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Characterizing various types of complete generalized metric spaces via fixed point theorems - A survey

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Abstract. We restore several known extensions of various classical characterizations of metric completeness, established via fixed point results, to the framework of quasi-metric spaces, G-metric spaces and partial metric spaces, respectively.

Keywords: Quasi-metric space, G-metric space, partial metric space, fixed point, completeness.

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1. Introduction and preliminaries

Throughout this paper, we shall denote by $\mathbb{N}$ the set of natural numbers. Almost all of our notation and terminology are standard.

In order to help the reader we remind some well-known concepts.

Let T be a self-map of a set X and α be a function from $X \times X$ to $[0, \infty)$. According to Samet et al. [40], T is α-admissible provided that $\alpha(Tx, Ty) \geqslant 1$ whenever $\alpha(x, y) \geqslant 1$. If, in addition, there is $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ we say that T is (α, x_0)-admissible.

By a (c)-comparison function we mean a non-decreasing function $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for all $t \geqslant 0$.

A self-map T of a metric space (X, d) is called a contraction if there is a constant $c \in (0, 1)$ such that $d(Tx, Ty) \leqslant cd(x, y)$ for all $x, y \in X$.

A regular function for a metric space (X, d) is a function $\alpha : X \times X \to [0, \infty)$ that satisfies the following property: Whenever $(x_n)_{n \in \mathbb{N}}$ is a sequence in X such that

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$\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and there exists $x \in X$ such that $d(x, x_n) \to 0$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N}$ (compare [40]).

In [17], Hu obtained the first characterization of metric completeness via contractions having a fixed point. Later, several authors characterized complete metric spaces via fixed point results. Next, we present a relevant part of these characterizations in a unified way.

Theorem 1.1 For a metric space (X, d) the following conditions are equivalent.

(1) (X, d) is complete.

(2) Every contraction on any closed subspace C of (X, d) has a (unique) fixed point $z \in C$.

(3) For every self-map T of X such that there is a constant $c \in (0, 1/2)$ satisfying

$$d(Tx, Ty) \leqslant c[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$, T has a (unique) fixed point.

(4) For every self-map T of X such that there is a lower semicontinuous function $\varphi : X \to [0, \infty)$ satisfying

$$d(x, Tx) \leqslant \varphi(x) - \varphi(Tx),$$

for all $x \in X$, T has a fixed point.

(5) For every self-map T of X such that there are a regular function α for which T is (α, x_0)-admissible and a (c)-comparison function ψ satisfying

$$\alpha(x, y)d(Tx, Ty) \leqslant \psi(d(x, y)),$$

for all $x, y \in X$, T has a fixed point.

(6) For every self-map T of X such that there is a constant $c \in (0, 1)$ satisfying

$$d(x, Tx) \leqslant 2d(x, y) \Rightarrow d(Tx, Ty) \leqslant cd(x, y),$$

for all $x, y \in X$, T has a (unique) fixed point.

Remark 1 *In [17], Hu proved the equivalence between conditions (1) and (2). Regarding Hu's characterization, we recall that Connell [13] gave an example of a non-complete metric space for which every contraction on it has a fixed point, so the celebrated Banach contraction principle does not characterize the metric completeness. In [19], Kannan proved that* (1) $\Rightarrow$ (3)*, and Caristi proved in [11] that* (1) $\Rightarrow$ (4)*. Subrahmanyam [43] and Kirk [24] respectively showed that* (3) $\Rightarrow$ (1) *and* (4) $\Rightarrow$ (1)*. On the other hand, Samet et al. [40] proved that* (1) $\Rightarrow$ (5)*, and Romaguera and Tirado showed in [35] that* (5) $\Rightarrow$ (1)*. Finally, Suzuki proved in [44] the equivalence between conditions (1) and (6).*

In [18], Kada et al. introduced and discussed the notion of w-distance in the realm of metric spaces. This structure provides an efficient tool to improve various important results as Ekeland's Variational Principle and Caristi's fixed point theorem, among others. In fact, several authors have obtained interesting fixed point theorems via w-distances, not only on metric spaces but also on other generalized metric structures (see the monograph [31] with the references therein). In Theorem 1.2 below we compile well-known and featured w-distance improvements of various parts of Theorem 1.1.

Let us recall that a w-distance on a metric space (X, d) is a function $w : X \times X \to [0, \infty)$ that satisfies the following conditions:

(w1) $w(x, y) \leqslant w(x, z) + w(z, y)$, for all $x, y, z \in X$;
(w2) for each $x \in X$, the function $w(x, \_) : X \to [0, \infty)$ is lower semicontinuous;
(w3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $w(x, y) \leqslant \delta$ and $w(x, z) \leqslant \delta$ imply $d(y, z) \leqslant \varepsilon$.

A w-distance w is symmetric provided that it fulfills the following condition: $w(x, y) = w(y, x)$, for all $x, y, z \in X$.

Examples of w-distances on metric spaces may be found, for instance, in [18, 31, 46]. In particular, every metric d on a set X is a w-distance on the metric space (X, d).

Theorem 1.2 For a metric space (X, d) the following conditions are equivalent.
(1) (X, d) is complete.
(2) For every self-map T of X such that there are a w-distance w on (X, d) and a constant $c \in (0, 1)$ satisfying

$$w(Tx, Ty) \leqslant cw(x, y),$$

for all $x, y \in X$, T has a (unique) fixed point $z \in X$. Moreover, $w(z, z) = 0$.
(2') For every self-map T of X such that there are a symmetric w-distance w on (X, d) and a constant $c \in (0, 1)$ satisfying

$$w(Tx, Ty) \leqslant cw(x, y),$$

for all $x, y \in X$, T has a (unique) fixed point $z \in X$. Moreover, $w(z, z) = 0$.
(3) For every self-map T of X such that there is a w-distance w on (X, d) and a constant $c \in (0, 1/2)$ satisfying

$$w(Tx, Ty) \leqslant c[w(Tx, x) + w(Ty, y)],$$

for all $x, y \in X$, T has a (unique) fixed point $z \in X$. Moreover, $w(z, z) = 0$.
(4) For every self-map T of X such that there are a w-distance w on (X, d) and a lower semicontinuous function $\varphi : X \to [0, \infty)$ satisfying

$$w(x, Tx) \leqslant \varphi(x) - \varphi(Tx),$$

for all $x \in X$, T has a fixed point $z \in X$ such that $w(z, z) = 0$.
(5) For every self-map T of X such that there are a w-distance w on (X, d), a regular function α for which T is (α, x_0)-admissible and a (c)-comparison function ψ satisfying

$$\alpha(x, y)w(Tx, Ty) \leqslant \psi(w(x, y)),$$

for all $x, y \in X$, T has a fixed point $z \in X$.

Remark 2 *The equivalence between (1), (2) and (2') was showed by Suzuki and Takahashi [46]. Suzuki [45] (see also [42]) showed the equivalence between (1) and (3), and Kada et al. [18] proved that (1) and (4) are equivalent. Finally,* (5) $\Rightarrow$ (1) *follows from the obvious fact that* (5) $\Rightarrow$ (2)*, while* (1) $\Rightarrow$ (5) *can be deduced from various well-known fixed point results. We present a sketch of its proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. For each $n \in \mathbb{N} \cup \{0\}$ put $x_n = T^n x_0$. Taking into account that T is α-admissible we get $w(x_n, x_{n+1}) \leqslant \alpha(x_{n-1}, x_n)w(x_n, x_{n+1}) \leqslant \psi(w(x_{n-1}, x_n))$, for all $n \in \mathbb{N}$. Now, by recursion we deduce that $w(x_n, x_{n+1}) \leqslant \psi^n(w(x_0, x_1))$ for all $n \in \mathbb{N}$. Combining the technique used in the first lines of page 2156 in [40] with condition (w3), we conclude that*

$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X,d). So, there is $z \in X$ such that $d(z,x_n) \to 0$ as $n \to \infty$. Applying (w2) we deduce from standard arguments that $w(x_n,z) \to 0$ as $n \to \infty$. Finally, since α is regular we have that $w(x_{n+1},Tz) \leqslant w(x_n,z)$ for all $n \in \mathbb{N}$, and by (w3), $z = Tz$.

In the light of Theorem 1.2 we emphasize that the problem of obtaining a nice w-distance generalization of Suzuki's characterization of metric completeness (Theorem 1.1, (1) $\Leftrightarrow$ (6)) presents interesting difficulties that deserve attention. Thus, in [38] it was given an example of a self-map T of a complete metric space (X,d), without fixed points, but for which there is a w-distance w and a constant $c \in (0,1)$ satisfying

$$w(x,Tx) \leqslant 2w(x,y) \implies w(Tx,Ty) \leqslant cw(x,y),$$

for all $x,y,z \in X$.

This fact motivated the following notion introduced in [38]: A w-distance w on a metric space (X,d) is called presymmetric if it satisfies the following property: Whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $d(x,x_n) \to 0$ and $w(x_n,x) \to 0$ as $n \to \infty$, for some $x \in X$, then there is a subsequence $(x_{j(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ fulfilling $w(x,x_{j(n)+1}) \leqslant w(x_{j(n)},x)$ for all $n \in \mathbb{N}$.

Note that every symmetric w-distance is presymmetric. (We point out that the idea of presymmetry has been recently applied to other structures [22, 23, 37, 39]).

Then, we have [38, Theorems 2 and 11]:

Theorem 1.3 For a metric space (X,d) the following conditions are equivalent.

(1) (X,d) is complete.

(2) For every self-map T of X such that there is a presymmetric w-distance w on (X,d) and a constant $c \in (0,1)$ satisfying

$$w(x,Tx) \leqslant 2w(x,y) \Rightarrow w(Tx,Ty) \leqslant cw(x,y),$$

for all $x,y \in X$, T has a (unique) fixed point $z \in X$. Moreover, $w(z,z) = 0$.

(2') For every self-map T of X such that there is a symmetric w-distance w on (X,d) and a constant $c \in (0,1)$ satisfying

$$w(x,Tx) \leqslant 2w(x,y) \Rightarrow w(Tx,Ty) \leqslant cw(x,y),$$

for all $x,y \in X$, T has a (unique) fixed point $z \in X$. Moreover, $w(z,z) = 0$.

This review paper deals with the problem of extending Theorems 1.1, 1.2 and 1.3 to some types of generalized metric spaces as quasi-metric spaces, G-metric spaces and partial metric spaces. In this direction we will establish several well-known results and raise some natural questions.

2. Characterizing complete quasi-metric spaces

In the realm of general topology, Wilson [47] introduced the concepts of a quasi-metric and a quasi-metric space. Later, numerous authors have explored the properties of quasi-metric spaces as well as its connection with other topological structures. In particular, it was proved that distinguished non-metrizable topological spaces, as the Sorgenfrey line, the Michael line, the Niemytzki plane and the Kofner plane, are quasi-metrizable (see, e.g., [15]). Partially due for its applications to theoretical computer science and the complexity analysis of algorithms (see [26] and [41] for pioneering contributions) the study of the fixed point theory for quasi-metric spaces and other related structures, as partial metric spaces, has attracted the interest of many researchers over the last decades (for recent contributions see, e.g., [5, 8, 16, 33] and the references therein). The references [15] and [12] provide excellent sources for the study of quasi-metric spaces.

In order to help non-specialist readers, we remind some pertinent notions and properties.

According to modern terminology, by a quasi-metric on a set X we mean a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$, and (ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

If d satisfies condition (ii) and the following condition stronger than (i), $d(x, y) = 0$ if and only if $x = y$, we say that d is a T_1 quasi-metric on X.

A (T_1) quasi-metric space is a pair (X, d) such that X is a set and d is a (T_1) quasi-metric on X.

Given a quasi-metric d on X, the function d^* given by $d^*(x, y) = d(y, x)$ is also a quasi-metric on X, called the conjugate of d, and the function d^s given by $d^s(x, y) = \max\{d(x, y), d^*(x, y)\}$ is a metric on X.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X, \varepsilon > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Thus, a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, d) is declared τ_d-convergent if it converges in the topological space (X, τ_d). Hence, $(x_n)_{n\in\mathbb{N}}$ is τ_d-convergent to $x \in X$ if and only if $d(x, x_n) \to 0$ as $n \to \infty$.

By a Hausdorff quasi-metric space we mean a quasi-metric space (X, d) such that τ_d is a Hausdorff topology on X, and by a doubly Hausdorff quasi-metric space we mean a quasi-metric space (X, d) such that τ_d and τ_{d^*} are Hausdorff topologies on X.

A topological space (X, τ) is quasi-metrizable provided that there is a quasi-metric d on X such that $\tau = \tau_d$ on X.

In the realm of quasi-metric spaces there are several different notions of Cauchy sequence and completeness, all of them coincide with the classical notions of Cauchy sequence and completeness when considering metric spaces. Here we will use the following ones.

Let (X, d) be a quasi-metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be:

Left-Cauchy if for each $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n_\varepsilon \leqslant n \leqslant m$.

Right-Cauchy if for each $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $n_\varepsilon \leqslant n \leqslant m$, equivalently, if $(x_n)_{n\in\mathbb{N}}$ is left K-Cauchy in (X, d^*).

Cauchy if it is left-Cauchy and right-Cauchy, equivalently, if it is Cauchy in the metric space (X, d^s).

Then, (X, d) is said to be:

Smyth-complete if every left-Cauchy sequence is τ_{d^s}-convergent.

Left-complete if every left-Cauchy sequence is τ_d-convergent.

Right-complete if every right-Cauchy sequence is τ_d-convergent.

Half-complete if every Cauchy sequence is τ_d-convergent.

*half-complete if every Cauchy sequence is τ_{d^*}-convergent.

The following implications are obvious:

Smyth-complete $\Rightarrow$ left-complete $\Rightarrow$ half-complete, Smyth-complete $\Rightarrow$ *half-complete,

and

right-complete $\Rightarrow$ half-complete.

The reverse implications do not hold in general. In this regard, we recall two typical examples.

***Example* 2.1** Let d be the quasi-metric on the set $\mathbb{R}^+$ of non-negative real numbers given by $d(x, y) = \max\{y - x, 0\}$ for all $x, y \in \mathbb{R}^+$. It is well known that $(\mathbb{R}^+, d)$ is Smyth complete and *half-complete but not right complete. Note that it is not a T_1 quasi-metric space.

***Example* 2.2** Let d be the quasi-metric on the set $\mathbb{R}$ of real numbers given by $d(x, y) = y - x$ if $x \leqslant y$ and $d(x, y) = 1$ if $x > y$, for all $x, y \in \mathbb{R}$. Then, $(\mathbb{R}, \tau_d)$ is the famous Sorgenfrey line. It is well known that $(\mathbb{R}, d)$ is right-complete and *half-complete but not left-complete. Moreover, it is a doubly Hausdorff quasi-metric space.

In [6] it was obtained the following full quasi-metric generalization of Kannan-Subrahmanyam characterization of metric completeness (Theorem 1.1, (1) $\Leftrightarrow$ (3)).

Theorem 2.3 For a quasi-metric space (X, d) the following conditions are equivalent.
(1) (X, d) is half-complete.
(2) For every self-map T of X such that there is a constant $c \in (0, 1/2)$ satisfying

$$d(Tx, Ty) \leqslant c[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$, T has a (unique) fixed point.

In [34] it was obtained the following full quasi-metric generalization of Caristi-Kirk's characterization of metric completeness (Theorem 1.1, (1) $\Leftrightarrow$ (4)).

Theorem 2.4 For a quasi-metric space (X, d) the following conditions are equivalent.
(1) (X, d) is Smyth-complete.
(2) For every self-map T of X such that there is a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous on (X, τ_{d^s}) and satisfies

$$d(x, Tx) \leqslant \varphi(x) - \varphi(Tx),$$

for all $x \in X$, T has a fixed point.

On the other hand, in [35] quasi-metric generalizations of Theorem 1.1, (1) $\Leftrightarrow$ (4), for Hausdorff and doubly Hausdorff quasi-metric spaces respectively were obtained (see Theorems 2.5 and 2.6 below).

In this context (compare [35]), the notion of a regular function for a quasi-metric space is defined exactly as in the metric case, whereas a co-regular function for a quasi-metric space (X, d) is a function $\alpha : X \times X \to [0, \infty)$ that satisfies the following property: Whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and there exists $x \in X$ such that $d(x, x_n) \to 0$ as $n \to \infty$, then $\alpha(x, x_n) \geqslant 1$ for all $n \in \mathbb{N}$.

Theorem 2.5 For a Hausdorff quasi-metric space (X, d) the following conditions are equivalent.
(1) (X, d) is left-complete.
(2) For every self-map T of X such that there are a co-regular function α for which T

is (α, x_0)-admissible and a (c)-comparison function ψ satisfying

$$\alpha(x,y)d(Tx,Ty) \leqslant \psi(d(x,y)),$$

for all $x, y \in X$, T has a fixed point.

Theorem 2.6 For a doubly Hausdorff quasi-metric space (X,d) the following conditions are equivalent.

(1) (X,d) is right-complete.

(2) For every self-map T of X such that there are a regular function α for which T is (α, x_0)-admissible and a (c)-comparison function ψ satisfying

$$\alpha(y,x)d(Tx,Ty) \leqslant \psi(d(x,y)),$$

for all $x, y \in X$, T has a fixed point.

In view of the two preceding theorems we state the following.

Question 2.7 Obtain a full quasi-metric extension of Theorem 1.1, (1) $\Leftrightarrow$ (4), valid for not necessarily Hausdorff (actually, for not necessarily T_1) quasi-metric spaces.

The problem of obtaining a full quasi-metric generalization of Suzuki's characterization of metric completeness (Theorem 1.1, (1) $\Leftrightarrow$ (6)) presents serious difficulties as the following example given in [32] shows.

***Example* 2.8** Let $X = \mathbb{N} \cup \{\infty\}$ and let d be the quasi-metric on X given by $d(x,x) = 0$ for all $x \in X$, $d(n,\infty) = 0$ for all $n \in \mathbb{N}$, $d(\infty, n) = 1/n$ for all $n \in \mathbb{N}$, and $d(n,m) = 1/m$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

Then, (X,d) and (X,d^*) are Smyth-complete.

Now let T be the self-map of X defined by $T\infty = 1$, and $Tn = 2n$ for all $n \in \mathbb{N}$.

Obviously, T has no fixed point. However, the following inequality holds for all $x, y \in X$ and any constant c such that $1/2 \leqslant c < 1$:

$$d(x,Tx) \leqslant 2d(x,y) \Rightarrow d(Tx,Ty) \leqslant cd(x,y).$$

As with the quasi-metric generalization of Suzuki's theorem, the problem of obtaining an adequate generalization of Hu's theorem (Theorem 1.1, (1) $\Leftrightarrow$ (2)) presents certain difficulty. In [14] the authors obtained a partial solution for the T_1 case, which we present joint its proof in Theorem 2.10 below. We will need the following concepts from [14].

A subset C of a quasi-metric space (X,d) is called doubly closed if C is closed with respect to τ_d and with respect to τ_{d^*}.

A (d,d^*)-contraction on a quasi-metric space (X,d) is a self-map T of X such that there is a constant $c \in (0,1)$ satisfying $d(Tx,Ty) \leqslant cd(y,x)$, for all $x, y \in X$.

A quasi-metric space (X,d) is called weakly complete if every Cauchy sequence in the metric space (X,d^s) converges for τ_d or for τ_{d^*}.

Next we present an example of a weakly complete quasi-metric space that is not half-complete and not *half-complete.

***Example* 2.9** Let $X = \{0,\infty\} \cup \mathbb{N} \cup \{\frac{1}{n+1} : n \in \mathbb{N}\}$. Define a function d on $X \times X \to [0,\infty)$ by $d(0,0) = d(\infty,\infty) = 0$, $d(\frac{1}{n+1}, m) = d(m, \frac{1}{n+1}) = 1$, $d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|$, $d(\frac{1}{n+1}, \frac{1}{m+1}) = \left|\frac{1}{n+1} - \frac{1}{m+1}\right|$ if $n, m \in \mathbb{N}$, $d(n,\infty) = \frac{1}{n}$, $d(0, \frac{1}{n+1}) = \frac{1}{n+1}$, and $d(\infty, n) = d(\frac{1}{n+1}, \infty) = d(\infty, \frac{1}{n+1}) = d(\frac{1}{n+1}, 0) = d(0,n) = d(n,0) = 1$, for all $n \in \mathbb{N}$. Then (X,d) is a Hausdorff quasi-metric space. Moreover, every non eventually constant Cauchy sequence is a subsequence of $(n)_{n\in\mathbb{N}}$ or of $(\frac{1}{n+1})_{n\in\mathbb{N}}$. Since $(n)_{n\in\mathbb{N}}$ is τ_{d^*}-convergent (but not τ_d-convergent) and $(\frac{1}{n+1})_{n\in\mathbb{N}}$ is τ_d-convergent (but not τ_{d^*}-convergent), we deduce that (X,d) is weakly complete but not half-complete or *half-complete.

Theorem 2.10 For a T_1 quasi-metric space (X,d) the following conditions are equivalent.

(1) (X,d) is weakly complete.

(2) Every (d,d^*)-contraction on any doubly closed subspace C of (X,d) has a (unique) fixed point $z\in C$.

Proof. (1) $\Rightarrow$ (2) Let T a (d,d^*)-contraction on the doubly closed subspace C of (X,d). Fix $x_0\in C$, then $(T^n x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence such that $\{T^n x_0 : n\in\mathbb{N}\}\subset C$. Since (X,d) is weakly complete then $(T^n x_0)_{n\in\mathbb{N}}$ is τ_d-convergent or τ_{d^*}-convergent. If $(T^n x_0)_{n\in\mathbb{N}}$ is τ_d-convergent there exists $z\in X$ such that $d(z,T^n x_0)\to 0$ as $n\to\infty$. Since C is doubly closed then $z\in C$. Since T is a (d,d^*)-contraction, there exists $c\in(0,1)$ such that $d(T^{n+1}x_0,Tz)\leqslant cd(y,T^n x_0)$ for all $n\in\mathbb{N}$. Consequently $d(T^{n+1}x_0,Tz)\to 0$ as $n\to\infty$. From the triangle inequality we deduce $d(z,Tz)=0$. Therefore $z=Tz$ because (X,d) is a T_1 quasi-metric space. If $(T^n x_0)_{n\in\mathbb{N}}$ is τ_{d^*}-convergent there exists $z\in X$ such that $d(T^n x_0,z)\to 0$ as $n\to\infty$. Since C is doubly closed then $z\in C$. Since T is a (d,d^*)-contraction, there exists $c\in(0,1)$ such that $d(Tz,T^{n+1}x_0)\leqslant cd(T^n x_0,z)$ for all $n\in\mathbb{N}$. Consequently, $d(Tz,T^{n+1}x_0)\to 0$ as $n\to\infty$. From the triangle inequality we deduce $d(Tz,z)=0$. Therefore $z=Tz$ because (X,d) is a T_1 quasi-metric space. Finally, we show that z is the unique fixed point of T. Suppose that $y\in X$ verifies that $y=Ty$. Then, we have $d(y,z)=d(Ty,Tz)\leqslant cd(z,y)\leqslant c^2 d(y,z)$, so $d(y,z)=0$ and, thus, $y=z$.

(2) $\Rightarrow$ (1) Suppose that there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d) of distinct terms that is not τ_d-convergent and not τ_{d^*}-convergent. Then, the set $C:=\{x_n : n\in\mathbb{N}\}$ is a doubly closed subset of X. For each x_n we define

$$l_n=\min\{d(x_n,\{x_m : m>n\}),d(\{x_m : m>n\},x_n)\}.$$

Then, $l_n>0$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d^s), given $r\in[0,1)$, for each $n\in\mathbb{N}$ there exists $k(n)>n$ such that $d^s(x_{n'},x_{m'})<rl_n$ for all $m',n'\geqslant k(n)$.

Now, we construct a (d,d^*)-contraction on C without fixed point. Define $T:C\to C$ by $Tx_n=x_{k(n)}$ for all $n\in\mathbb{N}$. Let $n,m\in\mathbb{N}$, and suppose, without loss of generality, that $m>n$. Then,

$$d^s(Tx_n,Tx_m)=d^s(x_{k(n)},x_{k(m)})<cl_n\leqslant c\min\{(d(x_n,x_m),d(x_m,x_n)\}.$$

Hence $d(Tx_n,Tx_m)\leqslant cd(x_m,x_n)$ and $d(Tx_m,Tx_n)\leqslant cd(x_n,x_m)$. We deduce that T is a (d,d^*)-contraction on the doubly closed subspace C. This concludes the proof. ■

Question 2.11 Obtain a full quasi-metric generalization of Hu's characterization of metric completeness for not necessarily T_1 quasi-metric spaces.

We conclude this section by dealing with the problem of obtaining quasi-metric generalizations of Theorems 1.2 and 1.3.

According to Park [30] by a w-distance on a quasi-metric space (X,d) we mean a function $w : X\times X\to[0,\infty)$ that satisfies the following conditions:

(wq1) $w(x,y)\leqslant w(x,z)+w(z,y)$, for all $x,y,z\in X$;

(wq2) for each $x\in X$, the function $w(x,\_) : X\to[0,\infty)$ is lower semicontinuous on (X,τ_{d^*});

(wq3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $w(x,y) \leqslant \delta$ and $w(x,z) \leqslant \delta$ imply $d(y,z) \leqslant \varepsilon$.

Examples of w-distances on quasi-metric spaces can be found, for instance, in [7, 21, 25, 30].

Contrary to what happens in the realm of metric spaces, there exist quasi-metric spaces (X,d) for which the quasi-metric d is not a w-distance on (X,d). In fact, if d is a w-distance on (X,d), then the topological space (X,τ_d) is metrizable.

According to [39], a w-distance w on a quasi-metric space (X,d) is called presymmetric if it satisfies the following property: Whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $d^*(x,x_n) \to 0$ and $w(x_n,x) \to 0$ as $n \to \infty$, for some $x \in X$, then there is a subsequence $(x_{j(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ fulfilling $w(x,x_{j(n)+1}) \leqslant w(x_{j(n)},x)$ for all $n \in \mathbb{N}$.

By a *-regular function for a quasi-metric space (X,d) we mean a function $\alpha : X\times X \to [0,\infty)$ that satisfies the following property: Whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and there exists $x \in X$ such that $d^*(x,x_n) \to 0$ as $n \to \infty$, then $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{N}$.

Then, we have

Theorem 2.12 For a quasi-metric space (X,d) the following conditions are equivalent.

(1) (X,d) is *half-complete.

(2) For every self-map T of X such that there are a w-distance w on (X,d) and a function $\varphi : X \to [0,\infty)$ which is lower semicontinuous on (X,τ_{d^*}) and satisfies

$$w(x,Tx) \leqslant \varphi(x) - \varphi(Tx),$$

for all $x \in X$, T has a fixed point $z \in X$ such that $w(z,z) = 0$.

(3) For every self-map T of X such that there are a w-distance w on (X,d), a *-regular function α for which T is (α,x_0)-admissible and a (c)-comparison function ψ satisfying

$$\alpha(x,y)w(Tx,Ty) \leqslant \psi(w(x,y)),$$

for all $x,y \in X$, T has a fixed point $z \in X$.

Remark 3 *(1) ⇒ (3) follows from a natural adaptation of the development presented at the end of Remark 2, and (3) ⇒ (1) is implicit in the proof of Theorem 5 in [37]. On the other hand, (1) ⇔ (2) was proved in [21].*

As far as we know, the following two issues have not been studied.

Question 2.13 Obtain a full quasi-metric generalization of (1) ⇔ (2) in Theorem 1.2.

Question 2.14 Obtain a full quasi-metric generalization of (1) ⇔ (3) in Theorem 1.2.

With respect to question 2.13, note that from Theorem 2.12, (1) ⇒ (3), it follows that if (X,d) is a *half-complete quasi-metric space, then for every self-map T of X such that there are a w-distance w on (X,d) and a constant $c \in (0,1)$ satisfying

$$w(Tx,Ty) \leqslant cw(x,y),$$

for all $x,y \in X$, T has a (unique) fixed point $z \in X$. Moreover, $w(z,z) = 0$.

We conclude this section raising the following natural question.

Question 2.15 Obtain a full quasi-metric generalization of Theorem 1.3.

Note that from [39, Corollary 4.4] it follows that if (X,d) is a *half-complete quasi-metric space, then for every self-map T of X such that there are a presymmetric w-distance w on (X,d) and a constant $c \in (0,1)$ satisfying

$$w(x,Tx) \leqslant 2w(x,y) \Rightarrow w(Tx,Ty) \leqslant cw(x,y),$$

for all $x,y \in X$, T has a (unique) fixed point $z \in X$. Moreover, $w(z,z) = 0$.

3. Characterizing G-complete metric spaces

Motivated by the existence of several flaws in the study of the topological structure of the so-called D-metric spaces, Mustafa and Sims [27, 28] introduced and explored the notion of G-metric space. Mustafa-Sims' papers initiated the development of a deep study of the fixed point theory for these spaces (cf. [2–4, 10, 21, 36] and the references there in). In particular, [3] will be our basic reference for G-metric spaces.

A G-metric on a set X is a function $G : X \times X \times X \to [0,\infty)$ such that for all $x,y,z,u \in X$, the following conditions are satisfied:

(G1) $G(x,y,z) = 0$ if $x = y = z$;

(G2) $G(x,x,y) > 0$ if $x \neq y$;

(G3) $G(x,x,y) \leqslant G(x,y,z)$ if $y \neq z$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...$ (symmetry in all 3);

(G5) $G(x,y,z) \leqslant G(x,u,u) + G(u,y,z)$ (rectangle inequality).

A G-metric space is a pair (X,G) where X is a set and G is a G-metric on X.

Several examples of G-metric spaces can be found in [3, p. 34-35].

The following concepts and properties may be found in [3, Chapter 3].

(a) Each G-metric G on a set X induces a topology τ_G on X which has as a base the family of open balls $\{B_G(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_G(x,\varepsilon) = \{y \in X : G(x,y,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Moreover, the topological space (X,τ_G) is metrizable.

(b) A sequence $(x_n)_{n\in\mathbb{N}}$ in a G-metric space (X,G) is called a G-Cauchy sequence if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $G(x_n,x_m,x_k) < \varepsilon$ for all $n,m,k \geqslant n_0$.

(c) A G-metric space (X,G) is complete provided that every G-Cauchy sequence is τ_G-convergent.

(d) Given a G-metric space (X,G), the function $d_G : X \times X \to [0,\infty)$ given by $d_G(x,y) = G(x,y,y)$ for all $x,y \in X$, is a quasi-metric on X and the quasi-metric space (X,d_G) is doubly Hausdorff.

(e) The topologies τ_G, τ_{d_G} and $\tau_{(d_G)^*}$ agree on X.

(f) (X,G) is complete if and only if (X,d_G) is *half-complete.

Next, we discuss the problem of extending Theorem 1.1 to the G-metric framework. In fact, we reestablish several known results which show that it is possible to obtain suitable generalizations in all cases.

By a G-contraction on a G-metric space (X,G) we mean a self-map T that fulfills the following condition for all $x,y,z \in X$, with $c \in (0,1)$ a constant:

$$G(Tx,Ty,Tz) \leqslant cG(x,y,z).$$

If the self-map T fulfills the following weaker condition, we say that T is a weak

G-contraction on (X, G):

$$G(Tx, Ty, Ty \leqslant cG(x, y, y),$$

for all $x, y \in X$, wit $c \in (0, 1)$ a constant.

Then, we have

Theorem 3.1 For a G-metric space (X, G) the following conditions are equivalent.

(1) (X, G) is G-complete.

(2) Every weak G-contraction on any closed subspace C of (X, G) has a (unique) fixed point $z \in C$.

(3) Every G-contraction on any closed subspace C of (X, G) has a (unique) fixed point $z \in C$.

Remark 4 *(1) ⇒ (2) was essentially proved by Agarwal et al. [3], (2) ⇒ (3) is obvious and (3) ⇒ (1) was showed in [36].*

Theorem 3.2 For a G-metric space (X, G) the following conditions are equivalent.

(1) (X, G) is G-complete.

(2) For every self-map T of X such that there is a constant $c \in (0, 1/2)$ satisfying

$$G(Tx, Ty, Ty) \leqslant c[G(x, Tx, Tx) + G(y, Ty, Ty)],$$

for all $x, y \in X$, T has a (unique) fixed point $z \in X$.

Theorem 3.3 For a G-metric space (X, G) the following conditions are equivalent.

(1) (X, G) is G-complete.

(2) For every self-map T of X such that there is a lower semicontinuous function $\varphi : X \to [0, \infty)$ satisfying

$$G(x, Tx, Tx)) \leqslant \varphi(x) - \varphi(Tx),$$

for all $x \in X$, T has a fixed point $z \in X$.

Now, let T be a self-map of a set X and $\beta : X \times X \times X \to [0, \infty)$ a function. T is said to be β-admissible [3] if, for each $x, y, z \in X$, we have

$$\beta(x, y, z) \geqslant 1 \Rightarrow \beta(Tx, Ty, Tz) \geqslant 1.$$

If T is β-admissible and there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geqslant 1$, we will say that T is (β, x_0) admissible.

A G-regular function for a G-metric space (X, G) is a function $\beta : X \times X \times X \to [0, \infty)$ that satisfies the following condition: whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\beta(x_n, x_{n+1}, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, and $(x_n)_{n\in\mathbb{N}}$ G-converges to $x \in X$, then $\beta(x_n, x, x) \geqslant 1$ for all $n \in \mathbb{N}$.

Then, we have

Theorem 3.4 For a G-metric space (X, G) the following conditions are equivalent.

(1) (X, G) is complete.

(2) For every self-map T of X such that there are a G-regular function β for which T is (β, x_0)-admissible and a (c)-comparison function ψ satisfying

$$\beta(x, y, y)G(Tx, Ty, Ty) \leqslant \psi(G(x, y, y)),$$

for all $x, y \in X$, T has a fixed point $z \in X$.

Theorem 3.5 For a G-metric space (X, G) the following conditions are equivalent.
(1) (X, G) is complete.
(2) For every self-map T of X such that there is a constant $c \in (0, 1)$ satisfying

$$G(x, Tx, Tx) \leqslant 3G(x, y, y) \Rightarrow G(Tx, Ty, Ty) \leqslant cG(x, y, y),$$

for all $x, y \in X$, T has a (unique) fixed point $z \in X$.

Remark 5 *Theorems 3.2, 3.3 and 3.5 were stated in [36]; moreover, the quasi-metric d_G played an important role in their proofs. Regarding Theorem 3.4, Agarwal et al. [3], proved (1) $\Rightarrow$ (2) while that (2) $\Rightarrow$ (1) was showed in [36].*

4. Characterizing complete partial metric spaces

In [26], Matthews introduced and discussed the notion of partial metric space when he was studying the modeling of certain denotational semantics in the theory of computation. In particular, he obtained a partial metric version of the Banach contraction principle. Since then, the problem of obtaining relevant fixed point theorems on partial metric spaces has received a lot of attention (Chapter 7 of the book [20] jointly with the references therein provides an updated and detailed analysis of the fixed point theory for partial metric spaces).

Let us recall [26] that a partial metric on a set X is a function $p : X \times X \to [0, \infty)$ that satisfies the following conditions for all $x, y, z \in X$:
(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leqslant p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then, from (p1) and (p2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

Matthews also established an elegant connection between partial metric spaces and a class of quasi-metric spaces, the so-called weightable quasi-metric spaces ([26, Theorems 4.1 and 4.2]).

He showed that each partial metric p on a set X induces a T_0 topology τ_p on X which has as a base the family open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $D_p : X \times X \to [0, \infty)$ given by

$$D_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$, is a metric on X.

Matthews also introduced a notion of completeness for partial metric spaces and proved that a partial metric space (X, p) is complete if and only if the metric space (X, D_p) is complete.

Following [9] by a w-distance on a partial metric space (X, p) we mean a function $w : X \times X \to [0, \infty)$ that satisfies the following conditions:

(wp1) $w(x, y) \leqslant w(x, z) + w(z, y)$, for all $x, y, z \in X$;

(wp2) for each $x \in X$, the function $w(x, \_) : X \to [0, \infty)$ is lower semicontinuous on (X, τ_{D_p});

(wp3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $w(x, y) \leqslant \delta$ and $w(x, z) \leqslant \delta$ imply $p(y, z) \leqslant \varepsilon + p(y, y)$.

Then, we have

Theorem 4.1 For a partial metric space (X, p) the following conditions are equivalent.

(1) (X, p) is complete.

(2) For every self-map T of X such that there are a w-distance w on (X, p) and a constant $c \in (0, 1)$ satisfying

$$w(Tx, Ty) \leqslant cw(x, y),$$

for all $x, y \in X$, T has a (unique) fixed point $z \in X$. Moreover, $w(z, z) = 0$.

(3) For every self-map T of X such that there is a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous on (X, τ_{D_p}) and satisfies

$$p(x, Tx) \leqslant \varphi(x) - \varphi(Tx),$$

for all $x \in X$, T has a fixed point.

(4) For every self-map T of X such that there is a constant $c \in (0, 1)$ satisfying

$$p(x, Tx) \leqslant 2p(x, y) \Rightarrow p(Tx, Ty) \leqslant cp(x, y),$$

for all $x, y \in X$, T has a (unique) fixed point.

Remark 6 *The equivalence between (1) and (2) was obtained in [9]. The equivalence between (1) and (3) was proved in [1], while the equivalence between (1) and (4) was shown in [29].*

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