated B_PSSs over $\check{\amalg}$.

□

Theorem 3.7. *Two $\tilde{\tilde{\tilde{W}}}$-closed subsets $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ of B_PSWS $(\check{\amalg},\tilde{\tilde{\tilde{W}}},\breve{\pi},\neg\breve{\pi})$ over $\check{\amalg}$ are $\tilde{\tilde{\tilde{W}}}$-separated B_PSSs if and only if they are disjoint B_PSSs.*

Proof. The first condition is obvious. Conversely, assume that $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ are disjoint $\tilde{\tilde{\tilde{W}}}$-closed. So, $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\Phi,\check{\xi},\breve{\pi})$ and $B_PSW\text{-}cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi}) = (\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$, $B_PSW\text{-}cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ and hence

$$(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW-cl\,(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = B_PSW-cl\,(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = (\Phi,\check{\xi},\breve{\pi})$$

showing that $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ are $\tilde{\tilde{\tilde{W}}}$-separated B_PSSs over $\check{\amalg}$. □

Remark 3.8. *Two disjoint $\tilde{\tilde{\tilde{W}}}$-open sets $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi})$ and $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ are not necessarily $\tilde{\tilde{\tilde{W}}}$-separated.*

Example 3.9. Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}, \breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and

$$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi})\}$$

be a B_PSWS over $\check{\amalg}$ where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, $(\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\amalg})$, defined as follows

$$\begin{aligned}
(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1, \hbar_4\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1, \hbar_4\})\}, \\
(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_3\}, \{\hbar_1, \hbar_4\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_1, \hbar_4\})\}, \\
(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \{\hbar_1, \hbar_4\}), (\epsilon_2, \{\hbar_2, \hbar_3\}, \{\hbar_1, \hbar_4\})\}.
\end{aligned}$$

Obviously, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are disjoint $\widetilde{\widetilde{\widetilde{W}}}$-open but not $\widetilde{\widetilde{\widetilde{W}}}$-separated as $B_PSW - cl\ (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = B_PSW - cl\ (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$, which implies that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\ \tilde{\tilde{\cap}}\ B_PSW - cl\ (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, B_PSW-$cl(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. Hence the conclusion.

Corollary 3.10. *If two $\widetilde{\widetilde{\widetilde{W}}}$-closed subsets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ of B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ over $\check{\amalg}$ are $S\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs, then they are disjoint B_PSSs.*

Proof. Follows directly from Proposition 3.2 and Theorem 3.7. □

Remark 3.11. *The example below shows that the converse of Corollary 3.10 is not true in general. Hence two disjoint $\widetilde{\widetilde{\widetilde{W}}}$-closed sets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are not necessarily $S\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs .*

Example 3.12. Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}, \breve{\pi} = \{\epsilon_1\}$ and

$$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$$

be a B_PSWS over $\check{\amalg}$ where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\amalg})$, defined as follows

$$\begin{aligned}
(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_2\})\}, \\
(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) &= \{(\epsilon_1, \{\hbar_3\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_1\})\}.
\end{aligned}$$

Obviously, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c$ are disjoint $\widetilde{\widetilde{\widetilde{W}}}$-closed but not $S\widetilde{\widetilde{\widetilde{W}}}$-strong separated as B_PSW-$cl(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$ $= B_PSW$-$cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c = (\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$, which implies that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c \tilde{\tilde{\cap}}\ B_PSW$-$cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c = (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$, B_PSW-$cl(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c\ \tilde{\tilde{\cap}}\ (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c = (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c$. Hence the conclusion.

Definition 3.13. *A B_PS subset $(\check{\zeta}, \check{\xi}, \breve{\pi})$ of B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ over $\check{\amalg}$ is called B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected over $\check{\amalg}$ if there exist $\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs of $(\check{\zeta}, \check{\xi}, \breve{\pi})$. Otherwise, a B_PSS $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is called B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected over $\check{\amalg}$.*

Remark 3.14. *The null B_PSS $(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi})$ is always B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected.*

Definition 3.15. *Let $\hbar_\upsilon^\epsilon$, $\hbar'^{\epsilon'}_{\upsilon'}$ $\tilde{\tilde{\in}}$ $B_PSP(\check{\amalg})_{(\breve{\pi}, \neg\breve{\pi})}$ of a B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Then, $\hbar_\upsilon^\epsilon$ and $\hbar'^{\epsilon'}_{\upsilon'}$ are called B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected points if they are contained in B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected set over $\check{\amalg}$.*

Proposition 3.16. *Let* $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *be a* B_PSWS *over* $\check{\text{II}}$ *and* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *be a* B_PS $\widetilde{\widetilde{\widetilde{W}}}$*-connected set such that* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$*, where* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs*. Then* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *or* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$*.*

Proof. From $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs, then $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, B_PSW\text{-}cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$ and $B_PSW\text{-}cl\,(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$. Since $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, then $(\check{\zeta}, \check{\xi}, \breve{\pi}) = (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, \big((\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\tilde{\tilde{\cup}}(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big) = \big((\check{\zeta}, \check{\xi}, \breve{\pi})\tilde{\tilde{\cap}}(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\big) \,\tilde{\tilde{\cup}}\, ((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}))$. We state that at least one of the B_PSSs $\big((\check{\zeta}, \check{\xi}, \breve{\pi})\tilde{\tilde{\cap}}(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\big)$ and $\big((\check{\zeta}, \check{\xi}, \breve{\pi})\tilde{\tilde{\cap}}(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big)$ is null B_PSS. Now, suppose that if possible non of these B_PSSs is null, hence,

$$(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \neq (\Phi, \check{\xi}, \breve{\pi}) \text{ and } (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \neq (\Phi, \check{\xi}, \breve{\pi}).$$

Thus,

$$\begin{aligned}
&((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})) \,\tilde{\tilde{\cap}}\, B_PSW - cl\big((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big)\\
&\quad \tilde{\tilde{\subseteq}} \big((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\big) \,\tilde{\tilde{\cap}}\, \big(B_PSW - cl(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, B_PSW - cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big)\\
&\quad = \big((\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, B_PSW - cl(\check{\zeta}, \check{\xi}, \breve{\pi})\big) \,\tilde{\tilde{\cap}}\, \big((\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, B_PSW - cl(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big)\\
&\quad = (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\Phi, \check{\xi}, \breve{\pi})\\
&\quad = (\Phi, \check{\xi}, \breve{\pi}).
\end{aligned}$$

Similarly,

$$B_PSW\text{-}cl\big((\check{\zeta}, \check{\xi}, \breve{\pi})\tilde{\tilde{\cap}}(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\big) \,\tilde{\tilde{\cap}}\, \big((\check{\zeta}, \check{\xi}, \breve{\pi})\tilde{\tilde{\cap}}(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\big) = (\Phi, \check{\xi}, \breve{\pi}).$$

Therefore, $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs. Thus, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ can be expressed as B_PS union of a pair of $\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs. So, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected. Which is a contradiction. Hence, at least one of the B_PSSs $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ is null B_PSS. Now, if $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$, then $(\check{\zeta}, \check{\xi}, \breve{\pi}) = (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ which implies that $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. If $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = (\Phi, \check{\xi}, \breve{\pi})$, then $(\check{\zeta}, \check{\xi}, \breve{\pi}) = (\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ implying that $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$. Therefore, either $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ or $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. □

Corollary 3.17. *If* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is a* B_PS $\widetilde{\widetilde{\widetilde{W}}}$*-connected subset of a* B_PSWS $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *such that* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *where* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are both* B_PSW*-closed and nonnull disjoint* B_PSSs*. Then,* $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ *and* $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ *are* $\widetilde{\widetilde{\widetilde{W}}}$*-separated* B_PSSs*.*

Proof. Follows directly from Proposition 3.16 and Theorem 3.7. □

Proposition 3.18. *Let* $(\check{\zeta}, \check{\xi}, \breve{\pi})$ *be* B_PS $\widetilde{\widetilde{\widetilde{W}}}$*-connected and* $(\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\in}}\, B_PSS(\check{\text{II}})$ *such that* $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})$*. Then* $(\check{\delta}, \check{\gamma}, \breve{\pi})$ *is* B_PS $\widetilde{\widetilde{\widetilde{W}}}$*-connected. Specifically,* $B_PSW\text{-}cl(\check{\zeta}, \check{\xi}, \breve{\pi})$ *is also* B_PS $\widetilde{\widetilde{\widetilde{W}}}$*-connected.*

Proof. Suppose that $(\check{\delta}, \check{\gamma}, \breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected. Then, there exist nonnull B_PSSs $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ in which

$$(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi}) \tilde{\tilde{\sqcap}}\, B_PSW\text{-}cl\,(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) = B_PSW\text{-}cl\,(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) = (\Phi, \breve{\xi}, \breve{\pi})$$

and $(\breve{\delta}, \breve{\gamma}, \breve{\pi}) = (\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi}) \tilde{\tilde{\sqcup}} (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$.

From $(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\delta}, \breve{\gamma}, \breve{\pi}) = (\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi}) \tilde{\tilde{\sqcup}} (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$, it follows from Proposition 3.16 that $(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ or $(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$. Let $(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ thus, $B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} B_PSW\text{-}cl(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ then,

$$B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) \tilde{\tilde{\subseteq}} B_PSW\text{-}cl(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) = (\Phi, \breve{\xi}, \breve{\pi}),$$

but $(\Phi, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$, therefore,

$$B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) = (\Phi, \breve{\xi}, \breve{\pi}).$$

So, $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi}) \tilde{\tilde{\sqcup}} (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) = (\breve{\delta}, \breve{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi})$ then, $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\delta}, \breve{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi})$ implies that $B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) = (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$. Hence, $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}) = (\Phi, \breve{\xi}, \breve{\pi})$. This is a contradiction because $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ is nonnull B_PSS. Therefore, $(\breve{\delta}, \breve{\gamma}, \breve{\pi})$ is $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected. Also, from $(\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\delta}, \breve{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi})$, implies that $B_PSW\text{-}cl(\breve{\zeta}, \breve{\xi}, \breve{\pi})$ is $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected. □

Proposition 3.19. *Let* $\{(\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi}) : \delta \in \Delta\}$ *be the family of* $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$*-connected sets such that* $\tilde{\tilde{\bigcap}}_{\delta\in\Delta} (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi}) \neq (\Phi, \breve{\xi}, \breve{\pi})$. *Then* $\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ *is* $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$*-connected.*

Proof. Assume $(\breve{\delta}, \breve{\gamma}, \breve{\pi}) = \tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ is not $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected. Thus, there exist two nonnull disjoint B_PSW-open sets $(\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi})$ and $(\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi})$ such that $(\breve{\delta}, \breve{\gamma}, \breve{\pi}) = (\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi}) \tilde{\tilde{\sqcup}} (\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi})$. For each $\delta \in \Delta$, $(\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ and $(\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ are disjoint B_PSW-open sets in $(\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ in which

$$\begin{aligned}&\big((\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi})\tilde{\tilde{\sqcap}}(\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})\big) \tilde{\tilde{\sqcup}} \big((\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi})\tilde{\tilde{\sqcap}}(\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})\big)\\ &= \big((\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi})\tilde{\tilde{\sqcup}}(\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi})\big) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi}) = (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi}).\end{aligned}$$

Now, from $(\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected set, one of the B_PSSs $(\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ and $(\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ is a null B_PSSs, say, $(\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi}) = (\Phi, \breve{\gamma}, \breve{\pi})$. Then, $(\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi}) \tilde{\tilde{\sqcap}}\, (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi}) = (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi})$ which implies that $(\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi})$ for all $\delta \in \Delta$ and hence $\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\breve{\zeta}_\delta, \breve{\xi}_\delta, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi})$, that is, $(\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi}) \tilde{\tilde{\sqcup}} (\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\delta}_2, \breve{\gamma}_2, \breve{\pi})$. This given, $(\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi}) = (\Phi, \breve{\gamma}, \breve{\pi})$. This is a contradiction because $(\breve{\delta}_1, \breve{\gamma}_1, \breve{\pi})$ is nonnull B_PSS. Hence, $(\breve{\delta}, \breve{\gamma}, \breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected. □

Proposition 3.20. *For any two* B_PSPs $\hbar_\upsilon^\epsilon$, $\hbar'^{\epsilon'}_{\upsilon'} \tilde{\tilde{\in}} (\breve{\zeta}, \breve{\xi}, \breve{\pi}) \tilde{\tilde{\in}} B_PSS(\breve{\Pi})$ *in a* B_PSWS $(\breve{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *are contained in some* $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$*-connected set* $(\breve{\delta}, \breve{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\zeta}, \breve{\xi}, \breve{\pi})$. *Then* $(\breve{\zeta}, \breve{\xi}, \breve{\pi})$ *is* $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$*-connected.*

Proof. Let $(\breve{\zeta}, \breve{\xi}, \breve{\pi})$ be a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-disconnected set. Thus, there is a $\widetilde{\widetilde{\widetilde{W}}}$-separated B_PSSs $(\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ and $(\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$ of $(\breve{\zeta}, \breve{\xi}, \breve{\pi})$. Then, there are two B_PSPs $\hbar_\upsilon^\epsilon$, $\hbar'^{\epsilon'}_{\upsilon'}$ in which $\hbar_\upsilon^\epsilon \tilde{\tilde{\in}} (\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ and $\hbar'^{\epsilon'}_{\upsilon'} \tilde{\tilde{\in}} (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$. Through the assumption, there is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected set $(\breve{\delta}, \breve{\gamma}, \breve{\pi})$ containing $\hbar_\upsilon^\epsilon$, $\hbar'^{\epsilon'}_{\upsilon'}$ such that

$$(\breve{\delta}, \breve{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\zeta}, \breve{\xi}, \breve{\pi}) = (\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi}) \tilde{\tilde{\sqcup}} (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi}).$$

Thus, by Proposition 3.16, we have $(\breve{\delta}, \breve{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\zeta}_1, \breve{\xi}_1, \breve{\pi})$ or $(\breve{\delta}, \breve{\gamma}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\breve{\zeta}_2, \breve{\xi}_2, \breve{\pi})$. This implies that

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \neq (\Phi, \check{\zeta}, \breve{\pi}).$$

This is contradiction since $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are $\widetilde{\widetilde{W}}$-separated B_PSSs. So, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is $B_PS\ \widetilde{\widetilde{W}}$-connected. □

Proposition 3.21. *Let $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) : \delta \in \Delta\}$ be the family of $B_PS\ \widetilde{\widetilde{W}}$-connected sets such that one of the members of this family intersects every other member. Then, $\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is $B_PS\ \widetilde{\widetilde{W}}$-connected.*

Proof. Let $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})$ be a fixed member of the given family such that $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \neq (\Phi, \check{\zeta}, \breve{\pi})$ for every $\delta \in \Delta$. Then, $(\check{\delta}_\delta, \breve{\gamma}_\delta, \breve{\pi}) = (\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is $B_PS\ \widetilde{\widetilde{W}}$-connected for each $\delta \in \Delta$, hence by Proposition 3.20. Now,

$$\begin{aligned}\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) &= \widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} \big((\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})\widetilde{\widetilde{\cup}}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big)\\ &= (\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \,\widetilde{\widetilde{\cup}}\, \big(\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big).\end{aligned}$$

Since $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})$ is one of the family $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) : \delta \in \Delta\}$ and

$$\begin{aligned}\widetilde{\widetilde{\bigcap}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) &= \widetilde{\widetilde{\bigcap}}_{\delta\in\Delta} \big((\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})\widetilde{\widetilde{\cup}}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big)\\ &= (\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \,\widetilde{\widetilde{\cap}}\, \big(\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta}(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big) \neq (\Phi, \check{\xi}, \breve{\pi}).\end{aligned}$$

From $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi})$ intersects every $(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$. Therefore, $(\check{\zeta}_{\delta_0}, \check{\xi}_{\delta_0}, \breve{\pi}) \neq (\Phi, \check{\xi}, \breve{\pi})$. Hence, by Proposition 3.19, $\widetilde{\widetilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is $B_PS\ \widetilde{\widetilde{W}}$-connected. □

Proposition 3.22. *For each two $\hbar_\upsilon^\epsilon$, $\hbar'^{\epsilon'}_{\upsilon'} \,\widetilde{\widetilde{\in}}\, B_PSP(\check{\amalg})_{(\breve{\pi},\neg\breve{\pi})}$ of a B_PSWS $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ are $B_PS\ \widetilde{\widetilde{W}}$-connected, then $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is $B_PS\ \widetilde{\widetilde{W}}$-connected.*

Proof. Let $\hbar_\upsilon^\epsilon$ be a fixed B_PSP in a B_PSWS $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$. Then, for each $\hbar_\upsilon^\epsilon$ B_PS different than $\hbar'^{\epsilon'}_{\upsilon'}$, we have $B_PS\ \widetilde{\widetilde{W}}$-connected, say, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ containing $\hbar_\upsilon^\epsilon$ and $\hbar'^{\epsilon'}_{\upsilon'}$. Since $\hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\bigcap}}_{\hbar_\upsilon^\epsilon \widetilde{\widetilde{\in}} (\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})} (\check{\zeta}, \check{\xi}, \breve{\pi})$, it follows from Proposition 3.19 that $\widetilde{\widetilde{\bigcup}}_{\hbar_\upsilon^\epsilon \widetilde{\widetilde{\in}} (\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})} (\check{\zeta}, \check{\xi}, \breve{\pi}) = (\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ is $B_PS\ \widetilde{\widetilde{W}}$-connected. □

4 $B_PS\ \widetilde{\widetilde{W}}$-Connected Spaces

In this section, the concept of bipolar soft weak connected ($B_PS\ \widetilde{\widetilde{W}}$-connected) structure is presented. Also, some properties and results of this new concept of B_PSWS are discussed.

Definition 4.1. *Let $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ be a B_PSWS. A $B_PS\ \widetilde{\widetilde{W}}$-separation of $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ is defined to be the nonnull disjoint B_PSW-open sets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ such that $\check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) = \check{\amalg}$ for each $\epsilon \in \breve{\pi}$.*

Definition 4.2. *A B_PSWS $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is called $B_PS\ \widetilde{\widetilde{W}}$-disconnected if $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ has $B_PS\ \widetilde{\widetilde{W}}$-separation. That is, there exist nonnull disjoint B_PSW-open sets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ such that $\check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) = \check{\amalg}$ for all $\epsilon \in \breve{\pi}$. Otherwise, $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is said to be $B_PS\ \widetilde{\widetilde{W}}$-connected.*

Remark 4.3. *Suppose that* $|\check{\text{II}}| = 1$*, there are only three* B_PSWS *in* $\check{\text{II}}$ *(that is,* $(\Phi, \tilde{\tilde{\check{\text{II}}}}, \breve{\pi})$*,* $(\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ *and* $(\Phi, \Phi, \breve{\pi})$*) are* B_PS $\widetilde{\widetilde{W}}$*-connected spaces, then we will have four* $\widetilde{\widetilde{W}}$*-structures* B_PSWS*:-*

(i) $\widetilde{\widetilde{W}}_1 = \{(\Phi, \tilde{\tilde{\check{\text{II}}}}, \breve{\pi}), (\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi}), (\Phi, \Phi, \breve{\pi})\}$.

(ii) $\widetilde{\widetilde{W}}_2 = \{(\Phi, \tilde{\tilde{\check{\text{II}}}}, \breve{\pi}), (\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})\}$.

(iii) $\widetilde{\widetilde{W}}_3 = \{(\Phi, \tilde{\tilde{\check{\text{II}}}}, \breve{\pi}), (\Phi, \Phi, \breve{\pi})\}$.

(iv) $\widetilde{\widetilde{W}}_4 = \{(\Phi, \tilde{\tilde{\check{\text{II}}}}, \breve{\pi})\}$.

Now, we suppose that $|\check{\text{II}}| > 1$, for the rest of our work.

Example 4.4. Let $\check{\text{II}} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $\neg\breve{\pi} = \{\neg\epsilon_1, \neg\epsilon_2, \neg\epsilon_3\}$. Suppose that $\widetilde{\widetilde{W}} = \{(\Phi, \tilde{\tilde{\check{\text{II}}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \,\tilde{\tilde{\in}}\, B_PSS(\check{\text{II}})$ defined as follows:

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_3\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_2, \hbar_3\}, \{\hbar_1, \hbar_4\}), (\epsilon_3, \{\hbar_1, \hbar_2\}, \{\hbar_3\})\} and$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_3, \hbar_4\}, \{\hbar_1, \hbar_2\}), (\epsilon_2, \{\hbar_1, \hbar_2, \hbar_3\}, \{\hbar_4\}), (\epsilon_3, \{\hbar_1, \hbar_4\}, \phi)\}.$$

Thus, $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}$-connected space since there does not exist B_PS $\widetilde{\widetilde{W}}$-separation of $(\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})$.

Example 4.5. Let $\check{\text{II}} = \{\hbar_1, \hbar_2, \hbar_3\}$ and $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$. So, the B_PSWS $\widetilde{\widetilde{W}}$ over $\check{\text{II}}$ is given by $\widetilde{\widetilde{W}} = \{(\Phi, \tilde{\tilde{\check{\text{II}}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \,\tilde{\tilde{\in}}\, B_PSS(\check{\text{II}})$ defined as follows:

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_1\}, \{\hbar_2\})\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_2, \hbar_3\}, \{\hbar_1\})\}.$$

Therefore, $(\check{\text{II}}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{W}}$-disconnected since $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ form a B_PS $\widetilde{\widetilde{W}}$-separation of $(\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})$.

Proposition 4.6. *Let* $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) : \delta \in \Delta\}$ *be a family of* B_PS $\widetilde{\widetilde{W}}$*-connected sets such that* $\tilde{\tilde{\bigcap}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \neq (\Phi, \check{\xi}, \breve{\pi})$*. Then* $\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ *is* B_PS $\widetilde{\widetilde{W}}$*-connected.*

Proof. Suppose $(\check{\delta}, \check{\gamma}, \breve{\pi}) = \tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is not B_PS $\widetilde{\widetilde{W}}$-connected. Then, there exist two nonnull disjoint B_PS $\widetilde{\widetilde{W}}$-open sets $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$ such that $(\check{\delta}, \check{\gamma}, \breve{\pi}) = (\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$. For each $\delta \in \Delta$, $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ are disjoint B_PS $\widetilde{\widetilde{W}}$-open sets in $(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ such that

$$\big((\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big) \,\tilde{\tilde{\cup}}\, \big((\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\big)$$
$$= \big((\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})\big) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) = (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}).$$

Now, $(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}$-connected sets, one of the B_PSSs $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ and $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ is a null B_PSSs, say, $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) = (\Phi, \check{\gamma}, \breve{\pi})$. Then, $(\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\cap}}\, (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) = (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$ which implies that $(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$ for all $\delta \in \Delta$ and hence $\tilde{\tilde{\bigcup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$, that is, $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) \,\tilde{\tilde{\cup}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\delta}_2, \check{\gamma}_2, \breve{\pi})$. This gives, $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi}) = (\Phi, \check{\gamma}, \breve{\pi})$. This is a contradiction, because $(\check{\delta}_1, \check{\gamma}_1, \breve{\pi})$ is nonnull B_PSS. Hence, $(\check{\delta}, \check{\gamma}, \breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}$-connected. □

Theorem 4.7. *A B_PSWS $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ over $\check{\text{II}}$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space if and only if there are two B_PSW-closed sets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ such that $\check{\xi}_1(\neg\epsilon) \neq \phi$, $\check{\xi}_2(\neg\epsilon) \neq \phi$ for some $\neg\epsilon \in \neg\breve{\pi}$, and $\check{\xi}_1(\neg\epsilon)$ $\cup$ $\check{\xi}_2(\neg\epsilon) = \check{\text{II}}$ for each $\neg\epsilon \in \neg\breve{\pi}$ and $\check{\xi}_1(\neg\epsilon) \cap \check{\xi}_2(\neg\epsilon) = \phi$ for each $\neg\epsilon \in \neg\breve{\pi}$.*

Proof. Suppose that $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected. Then, there exist B_PS $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})$, say, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$. Then,

$$\begin{aligned}
&\check{\zeta}_1(\epsilon) \cup \check{\zeta}_2(\epsilon) = \check{\text{II}} \text{ for all } \epsilon \in \breve{\pi},\\
&\check{\zeta}_1(\epsilon) \cap \check{\zeta}_2(\epsilon) = \phi \text{ for all } \epsilon \in \breve{\pi} \text{ and}\\
&\check{\zeta}_1(\epsilon) \neq \phi, \check{\zeta}_2(\epsilon) \neq \phi \text{ for some } \epsilon \in \breve{\pi}.
\end{aligned}$$

Since $\check{\zeta}_1(\epsilon) = \check{\xi}_1^c(\neg\epsilon)$ and $\check{\zeta}_2(\epsilon) = \check{\xi}_2^c(\neg\epsilon)$. Now, we get

$$\begin{aligned}
&\check{\xi}_1^c(\neg\epsilon) \cup \check{\xi}_2^c(\neg\epsilon) = \check{\text{II}} \text{ for all } \epsilon \in \breve{\pi},\\
&\check{\xi}_1^c(\neg\epsilon) \cap \check{\xi}_2^c(\neg\epsilon) = \phi \text{ for all } \epsilon \in \breve{\pi} \text{ and}\\
&\check{\xi}_1^c(\epsilon) \neq \phi, \check{\xi}_2(\epsilon) \neq \phi \text{ for some } \epsilon \in \breve{\pi}.
\end{aligned}$$

From, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}$, then $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c$ are B_PSW-closed sets.
Conversely, assuming that there are B_PSW-closed sets $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ such that

$$\begin{aligned}
&\check{\xi}_1(\neg\epsilon) \cup \check{\xi}_2(\neg\epsilon) = \check{\text{II}} \text{ for all } \neg\epsilon \in \neg\breve{\pi},\\
&\check{\xi}_1(\neg\epsilon) \cap \check{\xi}_2(\neg\epsilon) = \phi \text{ for all } \neg\epsilon \in \neg\breve{\pi} \text{ and}\\
&\check{\xi}_1(\neg\epsilon) \neq \phi, \check{\xi}_2(\neg\epsilon) \neq \phi \text{ for some } \neg\epsilon \in \neg\breve{\pi}.
\end{aligned}$$

Then $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c$ are B_PSW-open sets such that

$$\begin{aligned}
&\check{\zeta}_1^c(\epsilon) = \check{\xi}_1(\neg\epsilon) \neq \phi \text{ and } \check{\zeta}_2^c(\epsilon) = \check{\xi}_2(\neg\epsilon) \neq \phi \text{ for some } \epsilon \in \breve{\pi},\\
&\check{\zeta}_1^c(\epsilon) \cup \check{\zeta}_2^c(\epsilon) = \check{\xi}_1(\neg\epsilon) \cup \check{\xi}_2(\neg\epsilon) = \check{\text{II}} \text{ for all } \epsilon \in \breve{\pi} \text{ and}\\
&\check{\zeta}_1^c(\epsilon) \cap \check{\zeta}_2^c(\epsilon) = \check{\xi}_1(\neg\epsilon) \cap \check{\xi}_2(\neg\epsilon) = \phi \text{ for all } \epsilon \in \breve{\pi}.
\end{aligned}$$

Thus, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})^c$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})^c$ form B_PS $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})$. Thus, $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space. □

Theorem 4.8. *The B_PS intersection of a pair of B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected spaces over a common universal set is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected.*

Proof. Let $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ and $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ be two B_PS $\widetilde{\widetilde{\widetilde{W}}}_i$-connected spaces over $\check{\text{II}}$, $i = 1, 2$ and $\widetilde{\widetilde{\widetilde{W}}}$ $= \widetilde{\widetilde{\widetilde{W}}}_1 \tilde{\tilde{\cap}} \widetilde{\widetilde{\widetilde{W}}}_2$. We need to show that the space $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected. If we say that $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is not B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected. Then there exist two B_PSSs $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$,$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}$, which forms a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-separation of $(\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ in $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. From $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}$, then $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}_1$ and $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ $\tilde{\tilde{\in}}$ $\widetilde{\widetilde{\widetilde{W}}}_2$. This lead to $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ form a B_PS $\widetilde{\widetilde{\widetilde{W}}}_1$-separation of $(\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ in $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ and also $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ form a B_PS $\widetilde{\widetilde{\widetilde{W}}}_2$-separation of $(\tilde{\tilde{\check{\text{II}}}}, \Phi, \breve{\pi})$ in $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ which is the contradiction to given hypothesis. Therefore, $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected space over $\check{\text{II}}$. □

Remark 4.9. *The B_PS union of a pair of B_PS $\widetilde{\widetilde{W}}$-connected spaces over the common universal set may not be B_PS $\widetilde{\widetilde{W}}$-connected. As shown in the following example.*

Example 4.10. Let $\check{\Pi} = \{\hbar_1, \hbar_2\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$, $\widetilde{\widetilde{W}}_1 = \{(\Phi, \tilde{\tilde{\Pi}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi})\}$ and $\widetilde{\widetilde{W}}_2 = \{(\Phi, \tilde{\tilde{\Pi}}, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$, where

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \phi, \check{\Pi}), (\epsilon_2, \check{\Pi}, \phi)\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \check{\Pi}, \phi), (\epsilon_2, \phi, \check{\Pi})\}.$$

Clearly $(\check{\Pi}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$ and $(\check{\Pi}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$ are B_PS $\widetilde{\widetilde{W}}$-connected spaces over $\check{\Pi}$ where $\widetilde{\widetilde{W}} = \widetilde{\widetilde{W}}_1 \tilde{\cup} \widetilde{\widetilde{W}}_2$. We note that $\widetilde{\widetilde{W}}_1 \tilde{\cup} \widetilde{\widetilde{W}}_2 = \{(\Phi, \tilde{\tilde{\Pi}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$ is not a B_PS $\widetilde{\widetilde{W}}$-connected space over $\check{\Pi}$ since $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ form a B_PS $\widetilde{\widetilde{W}}$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \breve{\pi})$ in $\widetilde{\widetilde{W}}_1 \tilde{\cup} \widetilde{\widetilde{W}}_2$.

Proposition 4.11. *The B_PS union of a pair of B_PS $\widetilde{\widetilde{W}}$-disconnected spaces over the common universal set is B_PS $\widetilde{\widetilde{W}}$-disconnected.*

Proof. Obvious. □

Remark 4.12. *The B_PS intersection of a pair of B_PS $\widetilde{\widetilde{W}}$-disconnected spaces over the common universal set is not necessarily a B_PS $\widetilde{\widetilde{W}}$-disconnected space, as shown in the following example.*

Example 4.13. Let $\check{\Pi} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$,
$\widetilde{\widetilde{W}}_1 = \{(\Phi, \tilde{\tilde{\Pi}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$ and
$\widetilde{\widetilde{W}}_2 = \{(\Phi, \tilde{\tilde{\Pi}}, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi}), (\check{\zeta}_4, \check{\xi}_4, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, $(\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$, $(\check{\zeta}_4, \check{\xi}_4, \breve{\pi}) \tilde{\tilde{\in}} B_PSS(\check{\Pi})$ defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_1, \hbar_2\}, \{\hbar_3\})\},$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \phi), (\epsilon_2, \{\hbar_3\}, \{\hbar_1\})\},$$
$$(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_3\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_1, \hbar_3\}, \{\hbar_2\})\} \text{ and}$$
$$(\check{\zeta}_4, \check{\xi}_4, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1\})\}.$$

Clearly $(\check{\Pi}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$ and $(\check{\Pi}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$ are B_PS $\widetilde{\widetilde{W}}$-disconnected spaces over $\check{\Pi}$ where $\widetilde{\widetilde{W}} = \widetilde{\widetilde{W}}_1 \tilde{\tilde{\cap}} \widetilde{\widetilde{W}}_2$. We note that $\widetilde{\widetilde{W}}_1 \tilde{\tilde{\cap}} \widetilde{\widetilde{W}}_2 = \{(\Phi, \tilde{\tilde{\Pi}}, \breve{\pi})\}$ is not a B_PS $\widetilde{\widetilde{W}}$-disconnected space over $\check{\Pi}$ since there is no two B_PS $\widetilde{\widetilde{W}}$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \breve{\pi})$ in $\widetilde{\widetilde{W}}_1 \tilde{\tilde{\cap}} \widetilde{\widetilde{W}}_2$.

Proposition 4.14. *Let $(\check{\Pi}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ be a B_PSWS over $\check{\Pi}$. If there exist a nonnull, nonabsolute B_PSW-clopen set $(\check{\zeta}, \check{\xi}, \breve{\pi})$ over $\check{\Pi}$ with $\check{\zeta}(\epsilon) \cup \check{\zeta}^c(\epsilon) = \check{\Pi}$ for each $\epsilon \in \breve{\pi}$, then $(\check{\Pi}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{W}}$-disconnected.*

Proof. Since $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is a nonnull, nonabsolute B_PSW-clopen set, then $(\check{\zeta}, \check{\xi}, \breve{\pi})^c$ is a nonnull nonabsolute B_PSW-clopen set. By Proposition 2.12 and the assumption, we get

$$\check{\zeta}(\epsilon) \cup \check{\zeta}^c(\epsilon) = \check{\Pi}, \text{ for each } \epsilon \in \breve{\pi}, \text{ and } \check{\xi}(\neg\epsilon) \cap \check{\xi}^c(\neg\epsilon) = \phi, \text{ for each } \neg\epsilon \in \neg\breve{\pi},$$

and

$$\check{\zeta}(\epsilon) \cap \check{\zeta}^c(\epsilon) = \phi, \text{ for each } \epsilon \in \breve{\pi}, \text{ and } \check{\xi}(\neg\epsilon) \cup \check{\xi}^c(\neg\epsilon) = \check{\amalg}, \text{ for each } \neg\epsilon \in \neg\breve{\pi},$$

Therefore, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ and $(\check{\zeta}, \check{\xi}, \breve{\pi})^c$ form a B_PS $\widetilde{\widetilde{W}}$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$. Hence, $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}$-disconnected space. □

Remark 4.15. *If there exist a nonnull, nonabsolute B_PSW-open set, B_PSW-closed set, then $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ may not be a B_PS $\widetilde{\widetilde{W}}$-disconnected space. As shown in the following example.*

Example 4.16. Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and
$\widetilde{\widetilde{W}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \,\tilde{\tilde{\in}}\, B_PSS(\check{\amalg})$ defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1, \hbar_2\}, \{\hbar_3\}), (\epsilon_2, \{\hbar_1\}, \{\hbar_3\})\} and$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_3\}, \{\hbar_1, \hbar_2\}), (\epsilon_2, \{\hbar_3\}, \{\hbar_1\})\}.$$

Obviously, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ is nonnull, nonabsolute B_PSW-clopen but $(\check{\amalg}, \widetilde{\widetilde{W}}, \breve{\pi}, \neg\breve{\pi})$ is not a B_PS $\widetilde{\widetilde{W}}$-disconnected space since there does not exist B_PS $\widetilde{\widetilde{W}}$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$.

Proposition 4.17. *Let $(\check{\amalg}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$ and $(\check{\amalg}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$ be two B_PSWSs over $\check{\amalg}$. Then,*

- *(i) If $(\check{\amalg}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}_1$-connected such that $\widetilde{\widetilde{W}}_2 \,\tilde{\tilde{\subseteq}}\, \widetilde{\widetilde{W}}_1$, then $(\check{\amalg}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}_2$-connected.*
- *(ii) If $(\check{\amalg}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}_1$-disconnected such that $\widetilde{\widetilde{W}}_1 \,\tilde{\tilde{\subseteq}}\, \widetilde{\widetilde{W}}_2$, then $(\check{\amalg}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}_2$-disconnected.*

Proof.

- (i) Assume that $(\check{\amalg}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}_1$-connected such that $\widetilde{\widetilde{W}}_2 \,\tilde{\tilde{\subseteq}}\, \widetilde{\widetilde{W}}_1$. Assume the contrary that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are B_PS $\widetilde{\widetilde{W}}_2$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ in $(\check{\amalg}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$. Since $\widetilde{\widetilde{W}}_2 \,\tilde{\tilde{\subseteq}}\, \widetilde{\widetilde{W}}_1$, then $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are B_PS $\widetilde{\widetilde{W}}_1$-separation of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ in $(\check{\amalg}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$. This is contradiction. Therefore, $(\check{\amalg}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{W}}_2$-connected.
- (ii) Let $(\check{\amalg}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$ be a B_PS $\widetilde{\widetilde{W}}_1$-disconnected such that $\widetilde{\widetilde{W}}_1 \,\tilde{\tilde{\subseteq}}\, \widetilde{\widetilde{W}}_2$. Assume the contrary that $(\check{\amalg}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}_2$-connected space. Since $\widetilde{\widetilde{W}}_1 \,\tilde{\tilde{\subseteq}}\, \widetilde{\widetilde{W}}_2$, then by (i), we get $(\check{\amalg}, \widetilde{\widetilde{W}}_1, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{W}}_1$-connected. This is contradiction. Therefore, $(\check{\amalg}, \widetilde{\widetilde{W}}_2, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{W}}_2$-disconnected.

□

Proposition 4.18. *Let $\left((\check{\zeta}, \check{\xi}, \breve{\pi}), \widetilde{\widetilde{W}}_{(\check{\zeta}, \check{\xi}, \breve{\pi})}, \breve{\pi}, \neg\breve{\pi}\right)$ be B_PS $\widetilde{\widetilde{W}}$-connected, then $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is B_PS $\widetilde{\widetilde{W}}$-connected.*

Proof. Let $\left((\check{\zeta}, \check{\xi}, \breve{\pi}), \widetilde{\widetilde{W}}_{(\check{\zeta}, \check{\xi}, \breve{\pi})}, \breve{\pi}, \neg\breve{\pi}\right)$ be a B_PS $\widetilde{\widetilde{W}}$-connected space. Assume $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is B_PS $\widetilde{\widetilde{W}}$-disconnected, then there exist $\widetilde{\widetilde{W}}$-separated B_PSSs, say, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ of $(\check{\zeta}, \check{\xi}, \breve{\pi})$, so by Theorem 3.7 that $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$ and $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$ are B_PS $\widetilde{\widetilde{W}}$-separation of $(\check{\zeta}, \check{\xi}, \breve{\pi})$. This is a contradiction. Thus, $(\check{\zeta}, \check{\xi}, \breve{\pi})$ is a B_PS $\widetilde{\widetilde{W}}$-connected space. □

Definition 4.19. A property $\mathcal{P}$ of a B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is said to be a B_PS weak hereditary property ($B_PS\widetilde{\widetilde{\widetilde{W}}}$- hereditary property) if every B_PSWSS $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ of $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ also has the property $\mathcal{P}$.

Remark 4.20. *The B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected space (resp. B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space) is not necessarily a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-hereditary property. As shown in the following example.*

Example 4.21. Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and
$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\tilde{\check{\amalg}}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \tilde{\tilde{\in}} B_PSS(\check{\amalg})$, defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2, \hbar_3\}), (\epsilon_2, \{\hbar_1\}, \{\hbar_2, \hbar_3\})\} \textit{ and}$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1, \hbar_3\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1, \hbar_3\})\}.$$

Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected space.
Now let $\check{Y} = \{\hbar_1, \hbar_2\}$, then $\widetilde{\widetilde{\widetilde{W}}}_{\check{Y}} = \{(\Phi, \tilde{\tilde{\tilde{\check{Y}}}}, \breve{\pi}), (^{\check{Y}}\check{\zeta}_1, ^{\check{Y}}\check{\xi}_1, \breve{\pi}), (^{\check{Y}}\check{\zeta}_2, ^{\check{Y}}\check{\xi}_2, \breve{\pi})\}$, such that

$$(^{\check{Y}}\check{\zeta}_1, ^{\check{Y}}\check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_1\}, \{\hbar_2,\})\} \textit{ and}$$
$$(^{\check{Y}}\check{\zeta}_2, ^{\check{Y}}\check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2\}, \{\hbar_1\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1\})\}.$$

Clearly, $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected subspace of $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. While $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected space.

Example 4.22. Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}$, $\breve{\pi} = \{\epsilon_1, \epsilon_2\}$ and
$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\tilde{\check{\amalg}}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi})\}$, where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) \tilde{\tilde{\in}} B_PSS(\check{\amalg})$, defined as follows

$$(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \{\hbar_1\}, \{\hbar_2\}), (\epsilon_2, \{\hbar_2\}, \{\hbar_1, \hbar_3\})\} \textit{ and}$$
$$(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \{\hbar_2, \hbar_3\}, \phi), (\epsilon_2, \{\hbar_1, \hbar_3\}, \{\hbar_2\})\}.$$

Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-disconnected space.
Let $\check{Y} = \{\hbar_3\}$, then $\widetilde{\widetilde{\widetilde{W}}}_{\check{Y}} = \{(\Phi, \tilde{\tilde{\tilde{\check{Y}}}}, \breve{\pi}), (^{\check{Y}}\check{\zeta}_1, ^{\check{Y}}\check{\xi}_1, \breve{\pi}), (^{\check{Y}}\check{\zeta}_2, ^{\check{Y}}\check{\xi}_2, \breve{\pi})\}$, such that

$$(^{\check{Y}}\check{\zeta}_1, ^{\check{Y}}\check{\xi}_1, \breve{\pi}) = \{(\epsilon_1, \phi, \phi), (\epsilon_2, \phi, \check{Y})\},$$
$$(^{\check{Y}}\check{\zeta}_2, ^{\check{Y}}\check{\xi}_2, \breve{\pi}) = \{(\epsilon_1, \check{Y}, \phi), (\epsilon_2, \check{Y}, \phi)\}.$$

Clearly, $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected subspace of $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. While $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected space.

5 B_PS $\widetilde{\widetilde{\widetilde{W}}}$-Locally Connected Spaces and B_PS $\widetilde{\widetilde{\widetilde{W}}}$-Components

In this section, a new type of connected set is studied, known as $\widetilde{\widetilde{\widetilde{W}}}$-locally connected in B_PSWS. Furthermore, B_PS $\widetilde{\widetilde{\widetilde{W}}}$-component with some properties.

Definition 5.1. *A B_PS $\widetilde{\widetilde{\widetilde{W}}}$-component of B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ corresponding to $\hbar_{\upsilon}^{\epsilon}$ is the B_PS union of all B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected $(\check{\zeta}, \check{\xi}, \breve{\pi}) \tilde{\tilde{\subseteq}} (\tilde{\tilde{\tilde{\check{\amalg}}}}, \Phi, \breve{\pi})$ which contains $\hbar_{\upsilon}^{\epsilon}$. It is denoted by $B_PSW\text{-}Co(\hbar_{\upsilon}^{\epsilon})$ that is*

$$B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) = \widetilde{\widetilde{\bigcup}}\{(\check{\zeta},\check{\xi},\breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\tilde{\tilde{\check{\text{II}}}},\Phi,\breve{\pi}) : \hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, (\check{\zeta},\check{\xi},\breve{\pi}) \text{ and } (\check{\zeta},\check{\xi},\breve{\pi}) \text{ is } B_PS\ \widetilde{\widetilde{\widetilde{W}}}\text{-connected}\}.$$

Definition 5.2. *A B_PSWS $(\check{\text{II}},\widetilde{\widetilde{\widetilde{W}}},\breve{\pi},\neg\breve{\pi})$ is called B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally connected at $\hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, (\tilde{\tilde{\check{\text{II}}}},\Phi,\breve{\pi})$ if for every B_PS $\widetilde{\widetilde{\widetilde{W}}}$-open set $(\check{\zeta},\check{\xi},\breve{\pi})$ containing $\hbar_\upsilon^\epsilon$, there is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected open $(\check{\delta},\check{\gamma},\breve{\pi})$ containing $\hbar_\upsilon^\epsilon$ such that $\hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, (\check{\delta},\check{\gamma},\breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, (\check{\zeta},\check{\xi},\breve{\pi})$. A B_PSWS $(\check{\text{II}},\widetilde{\widetilde{\widetilde{W}}},\breve{\pi},\neg\breve{\pi})$ is said to be B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally connected if it is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally connected at every B_PSP $\hbar_\upsilon^\epsilon \,\widetilde{\widetilde{\in}}\, (\tilde{\tilde{\check{\text{II}}}},\Phi,\breve{\pi})$. Otherwise, it is said to be B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally disconnected.*

Remark 5.3. *B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally connectedness and B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connectedness are independent as shown below.*

Example 5.4. Let $\check{\text{II}} = \{\hbar_1,\hbar_2,\hbar_3\}$, $\breve{\pi} = \{\epsilon_1,\epsilon_2\}$ and $\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi,\tilde{\tilde{\check{\text{II}}}},\breve{\pi}), (\tilde{\tilde{\check{\text{II}}}},\Phi,\breve{\pi}), (\check{\zeta}_1,\check{\xi}_1,\breve{\pi}), (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}), (\check{\zeta}_3,\check{\xi}_3,\breve{\pi}), (\check{\zeta}_4,\check{\xi}_4,\breve{\pi})\}$, where $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi}), (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}), (\check{\zeta}_3,\check{\xi}_3,\breve{\pi}), (\check{\zeta}_4,\check{\xi}_4,\breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\text{II}})$, defined as follows

$$\begin{aligned}
&(\check{\zeta}_1,\check{\xi}_1,\breve{\pi}) = \{(\epsilon_1,\{\hbar_1,\hbar_2\},\phi)\},\\
&(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = \{(\epsilon_1,\{\hbar_2,\hbar_3\},\phi)\},\\
&(\check{\zeta}_3,\check{\xi}_3,\breve{\pi}) = \{(\epsilon_1,\check{\text{II}},\phi),(\epsilon_2,\phi,\check{\text{II}})\} \textit{ and}\\
&(\check{\zeta}_4,\check{\xi}_4,\breve{\pi}) = \{(\epsilon_1,\phi,\check{\text{II}}),(\epsilon_2,\check{\text{II}},\phi)\}.
\end{aligned}$$

Then $(\check{\text{II}},\widetilde{\widetilde{\widetilde{W}}},\breve{\pi},\neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally connected space but not B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected.

Example 5.5. Let $\check{\text{II}} = \{\hbar_1,\hbar_2,\hbar_3\}$, $\breve{\pi} = \{\epsilon_1,\epsilon_2\}$ and $\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi,\tilde{\tilde{\check{\text{II}}}},\breve{\pi}), (\tilde{\tilde{\check{\text{II}}}},\Phi,\breve{\pi}), (\check{\zeta}_1,\check{\xi}_1,\breve{\pi}), (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}), (\check{\zeta}_3,\check{\xi}_3,\breve{\pi}), (\check{\zeta}_4,\check{\xi}_4,\breve{\pi})\}$ where $(\check{\zeta}_1,\check{\xi}_1,\breve{\pi}), (\check{\zeta}_2,\check{\xi}_2,\breve{\pi}), (\check{\zeta}_3,\check{\xi}_3,\breve{\pi}), (\check{\zeta}_4,\check{\xi}_4,\breve{\pi}) \,\widetilde{\widetilde{\in}}\, B_PSS(\check{\text{II}})$, defined as follows

$$\begin{aligned}
&(\check{\zeta}_1,\check{\xi}_1,\breve{\pi}) = \{(\epsilon_1,\{\hbar_2\},\{\hbar_3\})\},\\
&(\check{\zeta}_2,\check{\xi}_2,\breve{\pi}) = \{(\epsilon_1,\{\hbar_1,\hbar_2\},\phi)\},\\
&(\check{\zeta}_3,\check{\xi}_3,\breve{\pi}) = \{(\epsilon_1,\{\hbar_2\},\{\hbar_3\}),(\epsilon_2,\{\hbar_2\},\{\hbar_1,\hbar_3\})\},\\
&(\check{\zeta}_4,\check{\xi}_4,\breve{\pi}) = \{(\epsilon_1,\{\hbar_1,\hbar_2\},\phi),(\epsilon_2,\{\hbar_2\},\{\hbar_1,\hbar_3\})\}.
\end{aligned}$$

Then, $(\check{\text{II}},\widetilde{\widetilde{\widetilde{W}}},\breve{\pi},\neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected space but not B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally connected because $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ is the B_PS $\widetilde{\widetilde{\widetilde{W}}}$-open set containing $\hbar^{\epsilon_1}_{1_{\hbar_2}}$, but there is no B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected open subset of $(\check{\zeta}_2,\check{\xi}_2,\breve{\pi})$ containing $\hbar^{\epsilon_1}_{1_{\hbar_2}}$.

Remark 5.6. *For a B_PSWS $(\check{\text{II}},\widetilde{\widetilde{\widetilde{W}}},\breve{\pi},\neg\breve{\pi})$, we have*

(i) According to Proposition 4.6, every B_PS $\widetilde{\widetilde{\widetilde{W}}}$-component of a B_PSP is the largest B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected set containing this B_PSP.

(ii) If $(\check{\text{II}},\widetilde{\widetilde{\widetilde{W}}},\breve{\pi},\neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected space, then $(\tilde{\tilde{\check{\text{II}}}},\Phi,\breve{\pi})$ is only the B_PS $\widetilde{\widetilde{\widetilde{W}}}$-component of each B_PSP.

Example 5.7. Consider the B_PSWS in Example 5.4, we have the following:

$B_PSW\text{-}Co(\hbar^{\epsilon_1}_{1_{\hbar_2}}) = B_PSW\text{-}Co(\hbar^{\epsilon_1}_{1_{\hbar_3}}) = B_PSW\text{-}Co(\hbar^{\epsilon_1}_{2_{\hbar_1}}) = B_PSW\text{-}Co(\hbar^{\epsilon_1}_{2_{\hbar_3}}) = B_PSW\text{-}Co(\hbar^{\epsilon_1}_{3_{\hbar_1}}) = B_PSW\text{-}Co(\hbar^{\epsilon_1}_{3_{\hbar_2}}) = (\check{\zeta}_3,\check{\xi}_3,\breve{\pi})$ and
$B_PSW\text{-}Co(\hbar^{\epsilon_2}_{1_{\hbar_2}}) = B_PSW\text{-}Co(\hbar^{\epsilon_2}_{1_{\hbar_3}}) = B_PSW\text{-}Co(\hbar^{\epsilon_2}_{2_{\hbar_1}}) = B_PSW\text{-}Co(\hbar^{\epsilon_2}_{2_{\hbar_3}}) = B_PSW\text{-}Co(\hbar^{\epsilon_2}_{3_{\hbar_1}}) = B_PSW\text{-}Co(\hbar^{\epsilon_2}_{3_{\hbar_2}}) = (\check{\zeta}_4,\check{\xi}_4,\breve{\pi})$.

Theorem 5.8. *A B_PSWS $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally connected if and only if the B_PS $\widetilde{\widetilde{\widetilde{W}}}$-components of B_PS $\widetilde{\widetilde{\widetilde{W}}}$-open sets are B_PS $\widetilde{\widetilde{\widetilde{W}}}$-open sets.*

Proof. Assume that the space $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-locally connected. Let $(\check{\zeta}, \check{\xi}, \breve{\pi})$ be B_PS $\widetilde{\widetilde{\widetilde{W}}}$-open and $B_PSW\text{-}Co$ be a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-component of $(\check{\zeta}, \check{\xi}, \breve{\pi})$. If $\hbar_\upsilon^\epsilon \,\tilde{\tilde{\in}}\, B_PSW\text{-}Co$ and since $\hbar_\upsilon^\epsilon \in (\check{\zeta}, \check{\xi}, \breve{\pi})$, there is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected open set $(\check{\delta}, \check{\gamma}, \breve{\pi})$ such that $\hbar_\upsilon^\epsilon \,\tilde{\tilde{\in}}\, (\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}, \check{\xi}, \breve{\pi})$. Now, as $B_PSW\text{-}Co$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-component of $\hbar_\upsilon^\epsilon$ and $(\check{\delta}, \check{\gamma}, \breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected, we have $\hbar_\upsilon^\epsilon \,\tilde{\tilde{\in}}\, (\check{\delta}, \check{\gamma}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}Co$. This shows that $B_PSW\text{-}Co$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-open.

Conversely, let $\hbar_\upsilon^\epsilon \,\tilde{\tilde{\in}}\, (\tilde{\tilde{\tilde{\Pi}}}, \Phi, \breve{\pi})$ be arbitrary and $(\check{\zeta}, \check{\xi}, \breve{\pi})$ be a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-open set containing $\hbar_\upsilon^\epsilon$. Suppose $B_PSW\text{-}Co$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-component of $(\check{\zeta}, \check{\xi}, \breve{\pi})$ such that $\hbar_\upsilon^\epsilon \,\tilde{\tilde{\in}}\, B_PSW\text{-}Co$. Now, $B_PSW\text{-}Co$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected open set with $\hbar_\upsilon^\epsilon \,\tilde{\tilde{\in}}\, B_PSW\text{-}Co \,\tilde{\tilde{\subseteq}}\, (\check{\zeta}, \check{\xi}, \breve{\pi})$. This proves the theorem. □

Theorem 5.9. *Let $(\check{\Pi}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ be a B_PSWS, then*

(i) The family of all distinct B_PS $\widetilde{\widetilde{\widetilde{W}}}$-components of a B_PSPs of $(\tilde{\tilde{\tilde{\Pi}}}, \Phi, \breve{\pi})$ forms a partition of $(\tilde{\tilde{\tilde{\Pi}}}, \Phi, \breve{\pi})$.

(ii) For every B_PS $\widetilde{\widetilde{\widetilde{W}}}$-component $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$, we have $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) = B_PSW\text{-}clB_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$.

Proof.

(*i*) Let $\{B_PSW\text{-}Co(\hbar_\upsilon^\epsilon)\colon\ \hbar_\upsilon^\epsilon \,\tilde{\tilde{\in}}\, (\tilde{\tilde{\tilde{\Pi}}}, \Phi, \breve{\pi})\}$ be a family of all distinct B_PS $\widetilde{\widetilde{\widetilde{W}}}$-components of $(\tilde{\tilde{\tilde{\Pi}}}, \Phi, \breve{\pi})$. Clearly, $(\tilde{\tilde{\tilde{\Pi}}}, \Phi, \breve{\pi}) = \tilde{\tilde{\bigcup}} \{B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) : \hbar_\upsilon^\epsilon \,\tilde{\tilde{\in}}\, (\tilde{\tilde{\tilde{\Pi}}}, \Phi, \breve{\pi})\}$. Suppose that there are two distinct B_PSPs $\hbar_\upsilon^\epsilon$ and $\hbar'^{\epsilon'}_{\upsilon'}$ such that $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) \,\tilde{\tilde{\cap}}\, B_PSW\text{-}Co(\hbar'^{\epsilon'}_{\upsilon'}) \neq (\Phi, \check{\xi}, \breve{\pi})$. By Proposition 4.6, $(\check{\zeta}, \check{\xi}, \breve{\pi}) = B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) \,\tilde{\tilde{\cup}}\, B_PSW\text{-}Co(\hbar'^{\epsilon'}_{\upsilon'})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected set. This contradicts that $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$ and $B_PSW\text{-}Co(\hbar'^{\epsilon'}_{\upsilon'})$ are the largest B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected sets containing $\hbar_\upsilon^\epsilon$ and $\hbar'^{\epsilon'}_{\upsilon'}$ respectively. Hence $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) \,\tilde{\tilde{\cap}}\, B_PSW\text{-}Co(\hbar'^{\epsilon'}_{\upsilon'}) = (\Phi, \check{\xi}, \breve{\pi})$.

(*ii*) Since $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected and $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) \,\tilde{\tilde{\subseteq}}\, B_PSW\text{-}clB_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$, it follows since Proposition 3.18 that $B_PSW\text{-}clB_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected set also. Since $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$ is the largest B_PS $\widetilde{\widetilde{\widetilde{W}}}$-connected set containing $\hbar_\upsilon^\epsilon$. Hence, $B_PSW\text{-}Co(\hbar_\upsilon^\epsilon) = B_PSW\text{-}clB_PSW\text{-}Co(\hbar_\upsilon^\epsilon)$.

□

6 B_PS $\widetilde{\widetilde{\widetilde{W}}}$-Compact Spaces

Because of the compactness property importance, this section researches it in B_PSWSs with some essential theorems.

Definition 6.1. *A family $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\colon\ (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}) \,\tilde{\tilde{\in}}\, \widetilde{\widetilde{\widetilde{W}}}\}_{\delta\in\Delta}$ of B_PSW-open sets on $\check{\Pi}$ is said to be a B_PSW-open cover of a B_PSS $(\check{\zeta}, \check{\xi}, \breve{\pi})$ if, $(\check{\zeta}, \check{\xi}, \breve{\pi}) \,\tilde{\tilde{\subseteq}}\, \tilde{\tilde{\cup}}_{\delta\in\Delta}\ (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})$. Furthermore, a B_PS subcover is a subfamily of $\{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ which is also a B_PSW-open cover.*

Definition 6.2. *A B_PS subset $(\check{\zeta}, \check{\xi}, \breve{\pi})$ of $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ is known as a bipolar soft weak compact set, denoted as a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact set, if each B_PSW-open cover of $(\check{\zeta}, \check{\xi}, \breve{\pi})$ has a finite B_PS subcover.*

Definition 6.3. *A B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is said to be B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact space if $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact subset of itself.*

Example 6.4. Let $\check{\amalg} = \mathbb{N}$ be the set of all natural numbers, $\breve{\pi} = \{\epsilon\}$ and $\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_n, \check{\xi}_n, \breve{\pi}) : n \in \mathbb{N}\}$ where,

$$
\begin{gathered}
(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon, \{1,2\}, \mathbb{N} \setminus \{1,2\})\}, \\
(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon, \{1,3\}, \mathbb{N} \setminus \{1,3\})\}, \\
(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon, \{1,4\}, \mathbb{N} \setminus \{1,4\})\}, \\
\vdots
\end{gathered}
$$

Then, a B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is not B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact, since $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi}) = \tilde{\tilde{\cup}}_{n \in \mathbb{N}} (\check{\zeta}_n, \check{\xi}_n, \breve{\pi})$. Whereas $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi}) \neq \tilde{\tilde{\cup}}_{i=1}^{n} (\check{\zeta}_i, \check{\xi}_i, \breve{\pi})$.

Remark 6.5. *Let $\check{\amalg}$ be a finite universe set and let $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi}) \tilde{\tilde{\notin}} \widetilde{\widetilde{\widetilde{W}}}$. If B_PS union of some B_PSW-open sets is $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$, then $(\tilde{\tilde{\check{\amalg}}}, \Phi, \breve{\pi})$ is B_PSW-compact space.*

Example 6.6. Let $\check{\amalg} = \{\hbar_1, \hbar_2, \hbar_3\}, \breve{\pi} = \{\epsilon\}$ and

$$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_2, \check{\xi}_2, \breve{\pi}), (\check{\zeta}_3, \check{\xi}_3, \breve{\pi})\}$$

be a B_PSWS over $\check{\amalg}$ where $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_2, \check{\xi}_2, \breve{\pi})$, $(\check{\zeta}_3, \check{\xi}_3, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\amalg})$, defined as follows

$$
\begin{aligned}
&(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon, \{\hbar_1\}, \{\hbar_2\})\}, \\
&(\check{\zeta}_2, \check{\xi}_2, \breve{\pi}) = \{(\epsilon, \{\hbar_2\}, \{\hbar_3\})\} \textit{ and} \\
&(\check{\zeta}_3, \check{\xi}_3, \breve{\pi}) = \{(\epsilon, \{\hbar_3\}, \{\hbar_1\})\}.
\end{aligned}
$$

Then, a B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact.

Remark 6.7. *Every B_PSW-closed subset of a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact space is not necessarily B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact.*

Example 6.8. Let $\check{\amalg} = \mathbb{N}$, $\breve{\pi} = \{\epsilon\}$ and

$$\widetilde{\widetilde{\widetilde{W}}} = \{(\Phi, \tilde{\tilde{\check{\amalg}}}, \breve{\pi}), (\check{\zeta}_0, \check{\xi}_0, \breve{\pi}), (\check{\zeta}_1, \check{\xi}_1, \breve{\pi}), (\check{\zeta}_n, \check{\xi}_n, \breve{\pi}) : n \in \mathbb{N} \setminus \{1\}\}$$

be a B_PSWS over $\check{\amalg}$ where $(\check{\zeta}_0, \check{\xi}_0, \breve{\pi})$, $(\check{\zeta}_1, \check{\xi}_1, \breve{\pi})$, $(\check{\zeta}_n, \check{\xi}_n, \breve{\pi})$ $\tilde{\tilde{\in}}$ $B_PSS(\check{\amalg})$, defined as follows

$$
\begin{aligned}
&(\check{\zeta}_0, \check{\xi}_0, \breve{\pi}) = \{(\epsilon, \{1,4,5,...\}, \{2,3\})\}, \\
&(\check{\zeta}_1, \check{\xi}_1, \breve{\pi}) = \{(\epsilon, \{1\}, \{2,3,4,5,...\})\} \textit{ and} \\
&(\check{\zeta}_n, \check{\xi}_n, \breve{\pi}) = \{(\epsilon, \{2,3,4,5,...,n\}, \{1\}) : n \in \mathbb{N} \setminus \{1\}\}.
\end{aligned}
$$

Then, a B_PSWS $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact. But the B_PSW-closed set $\{(\epsilon, \{2,3,4,5,...\}, \{1\})\}$ is not a B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact set, since the family $\{(\check{\zeta}_n, \check{\xi}_n, \breve{\pi}) : n \in \mathbb{N} \setminus \{1\}\}$ is a B_PSW-open cover of $\{(\epsilon, \{2,3,4,5,...\}, \{1\})\}$. That is

$$\{(\epsilon, \{2,3,4,5,...\}, \{1\})\} \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{\cup}}_{n\in\mathbb{N}\setminus\{1\}} (\check{\zeta}_n, \check{\xi}_n, \breve{\pi}).$$

Then this B_PSW-open cover has no finite B_PS subcover. That is

$$\{(\epsilon, \{2,3,4,5,...\}, \{1\})\} \,\widetilde{\widetilde{\nsubseteq}}\, \widetilde{\widetilde{\cup}}_{n=2}^{k} (\check{\zeta}_n, \check{\xi}_n, \breve{\pi}), \ for\ k \in \mathbb{N} \setminus \{1\}.$$

Proposition 6.9. *If* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *is a* B_PS $\widetilde{\widetilde{\widetilde{W}}}$*-compact space, then* $(\check{\amalg}, \widetilde{W}, \breve{\pi})$ *is an* S $\widetilde{W}$*-compact space.*

Proof. Straightforward. □

Proposition 6.10. *If* $(\check{\amalg}, \widetilde{W}, \breve{\pi})$ *is an* S $\widetilde{W}$*-compact space and* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *is a* B_PSWS *constructed since Theorem 2.24, then* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ *is a* B_PS $\widetilde{\widetilde{\widetilde{W}}}$*-compact space.*

Proof. Let $(\check{\amalg}, \widetilde{W}, \breve{\pi})$ be an S $\widetilde{W}$-compact space and $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ be a B_PSW-open cover of $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$. That is

$$(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{\cup}}_{\delta\in\Delta} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}).$$

Then, $\check{\amalg} = \cup\{\check{\zeta}_\delta(\epsilon)\}_{\delta\in\Delta}$ for all $\epsilon \in \breve{\pi}$. Since $(\check{\amalg}, \widetilde{W}, \breve{\pi})$ is an S $\widetilde{W}$-compact space, then $\check{\amalg} = \cup\{\check{\zeta}_{\delta_i}(\epsilon) : i = 1,2,..,n\}_{\delta_i\in\Delta}$. Since $\check{\xi}(\neg\epsilon) = \check{\amalg} \setminus \check{\zeta}(\epsilon)$ for all $\epsilon \in \breve{\pi}$, then $\Phi = \cap\{\check{\xi}_{\delta_i}(\neg\epsilon) : i = 1,2,...,n\}_{\delta_i\in\Delta}$. Hence, $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{\cup}}_{i=1}^{n} (\check{\zeta}_{\delta_i}, \check{\xi}_{\delta_i}, \breve{\pi})$. Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ is B_PS $\widetilde{\widetilde{\widetilde{W}}}$-compact. □

Theorem 6.11. *Let* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ *and* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ *be* B_PSWSs*. Then*

(*i*) *If* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ *is a* B_PS $\widetilde{\widetilde{\widetilde{W}}}_2$*-compact space on* $\check{\amalg}$ *and* $\widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{\widetilde{W}}}_2$*. Then* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ *is a* B_PS $\widetilde{\widetilde{\widetilde{W}}}_1$*-compact space on* $\check{\amalg}$*.*

(*ii*) *If* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ *is not* B_PS $\widetilde{\widetilde{\widetilde{W}}}_1$*-compact space on* $\check{\amalg}$ *and* $\widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{\widetilde{W}}}_2$*. Then* $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ *is also not* B_PS $\widetilde{\widetilde{\widetilde{W}}}_2$*-compact space on* $\check{\amalg}$*.*

Proof.

(*i*) Let $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ be a B_PSW_1-open cover of $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ in $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$. Since $\widetilde{\widetilde{\widetilde{W}}}_1 \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{\widetilde{W}}}_2$, then $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ is the B_PSW_2-open cover of $(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi})$ by the B_PSW_2-open sets of $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$. Since $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}_2$-compact space. Thus,

$$(\widetilde{\widetilde{\check{\amalg}}}, \Phi, \breve{\pi}) \,\widetilde{\widetilde{\subseteq}}\, \widetilde{\widetilde{\bigcup}}_{\delta=1}^{n} (\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi}), \text{ for some } \delta_1, \delta_2, ..., \delta_n \in \Delta.$$

Therefore, $(\check{\amalg}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ is a B_PS $\widetilde{\widetilde{\widetilde{W}}}_1$-compact space.

(*ii*) Let $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ be not $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_1$-compact space on $\check{\text{II}}$ and $\widetilde{\widetilde{\widetilde{W}}}_1 \tilde{\tilde{\subseteq}} \widetilde{\widetilde{\widetilde{W}}}_2$. Assume if possible $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_2$-compact space on $\check{\text{II}}$. By (*i*), $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}_1, \breve{\pi}, \neg\breve{\pi})$ is also a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_1$-compact. This is a contradiction. Hence, $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}_2, \breve{\pi}, \neg\breve{\pi})$ is not $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_2$-compact space on $\check{\text{II}}$.

□

Theorem 6.12. *Let $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ be a B_PSWSS of $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Then $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-compact space if and only if every B_PSW-open cover of $(\tilde{\tilde{\check{Y}}}, \Phi, \breve{\pi})$ by B_PSW-open set in $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$ contains a finite B_PS subcover.*

Proof. Let $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ be a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-compact space and $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ be a B_PSW-open cover of $(\tilde{\tilde{\check{Y}}}, \Phi, \breve{\pi})$ by B_PSW-open set in $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Now, $\check{Y} \subseteq \bigcup_{\delta\in\Delta} \big(\check{Y} \cap \check{\zeta}_\delta(\epsilon)\big)$ for each $\epsilon \in \breve{\pi}$ and $\phi \supseteq \bigcap_{\delta\in\Delta} \big(\check{Y} \cap \check{\xi}_\delta(\neg\epsilon)\big)$ for each $\neg\epsilon \in \neg\breve{\pi}$. Thus, $\check{\Delta}_Y = \{(^{\check{Y}}\check{\zeta}_\delta, ^{\check{Y}}\check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-open cover of $(\tilde{\tilde{\check{Y}}}, \Phi, \breve{\pi})$. Since $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-compact space, then there is a finite B_PS subcover, say, $\{(^{\check{Y}}\check{\zeta}_{\delta_i}, ^{\check{Y}}\check{\xi}_\delta, \breve{\pi})\}_{i=1}^n$ such that,

$$(\tilde{\tilde{\check{Y}}}, \Phi, \breve{\pi}) \tilde{\tilde{\subseteq}} \tilde{\tilde{\bigcup}}_{i=1}^n (^{\check{Y}}\check{\zeta}_{\delta_i}, ^{\check{Y}}\check{\xi}_{\delta_i}, \breve{\pi}), \text{ for some } \delta_1, \delta_2, ..., \delta_n \in \Delta.$$

Thus, implies that $\{(\check{\zeta}_{\delta_i}, \check{\xi}_{\delta_i}, \breve{\pi})\}_{i=1}^n$ is a finite B_PS subcover of $(\tilde{\tilde{\check{Y}}}, \Phi, \breve{\pi})$ by B_PSW-open set in $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Conversely, suppose $\check{\Delta}_{\check{Y}} = \{(^{\check{Y}}\check{\zeta}_\delta, ^{\check{Y}}\check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ which is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-open cover of $(\tilde{\tilde{\check{Y}}}, \Phi, \breve{\pi})$. Then, clearly $\check{\Delta} = \{(\check{\zeta}_\delta, \check{\xi}_\delta, \breve{\pi})\}_{\delta\in\Delta}$ is a B_PSW-open cover of $(\tilde{\tilde{\check{Y}}}, \Phi, \breve{\pi})$ by B_PSW-open set in $(\check{\text{II}}, \widetilde{\widetilde{\widetilde{W}}}, \breve{\pi}, \neg\breve{\pi})$. Thus, by given hypothesis we have $\{(\check{\zeta}_{\delta_i}, \check{\xi}_\delta, \breve{\pi})\}_{i=1}^n$ is a finite B_PS subcover of $(\tilde{\tilde{\check{Y}}}, \Phi, \breve{\pi})$. Therefore, $(\check{Y}, \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}, \breve{\pi}, \neg\breve{\pi})$ is a $B_PS\ \widetilde{\widetilde{\widetilde{W}}}_{\check{Y}}$-compact space. □

7 Conclusions And Future Research

The aim of this paper was to define a new bipolar soft weak structure named bipolar soft weak connectedness and bipolar soft weak compactness, and to introduce the principles of bipolar soft weak locally connected and bipolar soft weak component. The fundamental concepts of B_PSWS, which are related to bipolar soft sets, are continuously presented and explored, as well as the definitions and examples needed to explain the concepts. Additionally, the paper has invalidated some $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-locally connected space and $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-component features in $BSWSs$. We provided a definition, demonstrated the way the ideas of $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-locally connected spaces and $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected are distinct, and explored the ways in which the $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-connected subsets are $B_PS\ \widetilde{\widetilde{\widetilde{W}}}$-components. Therefore, the main definitions and results of compactness in B_PSWSs were demonstrated.

Future research on bipolar soft weak structures may focus on several key areas, including continuous mappings and separation axioms.

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Bekas S. M.Taher
Department of Mathematics
University of Zakho
Zakho, Iraq
E-mail: bekes81math@gmail.com

Baravan A. Asaad

Department of Mathematics
University of Zakho
Zakho, Iraq
Department of Computer Science
Cihan University
Duhok, Iraq
E-mail: baravan.asaad@uoz.edu.krd