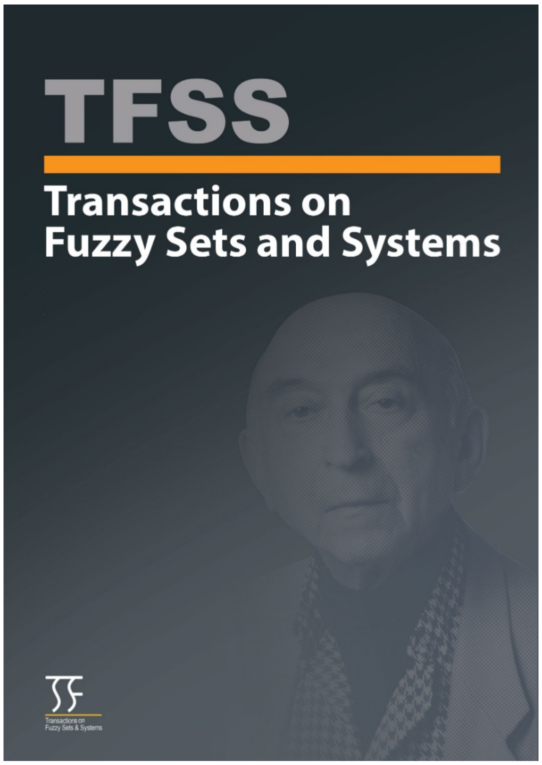
Transactions on Fuzzy Sets and Systems



Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.5, No.2, (2026), 276-305. DOI: https://doi.org/10.71602/tfss.2026.1207662

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.5, No.2, (2026), 276-305
DOI: https://doi.org/10.71602/tfss.2026.1207662

Article Type: Original Research Article

An Analytical Study for the System of Nonlinear Fuzzy Differential Equations with a Consequent Application

Payal Singh*, Kamal Hossain Gazi, Mohammed Rabih, K. Rangarajan, Sankar Prasad Mondal

Abstract. Realistic scenarios typically involve several interacting dependent variables, leading to nonlinearity in mathematical models. Nonlinear differential equations are a crucial mathematical phenomenon for describing real-world problems. On the other hand, fuzzy set theory is a valuable tool for modelling the uncertainty present in a system. So, nonlinear mathematical models under uncertainty need an explanatory study on a system of nonlinear fuzzy differential equations. In this paper, we analyse nonlinear fuzzy differential equations with possible applications. This paper advances the existing literature by introducing a definition of a fuzzy sequence and developing the existence and uniqueness theory for fuzzy solutions of nonlinear differential equations using the fixed-point theorem. The component-wise analysis of a fuzzy nonlinear system has been conducted. Furthermore, the analysed theory precedes an aptly fitted application of a fuzzy nonlinear dynamical system using a numerical scheme.

AMS Subject Classification 2020: 34A34; 47H10; 03E72
Keywords and Phrases: Fuzzy nonlinear differential equations, Fuzzy solution, Fuzzy sequence, Fixed point theorem, Numerical simulation.

1 Introduction

A differential equation is a significant tool for representing real-world issues in areas such as epidemic modelling, physics, engineering, and other scientific fields. Because of their adaptive nature, the majority of real-world problems are expressed in nonlinear terms. When all elements and complexities are considered in the mathematical modelling of the problem, the model becomes quite nonlinear. Each model component is interdependent and contributes to a more accurate knowledge of the situation. The general system of nonlinear differential equations is given as follows

$$\frac{dX}{dt} = \phi(t, X),\ \ X(t_0) = X_0 \tag{1}$$

where $\phi : I \times \mathbb{R}^n \to \mathbb{R}^n$ and $X = [X_1, X_2, \ldots, X_n]^T$.

When we simulate such a complex situation, imprecision may occur owing to human or machine mistakes. Before fuzzy set theory [1], uncertainty present in physical observation was mostly handled by probability theory. The difference between fuzzy set theory and probability theory may be understood by following the

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Received: 22 May 2025; **Revised**: 16 August 2025; **Accepted**: 17 August 2025; **Available Online**: 25 October 2025; **Published Online**: 7 November 2026.

How to cite: Singh P, Gazi KH, Rabih M, Rangarajan K, Mondal SP. An analytical study for the system of nonlinear Fuzzy differential equations with a consequent application. *Transactions on Fuzzy Sets and Systems.* 2026; 5(2): 276-305. DOI: https://doi.org/10.71602/tfss.2026.1207662

example. Lets consider the probability of buying fresh and red apples in a particular season is 70%, which means fresh and red apples are available with 7 out of 10 vendors. However, the number of red and fresh apples depends on an individuals perception. To understand the redness of an apple, fuzzy set theory is a more effective approach than probability theory. Stochastic equations are used in modelling real-world problems as mentioned in [2, 3].

However, in 1965, Zadeh proposed the fuzzy concept in his seminal paper [1], which compelled the world to think in this direction. Subsequently, considerable work has been undertaken in this direction. Keeping the importance of differential equations, Chang and Zadeh [4] proposed fuzzy derivatives in 1972. Using different derivatives in the sense of fuzzy uncertainty, Equation (1) is given as follows:

$$\frac{d\tilde{X}}{dt} = \tilde{\phi}(t, \tilde{X}),\ \tilde{X}(t_0) = \tilde{X}_0 \tag{2}$$

where $\tilde{\phi} : I \times \Omega^n \to \Omega^n$, Ω is the compilation of fuzzy numbers, the characteristics of Ω are discussed later in Section 2. In the following subsection, we present a brief discussion on the historical chronology of fuzzy logic and uncertain differential equations.

1.1 Literature Review of Fuzzy Set, Fuzzy Number, Fuzzy Logic and Fuzzy Differential Equations

Fuzzy sets were invented by Lotfi A. Zadeh [1] in 1965. Furthermore, Chang, S. L. et al. [4] discussed control systems and fuzzy mapping elaborately. Then, Zadeh, L. A. [5] introduced the concept of linguistic terms and their application in approximation and developed basic properties of fuzzy sets [6]. Additionally, Zadeh, L. A., works on fuzzy algorithms [7] and their extension. Further, the intuitionistic fuzzy set was introduced by Krassimir T. Atanassov [8] in 1988. Fuzzy set, intuitionistic fuzzy set and neutrosophic fuzzy set are applied to differential equations in numerous applications.

Ye, J. et al. [9] defined derivatives using the extension principle in a fuzzy environment and utilized them in applications. Puri, M. L. et al. [10] proposed an H-derivative of fuzzy valued functions according to the Hukuhara difference. Then, many researchers initially aimed to solve the fuzzy starting value problem simultaneously, using fuzzy initial conditions. This idea was applied in the context of fuzzy dynamical systems. Hukuhara differentiability, based on the H-difference, is the most fundamental approach that is frequently applied to solve fuzzy differential equations (FDEs). The disadvantage of employing the Hukuhara derivative is that the solution loses its fuzzy nature over time. To address this issue, the Generalised Hukuhara derivative [11] concept is proposed, which is widely used among numerous fuzzy derivatives. The generalised Hukuhara derivative enables the solving of fuzzy differential equations; however, with this derivative, the answer is acquired as a possible collection of solutions from which to choose the most appropriate solution in the circumstances. These are the basic drawbacks of generalised Hukuhara derivatives. To address this issue, the Modified Hukuhara derivative [12] is proposed, which provides a unique solution in fuzzy settings.

Most authors began by working on scalar differential equations with fuzzy initial conditions, employing various methodologies, including numerical, analytical, semi-analytical and transformational approaches. Various authors introduced the numerical approach to solve the FDE. Generally, the standard Euler approach is followed by a detailed error analysis. For example, a researcher applied the modified Euler approach to solve FDE. Furthermore, some researchers apply Taylor's approach of pth order to solve FDE, achieving an order of convergence of $O(h^p)$, which is superior to Euler's method.

Other numerical approaches, such as the Runge-Kutta and Corrector-Predictor methods, are used to solve FDE, respectively. Someone can use an FDE solution using the fifth order Runge-Kutta algorithm. Then, the researcher describes a numerical resolution to a second-order FDE using the Runge-Kutta method. Additionally, to analyze a solution to the FDE under generalised differentiability using the enhanced Euler technique under strongly generalised differentiability, although they examined the fuzzy initial condition here.

Allahviranloo, T. et al. [13] suggested Laplace transforms (LT) in the fuzzy field to solve the differential equations of the first order with generalised H-differentiability, but did not specify existence conditions. Salahshour, S. et al. [14] defined the conditions for the existence of the inverse Laplace transform. Additionally, researchers used FDE to solve fuzzy Laplace transformations. They solved fuzzy differential equations of the second and nth order with fuzzy initial conditions using the Generalised Hukuhara derivative.

Some writers have employed semi-analytical methods to evaluate the solutions of nonlinear differential equations, such as the Homotopy Perturbation Method (HPM) and the Adomian Decomposition Method (ADM). Then, the Variational Iteration Method (VIM) was described. Furthermore, the VIM method is applied to solve strongly nonlinear equations and 1st and 2nd order nonlinear initial value problems, respectively. Hamoud, A. A. et al. [15, 16] applied the modified Adomian decomposition technique and Homotopy analysis method to determine the solutions of the fuzzy Volterra-Fredholm integral equations, respectively. Additionally, some researchers used this method to solve FDEs in different systems. The fuzzy nonlinear differential equations were also solved by this methodology. Other approaches for solving FDEs, as well as key conclusions on continuity, solution existence and diverse applications.

For $\alpha-$level sets [17], authors have included complex fuzzy numbers. Additionally, the fuzzy differential system solution was provided in that format. Alamin, A. et al. [18] utilised the concept of fuzzy set theory to determine the solutions of the coupled homogeneous linear difference equation in intuitionistic fuzzy environments. Authors in [19] presented the idea of negative fuzzy numbers in a system of FDEs. The exchange of values when dealing with negative fuzzy numbers was also discussed. Ullah, Z. et al. [20] applied fuzzy Volterra integro-differential equations according to Laplace transformation and Adomian decomposition method (LADM) to solve the fuzzy differential equations. Then, Soma, N. et al. [21] utilised the existence and uniqueness conditions for boundary value problems under Granular differentiability in a fuzzy environment. Further, Nieto, J. J. et al. [22] have employed a variation of the constant formula to solve fuzzy differential equations.

Applying a range of numerical methods, such as Euler's method, Improved Euler's method, Runge-Kutta method and numerical methods under fuzzy inclusion, among others, writers have provided solutions to fuzzy differential equations along with their convergence analysis. Various authors solve the fuzzy initial value problem using the predictor-corrector method. The steadiness of the equation and the existence and uniqueness of fuzzy solutions were also covered in this solution. Numerous uses of fuzzy set theory are covered. Researchers discuss and solve the implicit impulsive initial value fuzzy problem. Fuzzy concepts were employed in ecological modelling and fuzzy parameters were employed in fuzzy differential equations, respectively. Additionally, fuzzy differential equations under the Hukuhara derivative, Generalised Hukuhara derivative and Zadeh's extension principle were covered by Wu, H. C. [23].

Song, S. et al. [24] proposed the fuzzy Riemann integral in detail. Bani Issa, B. S. et al. [25] applied the second kind of fuzzy integro-differential equations to evaluate the numerical computations. The existence and uniqueness conditions of a fuzzy solution to the Cauchy problem have been described by Bede, B. [26]. Then, the precise solution to fuzzy differential equations has been provided. First and higher-order linear FDEs have all been solved by various studies. Furthermore, fuzzy neural networks have been employed by authors in [12] to solve FDEs.

1.2 Motivation of This Study

Fuzzy nonlinear differential equations arise in various domains when systems exhibit nonlinear behaviour and uncertainty due to imprecise or missing data. The following are some motivations for investigating Fuzzy nonlinear differential equations:

a) Many real-world systems are inherently complex and involve interactions among multiple variables. Fuzzy nonlinear differential equations [27] provide a flexible framework for modelling such systems, allowing for the consideration of nonlinear dynamics and fuzzy logic to capture uncertainty.

b) Fuzzy nonlinear differential equations provide a flexible framework for modelling systems such as in [28].

c) A fuzzy concept [29] is adept at handling uncertainty by allowing the representation of vague or imprecise information. Fuzzy nonlinear differential equations extend this capability to differential equations, enabling the modelling of systems where exact numbers are unavailable or difficult to obtain.

d) Fuzzy nonlinear differential equations [30] are applied to solve optimisation and decision-making problems with uncertain objectives and constraints. They enable the modelling of complex optimisation problems with ambiguous objectives and constraints, resulting in more realistic and adaptable solutions.

e) Many systems in biology, ecology, economics and sociology exhibit nonlinear dynamics and uncertainty [31, 32]. Fuzzy nonlinear differential equations are used to model and analyze these systems, providing insights into their behaviour and facilitating decision-making in various domains.

1.3 Novelty and Key Findings of the Present Study

The novelty and key findings of this proposed study are discussed in detail in this section. The contributions of this paper can be regarded as novel from the following points of view:

1. Present the parametric form of a fuzzy nonlinear dynamical system, when input, output variables and mapping all are considered fuzzy numbers.

2. Propose a fuzzy sequence and use it in proving the existence and uniqueness of the fuzzy solution of nonlinear dynamical systems.

3. Demonstrate each component of a fuzzy nonlinear dynamical system, so we can visualize how each iteration behaves.

Furthermore, the key findings of the present study are as follows:

i. In this article, we emphasise the nonlinear term with uncertainty that arises in the study of nonlinear operators in fuzzy concepts.

ii. We also consider fuzzy mapping, as well as fuzzy variables and fuzzy sequences, which makes the system more complex but also more realistic.

iii. Present the definitions of fuzzy functions, fuzzy sequences and fuzzy continuity.

iv. In this article, we discuss how this Equation (2) behaves and also give the existence and uniqueness of the solution of Equation (2) using the fixed point theorem in a fuzzy set-up.

The succeeding section will present some existing impactful mathematical tools connected to the proposed theory in this paper.

1.4 Organization of This Study

This section represents the structure of this study in detail. The introduction and motivation of this research are presented in Section 1. Then, the mathematical preliminaries on fuzzy sets and associated properties are covered in Section 2. After that, the nonlinear differential equations under the fuzzy framework are discussed more elaborately in Section 3. Further, the application of fuzzy nonlinear differential equations is described in Section 4. Finally, the conclusion of the present study is drawn in Section 5.

2 Mathematical Prerequisite

This section provides a detailed discussion of the mathematical preliminaries. First, we describe the fuzzy set and then its various properties in detail.

2.1 Fuzzy Sets and Their Properties

The fuzzy set was introduced by Lotfi A. Zadeh [6] in 1965 to address the uncertainty inherent in systems. Every element in the fuzzy set [17] contains an ordered pair form where 1st one is the element itself, followed by its membership function ($\tilde{u}$). The definitions and properties of fuzzy sets are discussed as follows:

Definition 2.1. ***Fuzzy Set*** *[33]*
Let X be a non-empty universal set of discourse. Consider $\tilde{T}$ be a fuzzy set define on X and describe as

$$\tilde{T} = \{(\xi, \mu_{\tilde{T}}(\xi)) : \xi \in X\} \tag{3}$$

where $\mu_{\tilde{T}}(\xi)$ represent the membership function of fuzzy set $\tilde{T}$ and define on $\mu_{\tilde{T}}(\xi) : X \to [0, 1]$, for all arbitrary element $\xi \in X$.

The value of the membership function $\left(\mu_{\tilde{T}}(\xi)\right)$ zero represents the element ξ completely outside of the fuzzy set $\tilde{T}$, on the other hand, one represents the element ξ completely inside of the fuzzy set $\tilde{T}$ and any intermediate values of the membership function $\left(\mu_{\tilde{T}}(\xi)\right)$ represents the element ξ partially belong to the fuzzy set $\tilde{T}$. The membership value of any fuzzy set can't exceed $[0, 1]$.

Example 2.2. Assume the universal set $X = \{1, 2, \ldots, 10\}$ and fuzzy set $\tilde{U}$ define on it. Then the fuzzy set $\tilde{U}$ define as

$$\tilde{U} = \{(1, 0.5), (2, 0.3), (3, 0.6), (4, 0.8), (5, 0.9), (6, 1), (7, 0.4), (8, 0), (9, 0.5), (10, 0.8)\}$$

The membership value $\mu_{\tilde{U}}(\xi)$ of the element 2 is $\mu_{\tilde{U}}(2) = 0.3$, the element 5 is $\mu_{\tilde{U}}(5) = 0.9$, the element 6 is $\mu_{\tilde{U}}(6) = 1$ and the element 8 is $\mu_{\tilde{U}}(8) = 0$. That is, the element 6 fully belongs to the fuzzy set $\tilde{U}$, the element 8 fully does not belong to the fuzzy set $\tilde{U}$ and the elements 2 and 5 partially belong to the fuzzy set $\tilde{U}$, respectively.

Example 2.3. Consider $X = \mathbb{R}$ (set of real numbers) to be a universal set of discourse. Let $\tilde{S}$ be a fuzzy set define on $\mathbb{R}$ and defined as

$$\begin{aligned}\tilde{S} &= \left\{\left(x, \mu_{\tilde{S}}(x)\right) : x \in \mathbb{R}\right\}\\ &= \left\{\left(x, \frac{1}{1+x^2}\right) : x \in \mathbb{R}\right\}\end{aligned}$$

If $x = 0.5$, the membership value is $\mu_{\tilde{S}}(0.5) = 1/1.25 = 0.78125$, if $x = 0.2$ the membership value is $\mu_{\tilde{S}}(0.2) = 1/1.04 = 0.9615$ and for $x = 0$, the membership value is $\mu_{\tilde{S}}(0) = 1/1 = 1$.

Definition 2.4. $\alpha-$***cut*** *[17]*
Assume $\tilde{T}$ be a fuzzy set defined in Definition 2.1. An $\alpha-$cut of the fuzzy set $\tilde{T}$ is denoted as $[\tilde{T}]_\alpha$ and defined by a collection of all the elements of $\tilde{T}$, such that the membership values of the elements are greater than α, i.e.,

$$[\tilde{T}]_\alpha = \{\xi : \mu_{\tilde{T}}(\xi) \geq \alpha \;\&\; \xi \in X\} \tag{4}$$

If $\alpha = 0$, then the $\alpha-$cut of fuzzy set $(\tilde{T})$ represents as $[\tilde{T}]_0 = \{\xi : \mu_{\tilde{T}}(\xi) \geq 0 \ \& \ \xi \in X\}$, this is support of the fuzzy set $\tilde{T}$. Similarly, if $\alpha = 1$, then the $\alpha-$cut of fuzzy set $(\tilde{T})$ represents as $[\tilde{T}]_1 = \{\xi : \mu_{\tilde{T}}(\xi) = 1 \ \& \ \xi \in X\}$, this is core of the fuzzy set $\tilde{T}$.

Definition 2.5. ***Fuzzy Number*** *[33]*
Consider $\tilde{T}$ to be a fuzzy set defined on the set of real numbers as the universal set of discourse $(X = \mathbb{R})$. Then, the fuzzy set $\tilde{T}$ is said to be a fuzzy number if it satisfies the following conditions

1. *the fuzzy set $\tilde{T}$ is normal, i.e., there exist $\xi \in \mathbb{R}$ such that $\mu_{\tilde{T}}(\xi) = 1$,*
2. *the membership function $(\mu_{\tilde{T}}(\xi))$ is pice wise continuous,*
3. *the membership function $(\mu_{\tilde{T}}(\xi))$ is convex, i.e., $\mu_{\tilde{T}}(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min\left\{\mu_{\tilde{T}}(\xi_1), \mu_{\tilde{T}}(\xi_2)\right\}$ for all ξ_1, $xi_2 \in X$ and $\lambda \in [0,1]$,*
4. *support of membership function $(\mu_{\tilde{T}}(\xi))$, i.e., $\left\{\xi : \mu_{\tilde{T}}(\xi) > 0 \ \& \ \xi \in X\right\}$ is bounded.*

The fuzzy numbers are in numerous forms, including triangular fuzzy number (TFN) [34], trapezoidal fuzzy number (TrFN), pentagonal fuzzy number (PFN), hexagonal fuzzy number (HFN), Gaussian fuzzy number (GFN) and many more, which represent the uncertainty with different shapes, forms and levels of accuracy. The representation of imprecise real-world problems can be quantified more effectively by fuzzy numbers than by crisp values. The uncertainty and vagueness of the dataset and the proposed model can be handled and their importance lies in simplifying computations. Fuzzy numbers are widely used in numerous fields, including data analysis, control systems, decision-making, difference equations, differential equations and risk assessment, where measuring and calculating actual values are very difficult.

Definition 2.6. ***Parametric Form of Fuzzy Number*** *[34]*
Consider $\tilde{F}$ be a fuzzy number defined on the set of real numbers $(\mathbb{R})$. Then, fuzzy number $(\tilde{F})$ is in parametric form ($\alpha-$cut representation), given as follows

$$\begin{aligned}[\tilde{F}]_\alpha =& \{(\xi, \mu_{\tilde{T}}(\xi)) : \xi \in X\} \\ =& \left[\underline{f}, \overline{f}\right]\end{aligned} \tag{5}$$

where $\underline{f}$ is a lower bound with left increasing and $\overline{f}$ is an upper bound with right decreasing functions and satisfies the definition of an interval arithmetic.

Definition 2.7. ***Triangular Fuzzy Number (TFN)*** *[18]*
Triangular Fuzzy Number (TFN) is denoted as a triplet, $\tilde{T} = (d, e, f)$ and its membership function $(\mu_{\tilde{T}})$ is given as

$$\mu_{\tilde{T}}(x) = \begin{cases} \frac{x-d}{e-d} & ; \text{ if } d \leq x \leq e \\ \frac{f-x}{f-e} & ; \text{ if } e < x \leq f \\ 0 & ; \text{ otherwise} \end{cases} \tag{6}$$

where $d, e, f \in \mathbb{R}$ and $d \leq e \leq f$.

And, in Definition 2.7, after taking $\alpha-$cut of triangular fuzzy number $(\tilde{T})$, the obtained parametric form is,

$$[\tilde{T}]_\alpha = [d + (e-d)\alpha, f - (f-e)\alpha] \tag{7}$$

where $\alpha \in [0,1]$.

Definition 2.8. ***Arithmetic Operations of Fuzzy Numbers on $\alpha-$cut Representation*** *[33]*
Let $\tilde{S}$ and $\tilde{T}$ be two fuzzy numbers and $[\tilde{S}]_\alpha = [\underline{s}, \overline{s}]$ and $[\tilde{T}]_\alpha = [\underline{t}, \overline{t}]$ (where $\underline{s},\ \overline{s},\ \underline{t},\ \overline{t} \in \mathbb{R}$) are the two parametric forms of α with $\alpha \in [0,1]$ and k be any scalar. Then, the arithmetic operations between $\tilde{S}$ and $\tilde{T}$ is defined using their parametric forms ($\alpha-$cut forms of fuzzy numbers $\tilde{S}$ and $\tilde{T}$, where α is an arbitrary number, but after considering it, it is a fixed number), as follows

1. *Addition of two fuzzy numbers $\tilde{S}$ and $\tilde{T}$:*
$$[\tilde{S}]_\alpha \oplus [\tilde{T}]_\alpha = [\underline{s}, \overline{s}] \oplus [\underline{t}, \overline{t}] = [\underline{s} + \underline{t}, \overline{s} + \overline{t}] \tag{8}$$

2. *Multiplication of two fuzzy numbers $\tilde{S}$ and $\tilde{T}$:*
$$\begin{aligned}[\tilde{S}]_\alpha \otimes [\tilde{T}]_\alpha &= [\underline{s}, \overline{s}] \times [\underline{t}, \overline{t}] \\ &= \left[\min\left\{\underline{s}\underline{t}, \underline{s}\overline{t}, \overline{s}\underline{t}, \overline{s}\overline{t}\right\}, \max\left\{\underline{s}\underline{t}, \underline{s}\overline{t}, \overline{s}\underline{t}, \overline{s}\overline{t}\right\}\right]\end{aligned} \tag{9}$$

3. *Division of fuzzy number $\tilde{S}$ by $\tilde{T}$: $\frac{[\tilde{u}]_\alpha}{[\tilde{v}]_\alpha} = [\underline{u}, \overline{u}] \otimes [\frac{1}{\overline{v}}, \frac{1}{\underline{v}}]$,*
$$\frac{[\tilde{S}]_\alpha}{[\tilde{T}]_\alpha} = [\tilde{S}]_\alpha \oslash [\tilde{T}]_\alpha = [\underline{s}, \overline{s}] \otimes \left[\frac{1}{\overline{t}}, \frac{1}{\underline{t}}\right] \tag{10}$$
provided $[\tilde{T}]_\alpha$ does not contain 0 element.

Further, scalar multiplication of a fuzzy number $\tilde{S}$ by scaler number k, is given as

a). *If $k \geq 0$, then*
$$k \times [\tilde{S}]_\alpha = k\,[\underline{s}, \overline{s}] = [k\underline{s}, k\overline{s}] \tag{11}$$

b). *If $k < 0$, then*
$$k \times [\tilde{S}]_\alpha = k\,[\underline{s}, \overline{s}] = [k\overline{s}, k\underline{s}] \tag{12}$$

A fuzzy space [35] can be represented as a multidimensional mathematical space where every dimension can be represented by a fuzzy variable and each point in the space is represented by the point and its membership values instead of only the point.

Definition 2.9. ***Complete Fuzzy Metric Space*** *[36]*
Consider a fuzzy space Ω with Hausdorff distance [37] $\mathcal{D}$ defined on it. Let $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ and $\tilde{X}$ are three fuzzy numbers define on Ω $\left(\text{i.e., } [\tilde{U}]_\alpha,\ [\tilde{V}]_\alpha,\ [\tilde{W}]_\alpha,\ [\tilde{X}]_\alpha\right)$ and satisfies the following properties

A. $\mathcal{D}\left([\tilde{U}]_\alpha \oplus [\tilde{W}]_\alpha,\ [\tilde{V}]_\alpha \oplus [\tilde{W}]_\alpha\right) = \mathcal{D}\left([\tilde{U}]_\alpha,\ [\tilde{V}]_\alpha\right)$,

B. $\mathcal{D}\left([\tilde{U}]_\alpha, [\tilde{V}]_\alpha\right) = \mathcal{D}\left([\tilde{V}]_\alpha,\ [\tilde{U}]_\alpha\right)$,

C. $\mathcal{D}\left(k[\tilde{U}]_\alpha,\ k[\tilde{V}]_\alpha\right) = |k|\mathcal{D}\left([\tilde{U}]_\alpha,\ [\tilde{V}]_\alpha\right)$,

D. $\mathcal{D}\left([\tilde{U}]_\alpha \oplus [\tilde{V}]_\alpha,\ [\tilde{W}]_\alpha \oplus [\tilde{X}]_\alpha\right) \leq \mathcal{D}\left([\tilde{U}]_\alpha,\ [\tilde{W}]_\alpha\right) + \mathcal{D}\left([\tilde{V}]_\alpha,\ [\tilde{X}]_\alpha\right)$.

then $(\Omega,\ \mathcal{D})$ is complete metric space.

Definition 2.10. ***Fuzzy Equation*** *[38]*

A fuzzy equation is an equation that contains fuzzy numbers as variables or constants, sometimes both.

Example 2.11. Consider, $\mathbb{Y}$ be a universal set of discourse and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are three constant fuzzy numbers and $\tilde{X}$be a variable fuzzy numbers in $\mathbb{Y}$. Further, assume A, B, C are three crisp constants and X be a crisp variable defined on the set of real numbers ($\mathbb{R}$). Then the fuzzy equations define on $\mathbb{Y}$, as follows

$$\tilde{A}\tilde{X} + \tilde{B} = \tilde{C} \tag{13}$$

$$A\tilde{X} + B = \tilde{C} \tag{14}$$

$$A\tilde{X} + B = C \tag{15}$$

Remark 2.12. *From above Example 2.11, Equation (13), Equation (14) and Equation (15) are three Fuzzy Equations [38]. In Equation (13), all the variables and constants are fuzzy numbers; in Equation (14), the variable and one constant are fuzzy numbers; in Equation (15), only the variable is a fuzzy number, respectively. Further, Equation (16) is not a fuzzy equation, since neither variable nor any constants are fuzzy numbers.*

$$AX + B = C \tag{16}$$

Definition 2.13. ***Fuzzy Function*** *[39]*

Consider $\tilde{\Omega}$ and $\tilde{\Phi}$ are two non-empty universes of discourse. A fuzzy function $\tilde{f}$ define on $\tilde{\Omega}$ to $\tilde{\Phi}$ as follows

$$\tilde{f} : \tilde{\Omega} \to \tilde{\Phi} \tag{17}$$

and satisfies all functional properties [40].

Definition 2.14. ***Fuzzy Lipschitz*** *[26]*

A fuzzy function $\tilde{\phi} : I \times \Omega \to \Omega$ is said to be fuzzy Lipschitz [24], if $\tilde{\phi}$ satisfies,

$$\mathcal{D}\left(\tilde{\phi}(t,\tilde{U}), \tilde{\psi}(t,\tilde{V})\right) \le \mathcal{D}(\tilde{U},\tilde{V}),\ \forall \tilde{U},\tilde{V} \in \Omega \tag{18}$$

Definition 2.15. ***Fuzzy Continuity*** *[26]*

A fuzzy valued function $\tilde{\phi} : I \times \Omega \to \Omega$ is fuzzy continuous [24] at a point $(t_0, \tilde{U}_0)$ provided that for any fixed number $\alpha \in (0,1]$ and any $\epsilon > 0$, $\exists\ \delta(\alpha,\epsilon)$ such that,

$$\mathcal{D}\left(\tilde{\phi}(t,\tilde{U}), \tilde{\psi}(t,\tilde{V})\right) \le \epsilon \tag{19}$$

where $|t - t_0| < \delta$ and $\mathcal{D}(\tilde{U},\tilde{U}_0) < \delta(\alpha,\epsilon)$.

2.2 Fuzzy Derivatives

Fuzzy derivatives of the fuzzy function are defined in this section. Consider $\tilde{\phi}$ be a fuzzy mapping [41] is derivable if it satisfies the following:

Definition 2.16. ***Hukuhara Derivative***

Consider, a fuzzy mapping $\tilde{\phi}: I \to \Omega$ is said to be Hukuhara differentiable at τ_0, as in [10], if there exists an element $\tilde{\phi}(\tau_0) \in E$, such that for all small $h > 0$, $\tilde{\phi}(\tau_0 + h) \ominus \tilde{\phi}(\tau_0)$, $\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 - h)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{\phi}(\tau_0 + h) \ominus \tilde{\phi}(\tau_0)}{h} = \lim_{h \to 0-} \frac{\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 - h)}{h} = \dot{\tilde{\phi}}(\tau_0) \tag{20}$$

Remark 2.17. *Let $\tilde{U}, \tilde{V} \in \Omega$ and if there exists $\tilde{W} \in E$ such that $\tilde{U} = \tilde{V} \oplus \tilde{W}$, then $\tilde{W}$ is called the H-difference of $\tilde{U}$ and $\tilde{V}$ and it is denoted by $\tilde{U} \ominus \tilde{V}$. $\tilde{U} \ominus \tilde{V} \neq \tilde{U} + (-1)\tilde{V}$.*

Example 2.18. The Hukuhara derivative of $\tilde{\phi}(\tau) = \tau^3$ at τ_0. From the above definition mentioned in

$$\begin{aligned}
\lim_{h \to 0+} \frac{\tilde{\phi}(\tau_0 + h) \ominus \tilde{\phi}(\tau_0)}{h} &= \lim_{h \to 0+} \frac{(\tau_0 + h)^3 \ominus {\tau_0}^3}{h} \\
&= \lim_{h \to 0+} \frac{{\tau_0}^3 + h^3 + 3h\tau_0(\tau_0 + h) - {\tau_0}^3}{h} \\
&= \lim_{h \to 0+} \frac{h^3 + 3h\tau_0(\tau_0 + h)}{h} \\
&= 3\tau_0^2
\end{aligned}$$

Similarly, it can be easily shown,

$$\lim_{h \to 0+} \frac{\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 - h)}{h} = 3\tau_0^2.$$

Remark 2.19. *In the above example, mapping is only fuzzy; the variable $\tau \in I$ is a component of time. If it is a fuzzy variable, we find the derivative by taking $\alpha-$cut.*

Definition 2.20. ***Strongly Generalized Differentiability***

To overcome drawback of Hukuhara derivative, generalized Hukuhara derivative [11] is proposed, $\tilde{\phi}: I \to \Omega$ is said to be strongly generalized differentiable at τ_0 in Ω if,

(a) For all $h > 0$, sufficiently small $\exists\ \tilde{\phi}(\tau_0 + h) \ominus \tilde{\phi}(\tau_0)$, $\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 - h)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{\phi}(\tau_0 + h) \ominus \tilde{\phi}(\tau_0)}{h} = \lim_{h \to 0-} \frac{\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 - h)}{h} = \dot{\tilde{\phi}}(\tau_0)$$

.

(b) For all $h > 0$, sufficiently small $\exists\ \tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 + h)$, $\tilde{\phi}(\tau_0 - h) \ominus \tilde{\phi}(\tau_0)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 + h)}{h} = \lim_{h \to 0-} \frac{\tilde{\phi}(\tau_0 - h) \ominus \tilde{\phi}(\tau_0)}{-h} = \dot{\tilde{\phi}}(\tau_0)$$

.

(c) For all $h > 0$, sufficiently small $\exists\ \tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 + h)$, $\tilde{\phi}(\tau_0 - h) \ominus \tilde{\phi}(\tau_0)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 + h)}{-h} = \lim_{h \to 0-} \frac{\tilde{\phi}(\tau_0 - h) \ominus \tilde{\phi}(\tau_0)}{-h} = \dot{\tilde{\phi}}(\tau_0)$$

.

(d) For all $h > 0$, sufficiently small $\exists\ \tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 + h)$, $\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 - h)$ exist and the limits,

$$\lim_{h\to 0+} \frac{\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 + h)}{-h} = \lim_{h\to 0-} \frac{\tilde{\phi}(\tau_0) \ominus \tilde{\phi}(\tau_0 - h)}{h} = \dot{\tilde{\phi}}(\tau_0)$$

.

Definition 2.21. ***Generalized Hukuhara Derivative***

Let fuzzy mapping $\tilde{\phi} : I \to \Omega$ is said to be generalized Hukuhara differentiable [12] at $\tau_0 \in I$, and it is defined as below,

$$\lim_{h\to 0+} \frac{\tilde{\phi}(\tau_0 + h) \ominus_g \tilde{\phi}(\tau_0)}{h} = \dot{\tilde{\phi}}(\tau_0) \tag{21}$$

where $\ominus_g$ refers to gH-difference, which is defined as below

$$\tilde{U} \ominus_g \tilde{V} = [\underline{u}, \overline{u}] \ominus_g [\underline{v}, \overline{v}] = [\underline{w}, \overline{w}] = \begin{cases} \underline{u} = \underline{v} + \underline{w} \,\&\, \overline{u} = \overline{v} + \overline{w} \\ \quad or, \\ \underline{v} = \underline{u} - \overline{w} \,\&\, \overline{v} = \overline{u} - \underline{w} \end{cases} \tag{22}$$

So that, $[\underline{u}, \overline{u}] \ominus_g [\underline{v}, \overline{v}] = [\underline{w}, \overline{w}]$ always defined by,

$$[\underline{c}, \overline{c}] = [\min\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\}, \max\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\}] \tag{23}$$

Example 2.22. In this example, we consider fuzzy variable $\tilde{\phi}(\tilde{x}) = \tilde{x}^3$, $\tilde{x} > 0$, its Generalized Hukuhara derivative is defined as below,

$$\dot{\tilde{\phi}}(\tilde{x}) = \lim_{h\to 0+} \frac{\tilde{\phi}(\tilde{x} + h) \ominus_g \tilde{\phi}(\tilde{x})}{h}$$

Taking the $\alpha-$cut on both sides of the equations,

$$\begin{aligned} \left[\dot{\tilde{\phi}}\right]_\alpha (\tilde{x}) &= \lim_{h\to 0+} \frac{[\tilde{\phi}]_\alpha ([\tilde{x}]_\alpha + h) \ominus_g [\tilde{\phi}]_\alpha ([\tilde{x}]_\alpha)}{h} \\ &= \lim_{h\to 0+} \frac{\left[\underline{\phi}(\underline{x} + h), \overline{\phi}(\overline{x} + h)\right] \ominus_g \left[\underline{\phi}(\underline{x}), \overline{\phi}(\overline{x})\right]}{h} \end{aligned}$$

Using property of $\ominus_g$, we have,

$$\begin{aligned} &= \lim_{h\to 0+} \frac{\left[\min\left\{\underline{\phi}(\underline{x} + h) - \underline{\phi}(\underline{x}), \overline{\phi}(\overline{x} + h) - \overline{\phi}(\overline{x})\right\}, \max\left\{\underline{\phi}(\underline{x} + h) - \underline{\phi}(\underline{x}), \overline{\phi}(\overline{x} + h) - \overline{\phi}(\overline{x})\right\}\right]}{h} \\ &= \lim_{h\to 0+} \frac{\left[\min\left\{(\underline{x} + h)^3 - (\underline{x})^3, (\overline{x} + h)^3 - (\overline{x})^3\right\}, \max\left\{(\underline{x} + h)^3 - (\underline{x})^3, (\overline{x} + h)^3 - (\overline{x})^3\right\}\right]}{h} \end{aligned}$$

For a positive fuzzy variable, after simplifying the above equations, we have $= \left[3\underline{x}^2, 3\overline{x}^2\right]$.

Remark 2.23. ***Hukuhara Derivative:***

Initially, the authors began their work using the Hukuhara derivative. However, this derivative has a known drawback: as time progresses, the support (solution) may become unbounded. It is primarily suitable when the subtraction occurs from a larger interval to a smaller one.

Remark 2.24. ***Strongly Generalized Hukuhara Derivative:***

To overcome the drawback of the Hukuhara Derivative, the strongly generalized Hukuhara derivative is proposed. It is applied according to the situation. But it gives a possible set of solutions and one has to select a solution that best fits the problem.

Remark 2.25. ***Generalized Hukuhara Derivative:***

This is the best derivative among existing derivatives. It is applicable to all situations.

Definition 2.26. ***Fuzzy Integral***

As in [23], let $\tilde{f}$ is a fuzzy valued, closed and bounded function in $[a, b]$ then integral in parametric form is given as,

$$\int_a^b \left[\tilde{\phi}(u)\right]_\alpha du = \left[\int_a^b \underline{\phi} du, \int_a^b \overline{\phi} du\right] \tag{24}$$

We extend all results for Ω^n.

In the next section, we present results for fuzzy nonlinear dynamical systems, illustrating how the parametric form of the system is derived when the mapping is fuzzy, along with the corresponding argument. We present a fuzzy sequence in parametric form and utilize this definition in proving the existence and uniqueness of the solution in parametric form.

3 Nonlinear Differential Equations under Fuzzy Framework

Nonlinear fuzzy differential equations (NFDEs) [27] are mathematical models that incorporate nonlinear dynamics associated with fuzzy logic to manage the uncertainty and vagueness inherent in real-world problems. They are necessary when the initial conditions, parameters, data or settings of the system are ambiguous or imprecise. NFDEs increase the classical differential equations by considering fuzzy sets, enhancing them more appropriate for very complex and uncertain environments. They are crucial for precisely simulating model systems in which conventional techniques have failed due to uncertainty. These equations are broadly applicable in numerous fields, including economics, engineering, biotechnology and control systems. NFDEs provide a robust framework for examining and simulating uncertain dynamic systems, considering both nonlinear and fuzzy aspects.

Consider a system of fuzzy nonlinear differential equations as in Equation (2),

$$\frac{d\tilde{X}}{dt} = \tilde{\phi}\left(t, \tilde{X}\right), \tilde{X}\left(t_0\right) = \tilde{X}_0 \tag{25}$$

where, $\tilde{X} = \left[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n\right]^T$ and $I \times \Omega^n \to \Omega$.

Considering each vector of Equation (2), we obtain following system,

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{\phi}_1\left(t, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n\right); \tilde{x}_1\left(t_0\right) = \tilde{x}_{10} \\ \dot{\tilde{x}}_2 = \tilde{\phi}_2\left(t, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n\right); \tilde{x}_2\left(t_0\right) = \tilde{x}_{20} \\ \qquad \vdots \\ \dot{\tilde{x}}_n = \tilde{\phi}_n\left(t, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n\right); \tilde{x}_n\left(t_0\right) = \tilde{x}_{n0} \end{cases} \tag{26}$$

where, each $\tilde{\phi}_i : I \times \Omega^n \to \Omega$ is nonlinear fuzzy valued functions, $\forall i = 1, 2, \ldots, n$ and $\left[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n\right]^T \in \Omega^n$.

Using $\alpha-$cut on particular $\dot{\tilde{x}}_i$, we have,

$$\left[\dot{\tilde{x}}_i\right]_\alpha = \left[\tilde{\phi}_i\right]_\alpha \left(t, [\tilde{x}_1]_\alpha, [\tilde{x}_2]_\alpha, [\tilde{x}_3]_\alpha, \ldots, [\tilde{x}_n]_\alpha\right); \ \tilde{x}_i\left(t_0\right) = \tilde{x}_{i0} \tag{27}$$

From Equation (27), it is observed that $\tilde{\phi}_i$ is a fuzzy valued function and its arguments are also fuzzy. To visualize this concept, we write a parametric form of Equation (27) as in [42], as follows

$$\begin{aligned}\left[\min\left(\underline{\dot{x}}_i, \dot{\overline{x}}_i\right), \max\left(\underline{\dot{x}}_i, \dot{\overline{x}}_i\right)\right] &= \Big[\min\{\underline{\phi}_i([\tilde{x}_1]_\alpha, [\tilde{x}_2]_\alpha, \dots, [\tilde{x}_n]_\alpha), \overline{\phi}_i([\tilde{x}_1]_\alpha, [\tilde{x}_2]_\alpha, \dots, [\tilde{x}_n]_\alpha)\},\\
&\quad \max\{\underline{\phi}_i([\tilde{x}_1]_\alpha, [\tilde{x}_2]_\alpha, \dots, [\tilde{x}_n]_\alpha), \overline{\phi}_i([\tilde{x}_1]_\alpha, [\tilde{x}_2]_\alpha, \dots, [\tilde{x}_n]_\alpha)\}\Big]\\
&= \Big[\min\Big\{\underline{\phi}_i\left([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \dots, [\underline{x}_n, \overline{x}_n]\right),\\
&\qquad \overline{\phi}_i\left([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \dots, [\underline{x}_n, \overline{x}_n]\right)\Big\},\\
&\quad \max\Big\{\underline{\phi}_i\left([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \dots, [\underline{x}_n, \overline{x}_n]\right),\\
&\qquad \overline{\phi}_i\left([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \dots, [\underline{x}_n, \overline{x}_n]\right)\Big\}\Big]\\
&= \Big[\min\Big\{\min\underline{\phi}_i\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}, \max\overline{\phi}_i\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}\Big\},\\
&\quad \max\Big\{\min\underline{\phi}_i\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}, \max\overline{\phi}_i\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}\Big\}\Big]\end{aligned}$$

From the above equation, we have component wise,

$$\min\left(\underline{\dot{x}}_i, \dot{\overline{x}}_i\right) = \min\Big\{\min\underline{\phi}_i\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right), \max\overline{\phi}_i\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\Big\}; \min\left(\underline{x}_{i0}, \overline{x}_{i0}\right) \tag{28}$$

$$\max\left(\underline{\dot{x}}_i, \dot{\overline{x}}_i\right) = \max\Big\{\min\underline{\phi}_i\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right), \max\overline{\phi}_i\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\Big\}; \max\left(\underline{x}_{i0}, \overline{x}_{i0}\right) \tag{29}$$

where $\forall i \in N$. From Equation (28) and Equation (29), we may write parametric form of Equation (26), taking $i = 1, 2, \dots$ and the following equations are obtained,

$$\begin{cases}\min\left\{\underline{\dot{x}}_1, \dot{\overline{x}}_1\right\} = \min\Big\{\min\underline{\phi}_1\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}, \max\overline{\phi}_1\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\Big\}; & \min(\underline{x}_{10}, \overline{x}_{10})\\
\max\left\{\underline{\dot{x}}_1, \dot{\overline{x}}_1\right\} = \max\Big\{\min\underline{\phi}_1\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}, \max\overline{\phi}_1\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\Big\}; & \max(\underline{x}_{10}, \overline{x}_{10})\\
\min\left\{\underline{\dot{x}}_2, \dot{\overline{x}}_2\right\} = \min\Big\{\min\underline{\phi}_2\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}, \max\overline{\phi}_2\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\Big\}; & \min(\underline{x}_{20}, \overline{x}_{20})\\
\max\left\{\underline{\dot{x}}_2, \dot{\overline{x}}_2\right\} = \max\Big\{\min\underline{\phi}_2\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}, \max\overline{\phi}_2\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\Big\}; & \max(\underline{x}_{20}, \overline{x}_{20})\\
\qquad\vdots & \\
\min\left\{\underline{\dot{x}}_n, \dot{\overline{x}}_n\right\} = \min\Big\{\min\underline{\phi}_n\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}, \max\overline{\phi}_n\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\Big\}; & \min(\underline{x}_{n0}, \overline{x}_{n0})\\
\max\left\{\underline{\dot{x}}_n, \dot{\overline{x}}_n\right\} = \max\Big\{\min\underline{\phi}_n\left\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right\}, \max\overline{\phi}_n\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\Big\}; & \max(\underline{x}_{n0}, \overline{x}_{n0})\end{cases} \tag{30}$$

Here, Equation (30) is the parametric form of Equation (2). In compact form, we may write Equation (30) as follows,

$$\min\left(\underline{\dot{X}}, \dot{\overline{X}}\right) = \min\left\{\min\underline{\phi}\left(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right), \max\overline{\phi}\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\right\}; \min\left(\underline{X}_0, \overline{X}_0\right) \tag{31}$$

$$\max\left(\underline{\dot{X}}, \dot{\overline{X}}\right) = \max\left\{\min\underline{\phi}\left(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\right), \max\overline{\phi}\left\{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n\right\}\right\}; \max\left(\underline{X}_0, \overline{X}_0\right) \tag{32}$$

Remark 3.1. *This nonlinear differential equations model under a fuzzy environment is significant since it allows us to solve complex models of real-world systems where uncertainty, vagueness, and imprecision are inherent. Since real-world problems often contain nonlinearity and uncertainty, fuzzy sets possess the capability to effectively handle them. Further, it can analyse systems' vague measurements or linguistic inputs and design more robust controllers or predictors that tolerate ambiguity.*

3.1 Fuzzy Sequence

In this article, we establish the existence and uniqueness of solutions for a fuzzy nonlinear dynamical system using a fixed point theorem [43]. Some researchers analyze the Lacunary statistical convergence of fuzzy sequences in fuzzy fields. Furthermore, the authors discussed fuzzy sequences and fuzzy series of fuzzy numbers and their properties in detail. Some authors study the sequences of fuzzy numbers, their properties, convergent and bounded conditions. Then, determine the similarity measure and risk assessment of the fuzzy sequences. Additionally, different studies are conducted to evaluate the convergence of fuzzy sequences and their properties. On the other side, several researchers have solved the multi-criteria sequencing problem using a fuzzy sequential approach. To achieve this, we demonstrate that a fuzzy Cauchy sequence [44] converges within the same domain. Accordingly, we introduce a suitable definition of a fuzzy sequence to support this analysis.

Definition 3.2. ***Fuzzy Sequence*** *[44]*
A fuzzy sequence $\tilde{u}_n : N \to \Omega$ *is a mapping whose domain is the natural numbers and range is the set of fuzzy numbers* $\forall\ n \in N$.

Example 3.3. Let, $\tilde{u}_n = \frac{\tilde{1}}{n}$. To visualize this function in crisp set, taking parametric form of $\tilde{u}_n$ with $[\tilde{1}]_\alpha = [0.5\alpha + 0.5, 1.5 - 0.5\alpha]$

$$\begin{aligned} [\tilde{u}_n]_\alpha =& \frac{[\tilde{1}]_\alpha}{n} \\ [\min\{\underline{u}_n, \overline{u}_n\}, \max\{\underline{u}_n, \overline{u}_n\}] =& \frac{[0.5\alpha + 0.5, 1.5 - 0.5\alpha]}{n} \end{aligned} \tag{33}$$

Taking component-wise value of the above intervals,

$$\begin{cases} u_L(n) = \min\{\underline{u}_n, \overline{u}_n\} = \frac{0.5\alpha+0.5}{n}, \\ u_R(n) = \max\{\underline{u}_n, \overline{u}_n\} = \frac{1.5-0.5\alpha}{n} \end{cases} \tag{34}$$

when $\alpha \in [0, 1]$.

The visual diagram corresponding to Example 3.3 is given in Figure 1.

01.png 01.bb

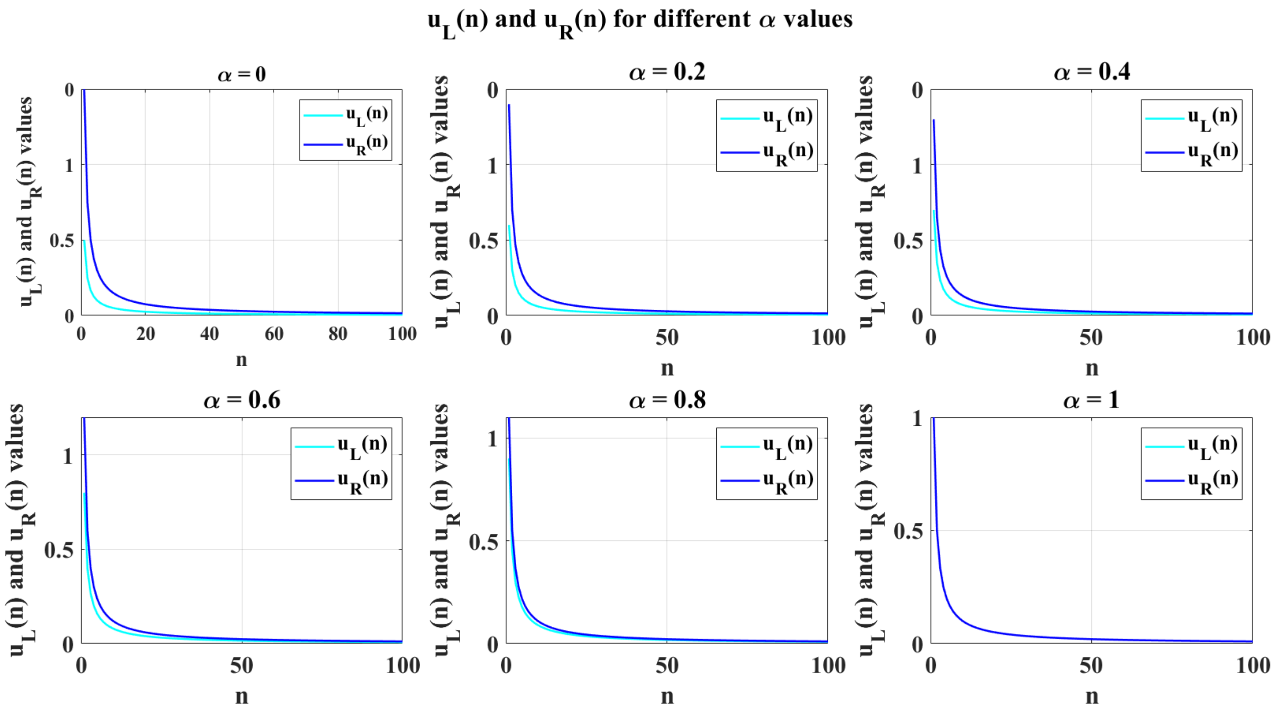


Figure 1: Graphical representation of Fuzzy sequence in parametric form

Remark 3.4. *The graphical representation of Example 3.3 is depicted in Figure 1 with different parameter* (α) *values. If parameter* (α) *increases from* 0 *to* 1*, simultaneously the width of the interval decreases as shown in the above graph.*

Remark 3.5. *To solve NFDE, fuzzy sequences play a crucial role in the solution. Fuzzy sequences possess the ability to handle the uncertainty inherent in a nonlinear differential equation, particularly within its solution space. Then, it can construct a fuzzy model for understanding the behaviour of systems under fuzzy conditions. After that, fuzzy sequence helps in identifying and analysing the advancement or approximation of those solutions, assisting in the identification of convergence or stability in the system. This NFDE model can be applied in various fields, including decision-making models, systems with uncertain parameters and control systems.*

3.2 Existence and Uniqueness Conditions for Fuzzy Nonlinear Differential Equation

The following theorem addresses the existence and uniqueness criteria for differential equations of nonlinear type in a fuzzy-based uncertain environment.

Theorem 3.6. *Let* Ω^n *be a complete fuzzy metric space and a mapping* $T \sim \frac{d}{dt}$ *is a contraction mapping, then Equation* (31) *has precisely one solution.*

Proof. Let Ω^n be a complete fuzzy metric space and let $\tilde{u}_n$ be a sequence. We need to show that $\tilde{u}_n$ is a Cauchy sequence so that it converges to a limit point and makes the space complete.

Lets take parametric form of $\tilde{u}_n$ is $[\underline{u}_n, \overline{u}_n]$. Then, from Equation (31), we have,

$$\frac{d}{dt} : \min(\underline{X}, \overline{X}) \to \min\left\{\min \underline{\phi}\left(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\right), \max \overline{\phi}\left(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\right)\right\} \tag{35}$$

$$T : \min(\underline{X}, \overline{X}) \to \min\left\{\min \underline{\phi}\left(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\right), \max \overline{\phi}\left(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\right)\right\} \tag{36}$$

Consider $\underline{u}_n$ converges to $\underline{u}$, which is a fixed point of T. T has no further fixed points.

We choose any $\underline{u}_0$,

$$\underline{u}_1 = T\underline{u}_0, \underline{u}_2 = T\underline{u}_1 = T^2\underline{u} = u_0, \ldots, \underline{u}_n = T^n\underline{u}_0 \tag{37}$$

$$\begin{aligned}
\mathcal{D}\left(\underline{u}_{m+1}, \underline{u}_m\right) &= \mathcal{D}(T\underline{u}_m, T\underline{u}_{m-1}) \\
&\leq \beta\mathcal{D}\left(\underline{u}_m, \underline{u}_{m-1}\right) \\
&\leq \beta\mathcal{D}\left(T\underline{u}_{m-1}, T\underline{u}_{m-2}\right) \\
&\leq \beta^2\mathcal{D}\left(\underline{u}_{m-1}, \underline{u}_{m-2}\right) \\
&\quad\vdots \\
&\leq \beta^m\mathcal{D}\left(\underline{u}_1, \underline{u}_0\right)
\end{aligned}$$

where β is a Lipschitz constant. Using the triangular inequality,

$$\begin{aligned}
\mathcal{D}\left(\underline{u}_m, \underline{u}_n\right) &\leq \mathcal{D}\left(\underline{u}_m, \underline{u}_{m+1}\right) + \mathcal{D}\left(\underline{u}_{m+1}, \underline{u}_{m+2}\right) + \cdots + \mathcal{D}(\underline{u}_{n-1}, \underline{u}_n) \\
&\leq \left(\beta^m + \beta^{m+1} + \cdots + \beta^{n-1}\right)\mathcal{D}(\underline{u}_0, \underline{u}_1) \\
&= \beta^m \frac{(1-\beta^{n-m})}{1-\beta}\mathcal{D}(\underline{u}_0, \underline{u}_1)
\end{aligned}$$

Since, $0 < \beta < 1$, the numerator value $1 - \beta^{n-m}$ is also less than 1. So, as $n \to \infty$,

$$\mathcal{D}\left(\underline{u}_m, \underline{u}_n\right) \leq \epsilon \tag{38}$$

Then, $\underline{u}_n$ is a Cauchy sequence, implies $\underline{u}_n$ converges to $\underline{u}$.

Similarly, $\overline{u}_n$ is a Cauchy sequence, implies $\overline{u}_n$ converges to $\overline{u}$.

Using the decomposition theorem as in Klir [45], $\widetilde{u}_n$ is a Cauchy sequence and converges to $\widetilde{u}$.

Thus, Equation (31) has precisely one solution. □

Remark 3.7. *The above theorem provides a guarantee of the solution for a system of fuzzy differential equations of a nonlinear type. The next section will explain the application of fuzzy nonlinear differential equations, which is the aptly fitted consequence of the proposed theory.*

Remark 3.8. *Contraction mapping is a stronger condition than Lipschitz continuity, i.e., if a nonlinear function satisfies the contraction condition, then it also satisfies the Lipschitz condition. For the existence and uniqueness of the solution, we show the Lipschitz condition in the following Example 3.6.*

4 Application of Fuzzy Nonlinear Differential Equations

Many applications of fuzzy nonlinear differential equations are given in [31, 32, 34] in different areas of epidemic modelling, inventory modelling, engineering and biological models, etc. In the application counterpart of the proposed theory in this paper, we employ the Susceptible-Infected-Recovered (SIR) model in a fuzzy setting.

4.1 Modelling of SIR in a Fuzzy Setting

The Susceptible-Infected-Recovered (SIR) model [46] provides insight into how a disease spreads in the environment, including the number of people infected and those recovered. The crisp model [28] of SIR gives vague or imprecise information. By taking uncertainty and imprecision in real-world data into account, fuzzy parameters improve the correctness of mathematical models. It is challenging to fully capture the complexity of disease dynamics due to the sensitivity of disease modelling to the estimation of various factors. Analysis of the transmission of infectious illnesses under uncertainty is based on the fuzzy SIR (Susceptible-Infectious-Recovered) model. This model, which provides a more reliable instrument for epidemiological studies, is frequently expanded to account for more complex and realistic disease dynamics. A fuzzy SEIR model of measles disease dynamics is examined in [47]. Fuzzy sets are used to represent the rates of illness transmission and recovery. This fuzzy SEIR model is solved using three different numerical techniques: the forward Euler method, the fourth-order Runge-Kutta and the nonstandard finite difference (NSFD) method. To address these uncertainties, an adaptive fuzzy SIR model that uses gradient descent optimisation and fuzzy logic is presented. In particular, they combine gradient descent and fuzzy logic, which adds an adaptive mechanism to deal with erratic infection and recovery rates in real time. The approximate solution of the fuzzy SIR model is obtained using the variational iteration method. The fuzzy SEIR model of amoebiasis infection is introduced, where fuzziness is added to the rates of disease transmission, disease-induced mortality and infection recovery. Dayan, F. et al. [48] applied an effective numerical-analytic approach for typhoid fever, which accounts for infection protection and utilises fuzzy parameters. In [49], the study aims to build a fuzzy parameter SIR model for COVID-19. Furthermore, a fuzzy non-standard finite difference (FNSFD) method for the SEIQR model is developed. So, the fuzzy modelling of SIR [46] is given as follows,

$$\frac{d\widetilde{S}}{dt} = \ominus\tilde{a} \otimes \tilde{S}\,(t) \otimes \tilde{I}\,(t) \tag{39}$$

$$\frac{d\widetilde{I}}{dt} = \tilde{a} \otimes \tilde{S}\,(t) \otimes \tilde{I}\,(t) \ominus \tilde{b} \otimes \tilde{I}\,(t) \tag{40}$$

$$\frac{d\widetilde{R}}{dt} = \tilde{b} \otimes \tilde{I}\,(t) \tag{41}$$

with initial conditions, $\tilde{S}\,(t_0) = \tilde{S}_0$, $\tilde{I}\,(t_0) = \tilde{I}_0$, $\tilde{R}\,(t_0) = \widetilde{R}_0$.

Theorem 4.1. *The above systems (Equations (39), (40) and (41)) with initial conditions,* $\tilde{S}\,(t_0) = \tilde{S}_0$, $\tilde{I}\,(t_0) = \tilde{I}_0$ *and* $\tilde{R}\,(t_0) = \widetilde{R}_0$ *has unique solution.*

Proof. In Theorem 3.6, it is already proposed and proved that if a nonlinear function satisfies the Lipschitz condition, then the system has a unique solution.

In the above example, fuzzy differential equations are nonlinear in nature. To obtain a unique solution, its nonlinear part $\tilde{S}(t) \otimes \tilde{I}(t)$ must satisfy the Lipschitz condition. Mathematically, it can be shown very easily.

Consider the nonlinear part of the above system, $\tilde{g}\left(t, \tilde{S}, \tilde{I}\right) = \tilde{S}(t) \otimes \tilde{I}(t)$. Then, $\widetilde{g}$ is continuous, which guarantees the existence of the solution.

Now, according to Definition 2.14, the other way to express Lipschitz continuity, the partial derivative of the dependent variable must be bounded.

$$\frac{\partial \tilde{g}\left(\tilde{S}, \tilde{I}\right)}{\partial \tilde{S}} = \tilde{I}(t) \tag{42}$$

$$\frac{\partial \tilde{g}\left(\tilde{S},\tilde{I}\right)}{\partial \tilde{I}} = \tilde{S}(t) \tag{43}$$

So, the above derivative values are always bounded; these values can never tend to infinity because they represent the number of infected and susceptible people here. Now, these fuzzy variables are taken from a complete fuzzy metric space, then the solution converges in the same metric space. Thus, this example has a unique solution. □

Example 4.2. Assuming values of $\tilde{a} = \widetilde{0.02}$, $\tilde{b} = \widetilde{0.03}$, $\widetilde{S}_0 = \widetilde{200}$, $\tilde{I}_0 = \widetilde{100}$ and $\widetilde{R}_0 = \widetilde{30}$. Then the solutions of the SIR model in the fuzzy setting are as follows:

Then the $\alpha-$cut representation of the above fuzzy numbers is as follows:

$$\begin{cases} \left[\widetilde{0.02}\right]_\alpha & = [0.01 + 0.01\alpha, 0.03 - 0.01\alpha] \\ \left[\widetilde{0.03}\right]_\alpha & = [0.02 + 0.01\alpha, 0.04 - 0.01\alpha] \\ \left[\widetilde{200}\right]_\alpha & = [190 + 10\alpha, 210 - 10\alpha] \\ \left[\widetilde{100}\right]_\alpha & = [90 + 10\alpha, 110 - 10\alpha] \\ \left[\widetilde{30}\right]_\alpha & = [25 + 5\alpha, 35 - 5\alpha] \end{cases} \tag{44}$$

Consider the first Equation (39), of the fuzzy SIR model,

$$\frac{d\tilde{S}}{dt} = \ominus \tilde{a} \otimes \tilde{S}\,(t) \otimes \tilde{I}\,(t) \tag{45}$$

Writing the parametric form of the above equation, using $\alpha-$cut,

$$\left[\frac{d\tilde{S}}{dt}\right]_\alpha = -\left[\widetilde{a}\right]_\alpha \otimes \left[\widetilde{S}(t)\right]_\alpha \otimes \left[\widetilde{I}(t)\right]_\alpha \tag{46}$$

Then the interval representations of Equation (46), as

$$\left[\frac{d\underline{S}}{dt}, \frac{d\overline{S}}{dt}\right] = -\left[\underline{aSI}, \overline{aSI}\right] \tag{47}$$

Comparing component wise of Equation (47), we get

$$\begin{cases} \frac{d\underline{S}}{dt} & = -\overline{aSI} \\ \frac{d\overline{S}}{dt} & = -\underline{aS}\ \underline{I} \end{cases} \tag{48}$$

Similarly, from Equation (40), we have

$$\begin{cases} \frac{d\underline{I}}{dt} & = \underline{aSI} - \underline{bI} \\ \frac{d\overline{I}}{dt} & = \overline{aSI} - \overline{bI} \end{cases} \tag{49}$$

and from Equation (41), we obtain

$$\begin{cases} \frac{d\underline{R}}{dt} & = \underline{bI} \\ \frac{d\overline{R}}{dt} & = \overline{bI} \end{cases} \tag{50}$$

Now, applying the numerical scheme, we obtain solutions, which are presented in Figure 2, Figure 3 and Figure 4 representing lower, upper alpha cuts and core, respectively.

02.png 02.bb

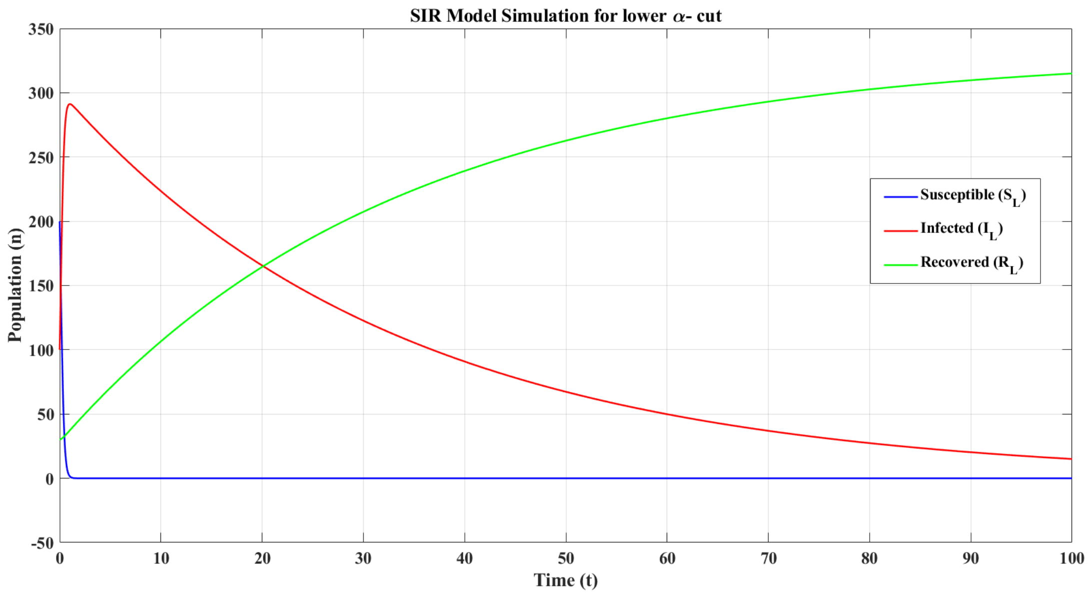


Figure 2: Dynamics of patients over time at support (lower alpha cut)

03.png 03.bb

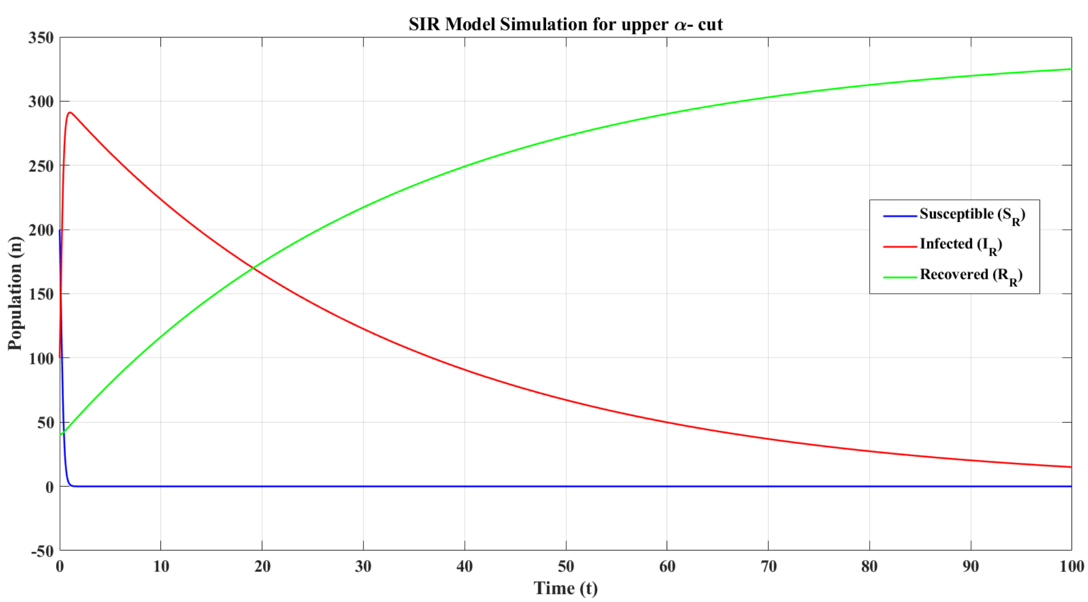


Figure 3: Dynamics of patients over time at support (Upper alpha cut)

04.png 04.bb

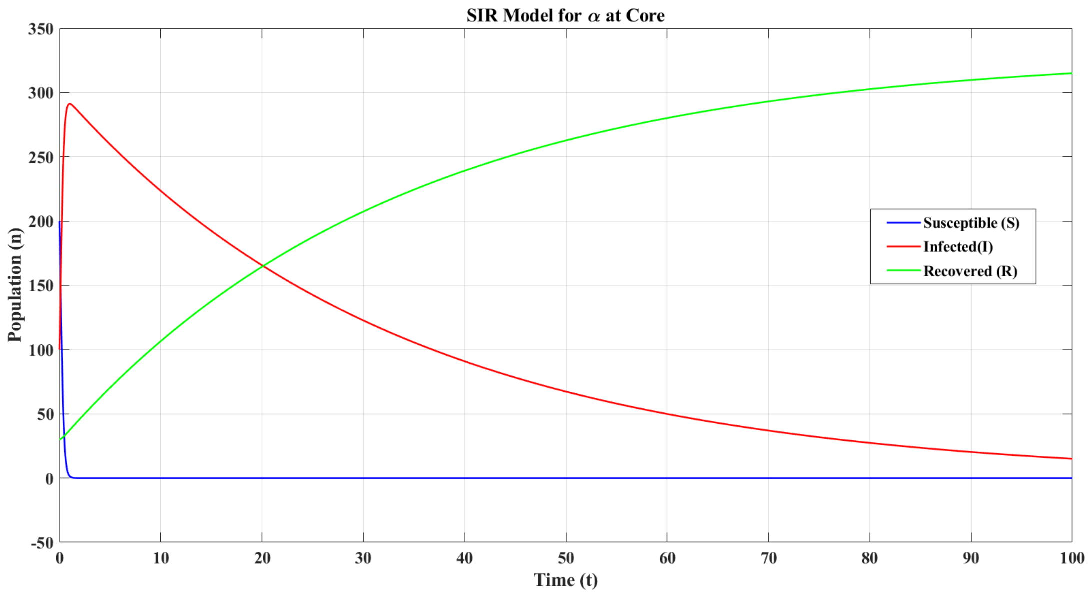


Figure 4: Dynamics of patients over time at the support at core

Remark 4.3. *Figure 2, Figure 3 and Figure 4 represent the lower $\alpha-$cut, upper $\alpha-$cut and core solution, respectively, of the SIR model of the Example 4.2. From the above three figures, we observe that, after a long time, the recovered (R) population optimised and the susceptible (S) and infected (I) populations are minimised and the population reach stable conditions.*

Example 4.4. Consider, another set initial values $\tilde{a} = \widetilde{0.01}$, $\tilde{b} = \widetilde{0.02}$, $\tilde{S}_0 = \widetilde{150}$, $\tilde{I}_0 = \widetilde{100}$ and $\tilde{R}_0 = \widetilde{50}$.

Then the $\alpha-$cut of the given fuzzy numbers are presented in Equation (51), as follows

$$\begin{cases} \left[\widetilde{0.01}\right]_\alpha & = [(0.005, 0.01, 0.015)]_\alpha = [0.005 + 0.005\alpha, 0.015 - 0.005\alpha] \\ \left[\widetilde{0.02}\right]_\alpha & = [(0.01, 0.02, 0.03)]_\alpha = [0.01 + 0.01\alpha, 0.03 - 0.01\alpha] \\ \left[\widetilde{150}\right]_\alpha & = [(145, 150, 155)]_\alpha = [145 + 5\alpha, 155 - 5\alpha] \\ \left[\widetilde{100}\right]_\alpha & = [(98, 100, 102)]_\alpha = [98 + 2\alpha, 102 - 2\alpha] \\ \left[\widetilde{50}\right]_\alpha & = [(48, 50, 52)]_\alpha = [48 + 2\alpha, 52 - 2\alpha] \end{cases} \tag{51}$$

Considering the three equations of the fuzzy SIR model (Equations (39), (40), (41)), we get three equations as follows

$$\begin{cases} \frac{d\overline{S}}{dt} & = -\overline{a}\overline{S}\overline{I} \\ \frac{d\underline{S}}{dt} & = -\underline{a}\underline{S}\ \underline{I} \end{cases} \tag{52}$$

,

$$\begin{cases} \frac{d\underline{I}}{dt} & = \underline{a}\underline{S}\underline{I} - \underline{b}\underline{I} \\ \frac{d\overline{I}}{dt} & = \overline{a}\overline{S}\overline{I} - \overline{b}\overline{I} \end{cases} \tag{53}$$

and

$$\begin{cases} \frac{d\underline{R}}{dt} & = \underline{b}\underline{I} \\ \frac{d\overline{R}}{dt} & = \overline{b}\overline{I} \end{cases} \tag{54}$$

Furthermore, applying the numerical scheme, we obtain the solution of the above set of equations graphically, represented in Figure 5, Figure 6 and Figure 7, respectively, in lower alpha cut, upper alpha cut and core forms.

05.png 05.bb

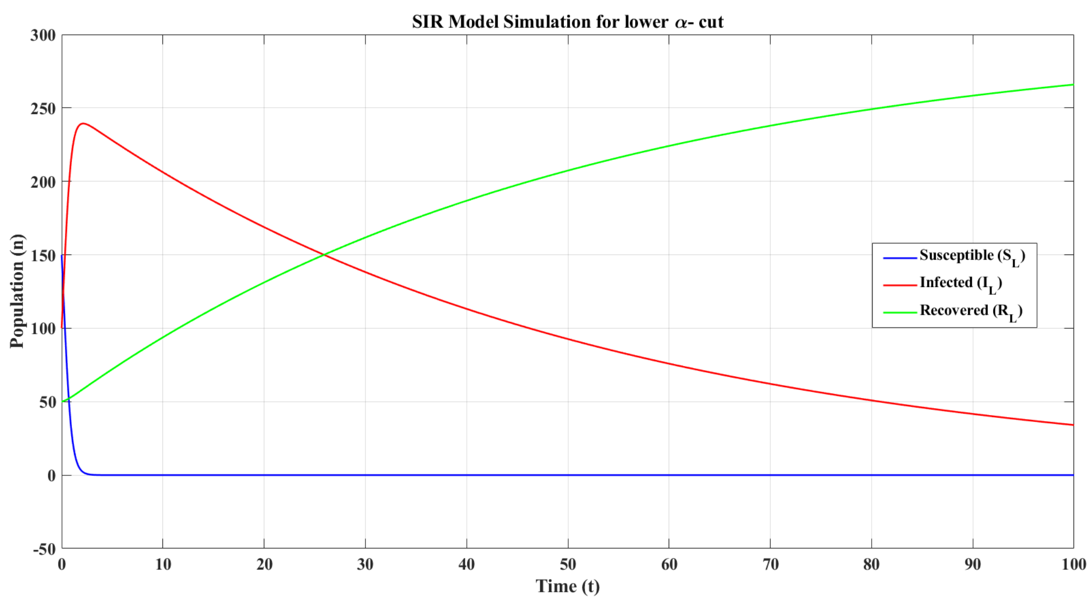


Figure 5: Evolution of populations at lower $\alpha-$cut

As small changes occur in parameters and initial conditions, the results are evident in Figure 5. The infected population and susceptible populations increase over time, while the recovered population becomes constant as time progresses at a lower alpha cut.

06.png 06.bb

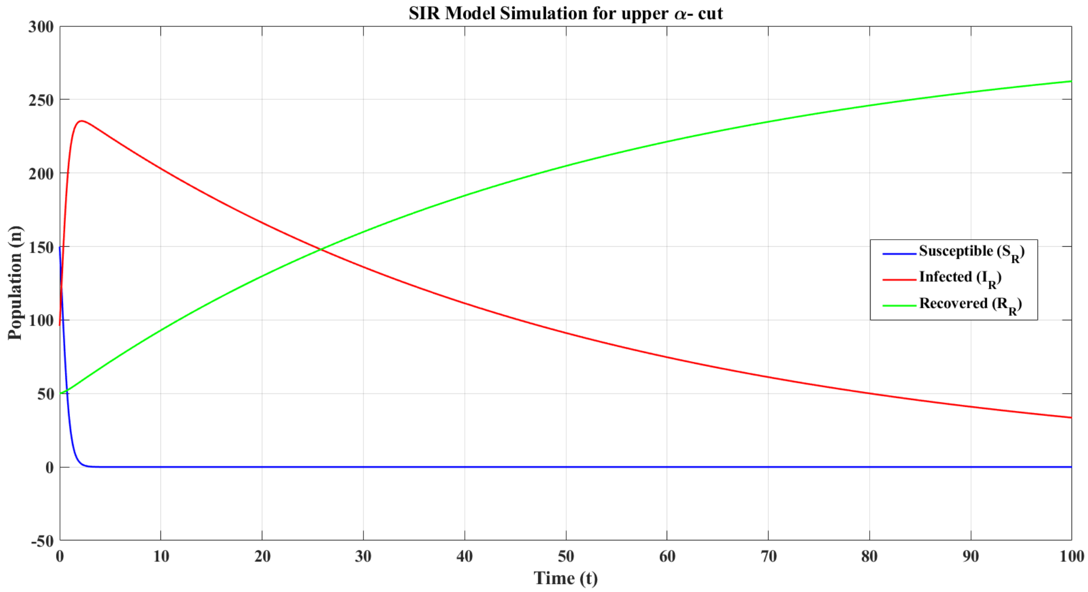


Figure 6: Evolution of populations at Upper $\alpha-$cut

As small changes occur in parameters and initial conditions, the results are evident in Figure 6. The infected population and susceptible populations increase over time, although the recovered population becomes constant as time increases at an upper alpha cut.

07.png 07.bb

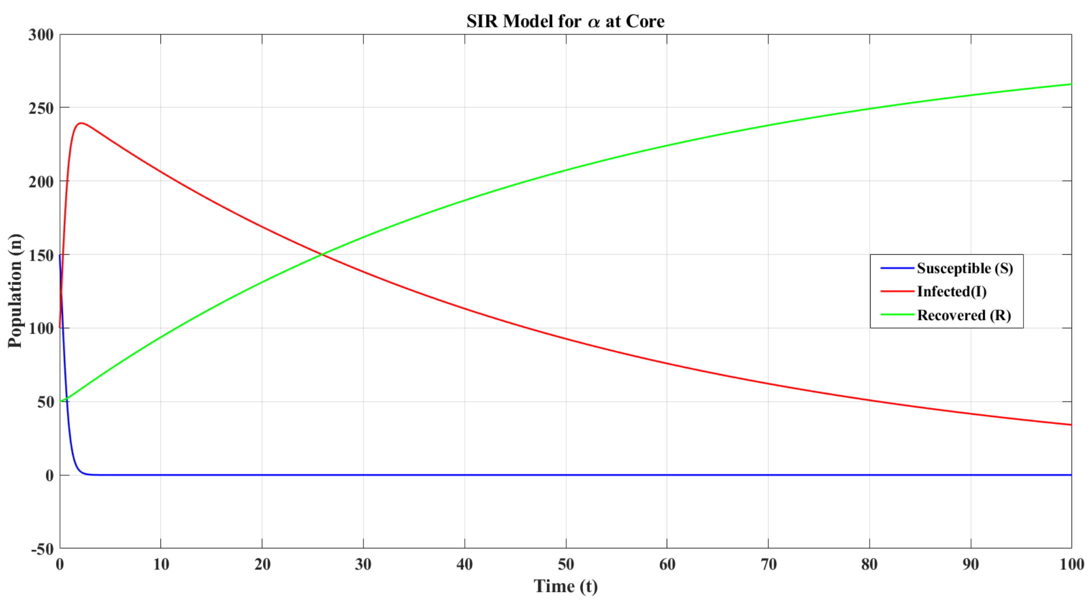


Figure 7: Dynamics of Fuzzy SIR model at core

Figure 7 illustrates the dynamics of susceptible, infected and recovered populations at the core. We can see from all graphs that small changes in parameters result in lots of variation in the number of populations.

Remark 4.5. *Figure 5, Figure 6 and Figure 7 represent the lower $\alpha-$cut, upper $\alpha-$cut and core solution, respectively, of the SIR model of the Example 4.4. Here, three colour curves represent the Susceptible (S), Infected (I) and Recovered (R) populations, respectively. After a certain time, the recovered (R) population is optimised and the susceptible (S) and infected (I) populations are minimised, as seen from all three graphs.*

Example 4.6. Now, assuming another set initial values as follows: $\tilde{a} = \widetilde{0.005}$, $\tilde{b} = \widetilde{0.002}$, $\tilde{S}_0 = \widetilde{500}$, $\tilde{I}_0 = \widetilde{300}$ and $\tilde{R}_0 = \widetilde{100}$.

Then the $\alpha-$cut of the given fuzzy numbers are presented in Equation (55), as follows

$$\begin{cases} [\tilde{a}]_\alpha = \left[\widetilde{0.005}\right]_\alpha & = [(0.004, 0.005, 0.006)]_\alpha = [0.004 + 0.001\alpha, 0.006 - 0.001\alpha] \\ [\tilde{b}]_\alpha = \left[\widetilde{0.002}\right]_\alpha & = [(0.001, 0.002, 0.003)]_\alpha = [0.001 + 0.001\alpha, 0.003 - 0.001\alpha] \\ [\tilde{S}_0]_\alpha = \left[\widetilde{500}\right]_\alpha & = [(498, 500, 502)]_\alpha = [498 + 2\alpha, 502 - 2\alpha] \\ [\tilde{I}_0]_\alpha = \left[\widetilde{300}\right]_\alpha & = [(298, 300, 302)]_\alpha = [298 + 2\alpha, 302 - 2\alpha] \\ [\tilde{R}_0]_\alpha = \left[\widetilde{100}\right]_\alpha & = [(98, 100, 102)]_\alpha = [98 + 2\alpha, 102 - 2\alpha] \end{cases} \tag{55}$$

Considering the three equations of the fuzzy SIR model (Equations (39), (40), (41)), we get three equations as follows

$$\begin{cases} \frac{d\underline{S}}{dt} & = -\overline{a}\overline{SI} \\ \frac{d\overline{S}}{dt} & = -\underline{aS}\ \underline{I} \end{cases} \tag{56}$$

,

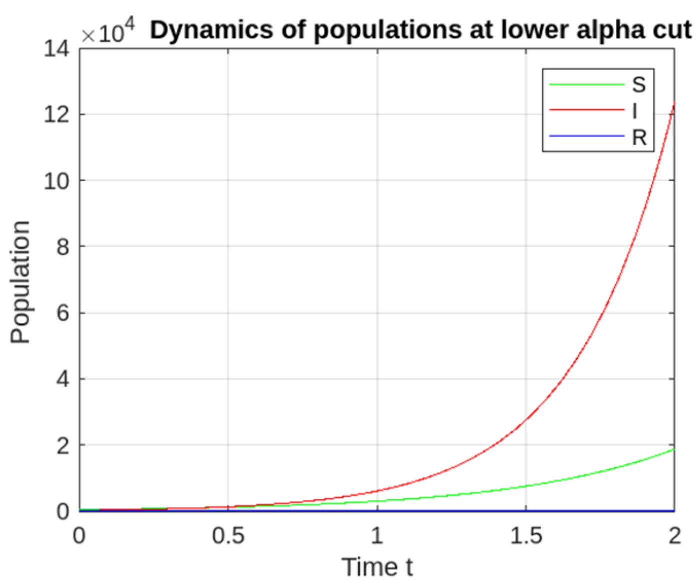
$$\begin{cases} \frac{d\underline{I}}{dt} & = \underline{aSI} - \underline{bI} \\ \frac{d\overline{I}}{dt} & = \overline{a}\overline{SI} - \overline{b}\overline{I} \end{cases} \tag{57}$$

and

$$\begin{cases} \frac{d\underline{R}}{dt} & = \underline{bI} \\ \frac{d\overline{R}}{dt} & = \overline{b}\overline{I} \end{cases} \tag{58}$$

Applying the numerical scheme, the obtained solution is displayed in the graphs below.

Furthermore, applying the numerical scheme, we obtain the solution of the above set of equations graphically, as represented in Figure 8, Figure 9 and Figure 10, respectively, in lower alpha cut, upper alpha cut and core forms.



08.png 08.bb

Figure 8: Dynamics of population at lower $\alpha-$cut

09.png 09.bb

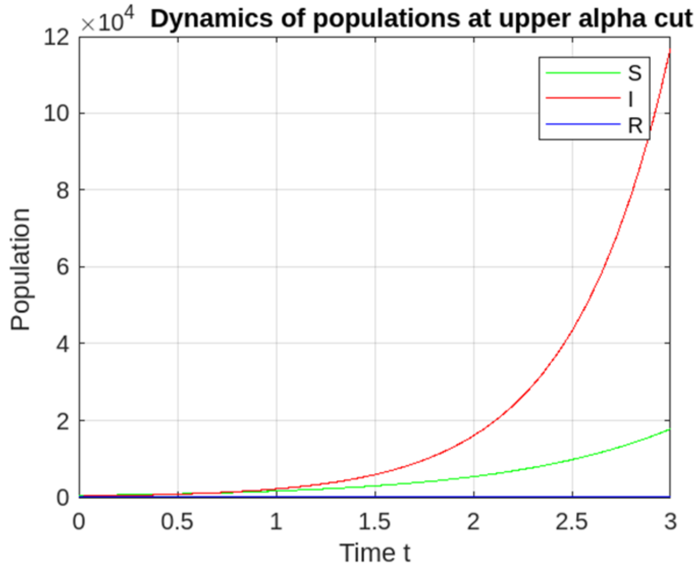


Figure 9: Dynamics of population at upper $\alpha-$cut

10.png 10.bb

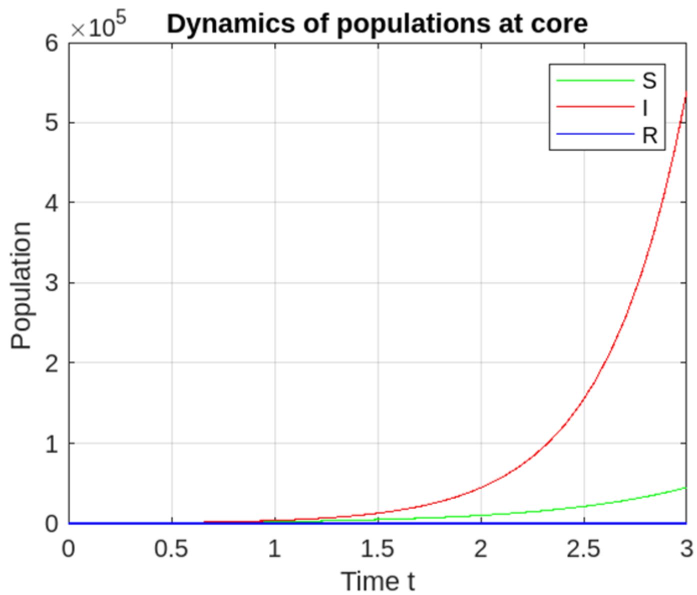


Figure 10: Dynamics of population at core

In Figures 8, 9 and 10, the number of susceptible and recovered populations is decreasing and the infected population is increasing. In this model, the value of parameters is considered small enough and initial populations are considered bigger in comparison to the value of parameters.

5 Conclusion

Dealing with nonlinear differential equations is significantly more challenging than with linear differential equations in an uncertain setting, particularly when fuzziness is present. However, most of the real-world physical phenomena are ruled by nonlinear dynamics. Furthermore, the uncertainty is an avoidable issue associated with the physical and managerial modelling. In this context, the proposed theory is both contemporary and significant, also providing the existence and uniqueness results for the fuzzy differential equation of nonlinear type. In a fuzzy nonlinear system, input, output variables and the mappings are all fuzzy numbers and sets. Therefore, we propose a fuzzy sequence with a specific perspective and utilise it in the fixed-point theorem to establish the existence and uniqueness result. Furthermore, an application of the SIR model has been analysed in the light of the proposed approach, both of tabular and graphical forms.

Despite its significant contribution, this paper has some limitations. The first limitation is that the proposed model is analysed based on hypothetical data, focusing on the theoretical perspective. Numerical results based on specific raw data from the medical and other convenient sectors may provide a more accurate landscape in this regard. We have used a single approach for solving nonlinear fuzzy differential equations, avoiding the comparison with another possible approach due to the absence of trusted data. Motivated by this work, further research can be conducted, exploring alternative methods for solving nonlinear fuzzy systems in the future.

5.1 Future Research Scope

The real-world problems can be modelled in the form of a system of fuzzy nonlinear differential equations. One can apply the theory mentioned regarding the system of nonlinear fuzzy differential equations to model real-world problems related to structure, finance (including demand and supply), inventory and other relevant areas. One can also extend the result to a system of fractional nonlinear fuzzy differential equations.

Acknowledgements: We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

Conflict of Interest: The authors declare no conflict of interest.

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Appendix A:

$$\begin{bmatrix} \tilde{a} \otimes \left(1 \ominus \frac{\tilde{N}}{\tilde{b}}\right) \ominus \tilde{d} \ominus \tilde{e} \otimes \tilde{C} \ominus \tilde{f} \otimes \tilde{I} \oplus \tilde{g}_1 \otimes \tilde{M} & \ominus \tilde{e} \otimes \tilde{N} & \ominus \tilde{f} \otimes \tilde{I} & \tilde{g}_1 \otimes \tilde{N} \\ \ominus \tilde{e} \otimes \tilde{C} & \tilde{h} \otimes \left(1 \ominus \frac{2\tilde{C}}{\tilde{j}}\right) \ominus \tilde{k} \ominus \tilde{l} \otimes \tilde{N} \ominus \tilde{o} \otimes \tilde{I} \oplus \tilde{p}_1 \otimes \tilde{M} & \tilde{o} \otimes \tilde{C} & \tilde{p}_1 \otimes \tilde{C} \\ 0 & -\tilde{u} \otimes \tilde{I} & \tilde{q} \otimes \left(1 \ominus \frac{2\tilde{I}}{\tilde{r}}\right) \ominus \tilde{s} \ominus \tilde{u} \otimes \tilde{C} \oplus \tilde{v}_1 \otimes \tilde{M} & \tilde{v}_1 \otimes \tilde{I} \\ \ominus \tilde{g}_2 \otimes \tilde{M} & \ominus \tilde{p}_2 \otimes \tilde{M} & \ominus \tilde{v}_2 \otimes \tilde{M} & \ominus \tilde{x} \ominus \tilde{g}_2 \otimes \tilde{N} \ominus \tilde{p}_2 \otimes \tilde{C} \ominus \tilde{v}_2 \otimes \tilde{I} \end{bmatrix} \quad (59)$$

$$\begin{bmatrix} \tilde{a} \otimes \left(1 \ominus \frac{\tilde{N}_e}{\tilde{b}}\right) \ominus \tilde{d} \ominus \tilde{e} \otimes \tilde{C}_e \ominus \tilde{f} \otimes \tilde{I} \oplus \tilde{g}_1 \otimes \tilde{M}_e & \ominus \tilde{e} \otimes \tilde{N}_e & \ominus \tilde{f} \otimes \tilde{I}_e & \tilde{g}_1 \otimes \tilde{N}_e \\ \ominus \tilde{e} \otimes \tilde{C}_e & \tilde{h} \otimes \left(1 \ominus \frac{2\tilde{C}_e}{\tilde{j}}\right) \ominus \tilde{k} \ominus \tilde{l} \otimes \tilde{N}_e \otimes \tilde{C}_e \ominus \tilde{o} \otimes \tilde{I}_e \oplus \tilde{p}_1 \otimes \tilde{M}_e & \tilde{o} \otimes \tilde{C}_e & \tilde{p}_1 \otimes \tilde{C}_e \\ 0 & -\tilde{u} \otimes \tilde{I}_e & \tilde{q} \otimes \left(1 \ominus \frac{2\tilde{I}_e}{\tilde{r}}\right) \ominus \tilde{s} \ominus \tilde{u} \otimes \tilde{C}_e \oplus \tilde{v}_1 \otimes \tilde{M}_e & \tilde{v}_1 \otimes \tilde{I}_e \\ \ominus \tilde{g}_2 \otimes \tilde{M}_e & \ominus \tilde{p}_2 \otimes \tilde{M}_e & \ominus \tilde{v}_2 \otimes \tilde{M}_e & \ominus \tilde{x} \ominus \tilde{g}_2 \otimes \tilde{N}_e \ominus \tilde{p}_2 \otimes \tilde{C}_e \ominus \tilde{v}_2 \otimes \tilde{I}_e \end{bmatrix} \quad (60)$$

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