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Research article

Nonlocal finite element model with eight degrees of freedom and sinusoidal shear deformation theory for bending and buckling analysis of functionally graded nanobeams

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Abstract

In the present study, an efficient nonlocal finite element model is used to investigate the buckling and bending behavior of functionally graded (FG) nanobeams. A two-node element with eight degrees of freedom is formulated according to the sinusoidal higher-order shear deformation theory. This theory assumes an accurate sinusoidal distribution of transverse shear stress in the thickness direction to provide stress-free boundary conditions without requiring shear correction factors on the top and bottom surfaces of the nanobeams. To apply size effects, the nonlocal Eringen elasticity theory is used. The material properties of FG nanobeams vary in the thickness direction as a continuous power function. Numerical results indicate the acceptable accuracy of the nonlocal finite element model. Additionally, the effects of various parameters, such as FG material power law index, length-to-thickness aspect ratio, and nonlocal parameter, on the critical buckling load and deflection of FG nanobeams are investigated.

Keywords: FG nanobeam, Nonlocal finite element, Sinusoidal Shear Deformation Theory, Static analysis.

1- Introduction

Nowadays, nanostructures such as nanorods, nanobeams, and nanoshells have diverse applications due to their outstanding electrical, chemical, thermal, and mechanical properties.

Micro/nano electromechanical systems (MEMS/NEMS) and nanoactuators are among these applications in which size effects are very important. Therefore, considering size effects in analyzing the mechanical behavior of these nanostructures is essential for better understanding their behavior and achieving appropriate designs.

It should be noted that since the classical continuum mechanics theory ignores size effects, it is not suitable for nanostructures. To overcome this issue, non-classical continuum theories, such as nonlocal elasticity theory [1, 2], strain gradient theory (SGT) [3, 4], modified coupled stress theory (MCST) [5], and nonlocal strain gradient theory (NSGT) [6], have been developed based on material size-dependent parameters.

However, investigating the effects of size dependence on the mechanical behavior of functionally graded (FG) materials with micro/nano structures should include the internal and external dimensions, which are always of fundamental importance. Unlike the classical elasticity theory, in nonlocal elasticity theory, stress at the reference point depends not only on the strain at the reference point but also on the strain at the points of the entire domain [2].

Many studies have been conducted so far based on the Eringen nonlocal elasticity theory to accurately predict the static behavior, free vibration, and buckling of homogeneous FG nanobeams. Ghayesh and Farajpour [7] reviewed studies on micro- and nano-scale structures made of FG materials. Thai and Vo [8] investigated the vibrations, buckling, and bending of nanobeams using an analytical solution based on the nonlocal sinusoidal shear deformation theory (SSDT). Thai [9] proposed a new nonlocal third-order shear deformation theory (TSDT) for the analysis of vibrations, buckling and bending of simply supported nanobeams using Eringen's nonlocal differential constitutive relations.

Analytical solutions are generally limited to simple geometries, specific properties of FG material, boundary conditions, and loadings. Therefore, numerical methods such as finite element, isogeometric analysis (IGA), and meshless method (MM) have been used to analyze the complex behavior of size-dependent FG nanostructures. The capacity and effectiveness of these methods have been investigated in a wide range of complex applications [10-12].

In existing literature, studies on nanobeams utilizing the finite element method (FEM) remain limited, despite the method's advantages in utilizing complex geometry, loading, and boundary conditions as well as arbitrary grading properties. To the best of the authors' knowledge, no publication currently provides a detailed examination of the static bending and buckling responses of functionally graded (FG) nanobeams with arbitrary FG material distributions using a finite element model based on sinusoidal higher-order nonlocal beam theories. Consequently, the primary objective and novelty of this paper is to propose an efficient finite element model to explore the bending and buckling behavior of FG nanobeams.

In the present study, an efficient finite element model is used to investigate the buckling and bending behavior of an FG nanobeam. The analysis is performed using a two-node beam element (with four degrees of freedom at each node) and sinusoidal shear deformation theory. The proposed model provides an accurate sinusoidal distribution of shear stress at each section of the beam without requiring a shear correction factor. The material properties of the nanobeam are considered to be functionally graded along the thickness direction and their variations are assumed to follow power law.

2- Governing equations

2-1- Material properties

A nanobeam with thickness h, length L, and width b made of two distinct materials (metal and ceramic) is considered. The coordinate system for the FG nanobeam is shown in Fig. 1.

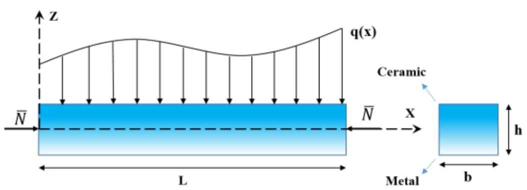

Fig. 1 Geometry and coordinate system of the FG nanobeam

The material properties of the FG nanobeam vary along the thickness direction according to a power function as follows (Fig. 2):

$$P(z) = (P_t - P_b)\left(\frac{z}{h} + \frac{1}{2}\right)^p + P_b \quad (1)$$

Where P_b and P_t are the material properties corresponding respectively to the bottom and top surfaces of the FG nanobeam; p is the material distribution parameter, which is greater than or equal to zero. For the sake of simplicity, the Poisson's ratio of the beam is assumed to be constant in this study.

2-2- Nonlocal elasticity theory

Nonlocal theories are based on size-dependent continuum mechanics that accounts for small-scale effects in the constitutive equations. Unlike the classical elasticity theory, in nonlocal elasticity theory, stress at a reference point depends not only on the strain at that point but also on the strain at all points of the body [2].

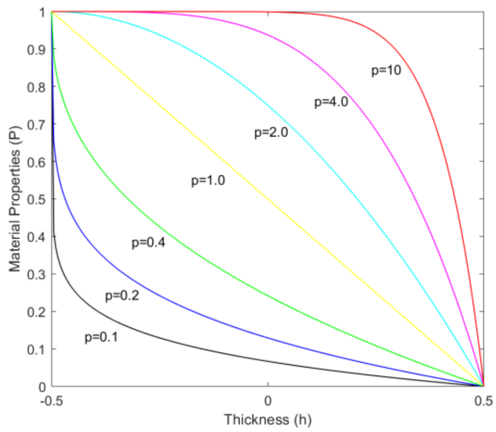

Fig. 2 Functionally graded distribution of the material properties along the thickness

The nonlocal stress tensor, S_{ij}, at a point can be written as:

$$(1 - \mu\nabla^2)S_{ij} = S_{ij}^L \quad (2)$$

where ∇^2 is the Laplacian operator in the two-dimensional Cartesian coordinate system, S_{ij}^L is the classical stress tensor at some point related to the strain tensor by Hooke's law, and $\mu = (e_0 a)^2$ is the nonlocal parameter that includes small-scale effects (e_0 is the characteristic constant for each material and a is the internal characteristic length). The value of the nonlocal parameter (μ) is important when the nonlocal elasticity theory is used.

For an isotropic FG nanobeam, the nonlocal constitutive relation in Eq. (2) can be rewritten as follows [13]:

$$\begin{Bmatrix} S_{xx} \\ S_{xz} \end{Bmatrix} - \mu\nabla^2 \begin{Bmatrix} S_{xx} \\ S_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix} \quad (3)$$

where S_{xx} and S_{xz} are the axial stress and the transverse shear stress, and C_{ij} is the stiffness coefficient(s) that is correlated with geometric constants. Moreover, by setting $e_0 a = 0$, the constitutive relation for the classical (local) theory is obtained.

$$C_{11} = E(z), \quad C_{55} = \frac{E(z)}{2(1+\nu)} \quad (4)$$

2-3- Sinusoidal Beam Theory Based on Nonlocal Elasticity

In this study, a quasi-two-dimensional sinusoidal shear deformation theory is considered for FG nanobeams considering transverse shear deformation. The displacement field in this theory can be derived as follows:

$$u = u_0 - z\frac{\partial w_0}{\partial x} + f(z)\phi$$
$$w = w_0 \quad (5)$$

where u_0 and w_0 are the axial and transverse displacement of the neutral axis of the beam web, and ϕ is the rotation of the cross section perpendicular to the neutral axis as a result of transverse shear deformation. The following points can be derived from the above equation:

- The axial displacement consists of extension, bending and shear components;
- The bending component of axial displacement is similar to that given by the Euler–Bernoulli beam theory;

A sinusoidal shear deformation function that does not require a shear correction factor is used as follows [8]:

$$f(z)=\left(\frac{h}{\pi}\right)\sin\left(\frac{\pi z}{h}\right) \tag{6}$$

This equation satisfies the condition that the shear stress on the top and bottom surfaces of the beam is negligible [8]. The non-zero strains in the beam theory can be expressed as:

$$\begin{aligned}\varepsilon_{xx}&=\frac{\partial u_0}{\partial x}-z\frac{\partial^2 w_0}{\partial x^2}+f(z)\frac{\partial \phi}{\partial x}\\ \gamma_{xz}&=g(z)\phi\end{aligned} \tag{7}$$

By rewriting the short form of the strain components, one obtains

$$\begin{aligned}\varepsilon_{xx}&=\varepsilon_{xx}^{(0)}+z\varepsilon_{xx}^{(1)}+f(z)\varepsilon_{xx}^{(2)}\\ \gamma_{xz}&=g(z)\gamma_{xz}^{(0)}\end{aligned} \tag{8}$$

where

$$g(z)=f'(z) \tag{9}$$

Now, using the principle of minimum total potential energy, the governing equations can be written as follows [14-16]:

$$\delta\Pi=\delta(U-V)=0 \tag{10}$$

where Π is the total potential energy, δU is the partial change in strain energy, and δV is the variation of the work done by external forces. The change in strain energy is defined in terms of stress resultants as follows:

$$\begin{aligned}\delta U&=\int_0^L\int_{-h/2}^{h/2}(\delta\varepsilon^T S)dxdz\\ &=\int_0^L\left(N_x\frac{\partial\delta u_0}{\partial x}-M_b\frac{\partial^2\delta w_0}{\partial x^2}+M_s\frac{\partial\delta\phi}{\partial x}+Q_{xz}\delta\phi\right)dx\end{aligned} \tag{11}$$

Moreover, Q_{xz},M_s,M_b and N_x are respectively the axial force, bending moment, shear moment, and shear force, which are given by the following equations:

$$\begin{aligned}(N_x,M_b,M_s)&=b\int_{-1/2}^{1/2}(1,z,f(z))S_x dz\\ Q_{xz}&=\int_{-1/2}^{1/2}g(z)S_{xz}dz\end{aligned} \tag{12}$$

The first variation of the work done by the compressive force is equal to

$$\delta V=\int_0^L q\delta w_0 dx+\int_0^L N_0\frac{\partial\delta w_0}{\partial x}\frac{\partial\delta w_0}{\partial x}dx \tag{13}$$

where q and N_0 are the transverse and axial loads, respectively. By substituting Eqs. (11)-(13) into Eq. (10) and integrating by parts and collecting the coefficients of $\delta\phi$, δw_0 and δu_0, the equations of motion for the sinusoidal nanobeam can be written as follows:

$$\begin{aligned}N_x-\mu\frac{\partial^2 N_x}{\partial x^2}&=A_{11}\frac{\partial u_0}{\partial x}-B_{11}\frac{\partial^2 w_0}{\partial x^2}+B_{11}^s\frac{\partial\phi}{\partial x}\\ M_b-\mu\frac{\partial^2 M_b}{\partial x^2}&=B_{11}\frac{\partial u_0}{\partial x}-D_{11}\frac{\partial^2 w_0}{\partial x^2}+D_{11}^s\frac{\partial\phi}{\partial x}\\ M_s-\mu\frac{\partial^2 M_s}{\partial x^2}&=B_{11}^s\frac{\partial u_0}{\partial x}-D_{11}\frac{\partial^2 w_0}{\partial x^2}+H_{11}^s\frac{\partial\phi}{\partial x}\\ Q_{xz}-\mu\frac{\partial^2 Q_{xz}}{\partial x^2}&=A_{55}^s\phi\end{aligned} \tag{14}$$

where the cross-sectional coefficients are expressed as follows:

$$\begin{aligned}(A_{11},B_{11},D_{11},B_{11}^s,D_{11}^s,H_{11}^s)&=\\ &\int_{-h/2}^{h/2}C_{11}(1,z,z^2,f(z),zf(z),f(z)^2)dz\\ A_{55}^s&=\int_{-h/2}^{h/2}C_{55}g(z)^2dz\end{aligned} \tag{15}$$

Using the total potential energy variation, the weak form of the governing equations is obtained as:

$$\int_0^L [\delta\varepsilon^{\circ T} A_{11}\varepsilon^{\circ} + \delta\varepsilon^{\circ T} B_{11}\varepsilon^{1} + \delta\varepsilon^{\circ T} B_{11}^{s}\varepsilon^{2} + \delta\varepsilon^{1T} B_{11}\varepsilon^{\circ} + \delta\varepsilon^{1T} D_{11}\varepsilon^{1} + \delta\varepsilon^{1T} D_{11}^{s}\varepsilon^{2} + \delta\varepsilon^{2T} B_{11}^{s}\varepsilon^{\circ} + \delta\varepsilon^{2T} D_{11}^{s}\varepsilon^{1} + \delta\varepsilon^{2T} H_{11}^{s}\varepsilon^{2} + \delta\gamma^{\circ T} A_{55}^{s}\gamma^{\circ} - (1-\mu\frac{\partial^2}{\partial x^2})(q\delta w_0 + N_0\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial x}\delta w)]dx \quad (16)$$

3- Finite element formulation

Recently, an efficient nonlocal finite element model has been used to study the buckling and bending behavior of FG nanobeams based on the modified high-order shear deformation theory. As can be seen in Fig. 3, the element has two nodes and eight degrees of freedom. The vector components of nodal displacement are given as follows:

$$d = \left\{ u_0^{(i)}, w_0^{(i)}, \left(\frac{\partial w_0}{\partial x}\right)^{(i)}, \varphi^{(i)} \right\}^T, \quad i=1,2 \quad (17)$$

The unknown components u_0 and ϕ are approximated using the linear Lagrange interpolation function with continuity C°. However, the cubic Hermite interpolation functions with continuity order C^1 are used to approximate the component w_0.

The generalized displacements in each element can be written as:

$$u_0 = N u_0^e \;,\; \phi = N\phi^e \;,\; w_0 = \bar{N} w_0^e \quad (18)$$

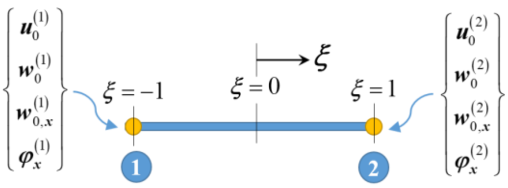

Fig. 3 Two-node element of the nanobeam with corresponding DOFs

The classical interpolation functions are given as:

$$N = [N_1 \quad N_2], \quad \bar{N} = [\bar{N}_1 \quad \bar{N}_2 \quad \bar{N}_3 \quad \bar{N}_4] \quad (19)$$

where

$$N_i = \frac{1}{2}(1+\xi_0) \quad i=1,2 \quad \xi_0 = \xi\xi_i$$
$$\bar{N}_i = \frac{1}{4}(\xi_0+1)(2+\xi_0-\xi^2) \quad i=1,3 \quad (20)$$
$$\bar{N}_i = \frac{1}{4}\xi_0.(\xi_0+1)^2(\xi_0-1) \quad i=2,4$$

In the above functions, ξ is the local coordinate of the element, as shown in Fig. 3. By inserting (18) into the generalized strain vectors of (7), one obtains:

$$\{\varepsilon^0\}^{(e)} = [B_0]^{(e)}\{\delta_i\}^{(e)}$$
$$\{\varepsilon^1\}^{(e)} = [B_1]^{(e)}\{\delta_i\}^{(e)}$$
$$\{\varepsilon^2\}^{(e)} = [B_2]^{(e)}\{\delta_i\}^{(e)} \quad (21)$$
$$\{\gamma^0\}^{(e)} = [B_3]^{(e)}\{\delta_i\}^{(e)}$$

where $\{\delta_i\}_{(e)}$ is the degree-of-freedom vector of the nanobeam element, as defined in (17); $[B_i]$ is an 8 × 1 matrix, representing the shape functions $\bar{N}, N$ and their derivatives, such that

$$[B_0] = \begin{bmatrix} \frac{dN_1}{dx} & 0 & 0 & 0 & \frac{dN_2}{dx} & 0 & 0 & 0 \end{bmatrix}$$
$$[B_1] = \begin{bmatrix} 0 & \frac{-d^2\bar{N}_1}{dx^2} & \frac{-d^2\bar{N}_2}{dx^2} & 0 & 0 & \frac{-d^2\bar{N}_3}{dx^2} & \frac{-d^2\bar{N}_4}{dx^2} & 0 \end{bmatrix}$$
$$[B_2] = \begin{bmatrix} 0 & 0 & 0 & \frac{dN_1}{dx} & 0 & 0 & 0 & \frac{dN_2}{dx} \end{bmatrix}$$
$$[B_3] = [0 \quad 0 \quad 0 \quad N_1 \quad 0 \quad 0 \quad 0 \quad N_2] \quad (22)$$
$$[\bar{N}_b] = [0 \quad \bar{N}_1 \quad \bar{N}_2 \quad 0 \quad 0 \quad \bar{N}_3 \quad \bar{N}_4 \quad 0]$$

Using the strain-displacement relations of the above equation, one can rewrite (16) as follows:

$$0 = \delta d^T \int_0^L (B_0^T A_{11} B_0 + B_0^T B_{11} B_1 + B_0^T B_{11}^s B^2 + B_1^T B_{11} B_0 + B_1^T D_{11} B_1 + B_1^T D_{11}^s B_2 + B_2^T B_{11}^s B_0 + B_2^T D_{11}^s B_1 + B_2^T H_{11}^s B_2 + B_s^T A_{55}^s B_s)dx \; \delta d - q\int_0^L \delta w \, dx \quad (23)$$

The final governing equation system is expressed in the matrix form for static and buckling analyses of the nanobeam as:

Static analysis: assuming the application of a transverse load, q, on the beam top surface, the following equation is obtained:

$$[k]\{d\} = \{F\} \tag{24}$$

Buckling analysis: an axial load, N_0, is applied to the beam centerline and the following equation is obtained:

$$([k] - N_0[k_g])\{d\} = \{0\} \tag{25}$$

where $[k]$ is the reference stiffness matrix, $[k_g]$ is the reference geometric stiffness matrix, $\{F\}$ is the force vector, and $\{d\}$ is the degrees-of-freedom vector in the reference system of the FG nanobeam. They can be obtained through the assembly of element-related matrices, which are defined as follows

$$\begin{aligned}
[K_e] = \int_0^L (& B_0^T A_{11} B_0 + B_0^T B_{11} B_1 + B_0^T B_{11}^s B^2 + B_1^T B_{11} B_0 \\
& + B_1^T D_{11} B_1 + B_1^T D_{11}^s B_2 + B_2^T B_{11}^s B_0 + B_2^T D_{11}^s B_1 \\
& + B_2^T H_{11}^s B_2 + B_s^T A_{55}^s B_s)dx \\
[K_{ge}] = \int_0^L & \left(\left\{ \frac{d\bar{N}_b}{dx} \right\}^T N_0 \left\{ \frac{d\bar{N}_b}{dx} \right\} + \mu \left(\left\{ \frac{d^2\bar{N}_b}{dx^2} \right\}^T N_0 \left\{ \frac{d^2\bar{N}_b}{dx^2} \right\} \right) \right) dx \\
\{F_e\} = q\int_0^L & \left(\{\bar{N}_b\}^T - \mu \left\{ \frac{d^2\bar{N}_b}{dx^2} \right\}^T \right) dx
\end{aligned} \tag{26}$$

Simply supported (*SS*) boundary conditions are considered at the two ends of the nanobeam as presented in Table 1.

Table 1: Boundary conditions of the FG nanobeam

Left boundary ($x = 0$)	Right boundary ($x = L$)
$u_0 \neq 0,\ w_0 = 0,\ w_{0,x} \neq 0,\ \varphi \neq 0$	$u_0 \neq 0,\ w_0 = 0,\ w_{0,x} \neq 0,\ \varphi \neq 0$

4- Numerical results

Two general steps are presented in this section. In the first step, the proposed nonlocal finite element model is validated in comparison with previously published results, and in the second step, the effect of nonlocal parameter, FG material strength, and different aspect ratios on the bending and buckling behavior of the FG nanobeam are investigated.

In this study, an FG nanobeam is considered with $E_t = 0.25$ TPa, $E_b = 1$ TPa, and $\nu_t = \nu_b = 0.3$. For simplicity, the following dimensionless parameters are used for the nanobeam displacement and critical buckling load:

$$\bar{w} = 100w\frac{E_t I}{qL^4}, \qquad \bar{N} = N_{cr}\frac{L^2}{E_t I} \tag{27}$$

To confirm the accuracy of the proposed nonlocal finite element method, the present results are compared with those in the literature. Table 2 gives the maximum dimensionless displacement ($\bar{w}$) for the FG nanobeam under uniform transverse load.

A wide range of values of nonlocal parameter (e_0a), FG material strength (p), and aspect ratios (L/h) are compared with the results of the Timoshenko beam theory (TBT) and sinusoidal beam theory (SBT).

Table 2: Non-dimensional transverse deflection ($\bar{w}$) of the nanobeam under uniform load ($N_{el} = 32$)

L/h	p	Nonlocal parameter, e_0a (nm): 0			0.5			1			1.5		
		TBT [13]	SBT [13]	Present (FE-HSDT)	TBT [13]	SBT [13]	Present (FE-HSDT)	TBT [13]	SBT [13]	Present (FE-HSDT)	TBT [13]	SBT [13]	Present (FE-HSDT)
10	0	5.3383	5.3381	5.3382	5.4659	5.4659	5.4658	5.8487	5.8487	5.8486	6.4867	6.4867	6.4866
	0.3	3.2169	3.2178	3.2196	3.2938	3.2951	3.2967	3.5245	3.5258	3.5275	3.9090	3.9104	3.9123
	1	2.4194	2.4193	2.4192	2.4772	2.4773	2.4772	2.6508	2.6509	2.6506	2.9401	2.9401	2.9399
	3	1.9249	1.9234	1.9233	1.9710	1.9694	1.9693	2.1091	2.1074	2.1070	2.3393	2.3375	2.3373
	10	1.5799	1.5790	1.5790	1.6176	1.6168	1.6168	1.7310	1.7301	1.7300	1.9190	1.9189	1.9189
30	0	5.2227	5.2228	5.2228	5.2366	5.2367	5.2367	5.2784	5.2785	5.2785	5.3480	5.3481	5.3481
	0.3	3.1486	3.1475	3.1490	3.1570	3.1559	3.1574	3.1822	3.1811	3.1826	3.2241	3.2230	3.2247
	1	2.3732	2.3732	2.3730	2.3795	2.3795	2.3793	2.3985	2.3985	2.3983	2.4301	2.4302	2.4301
	3	1.8894	1.8892	1.8889	1.8944	1.8943	1.8940	1.9095	1.9094	1.9090	1.9347	1.9346	1.9343
	10	1.5489	1.5488	1.5488	1.5530	1.5529	1.5529	1.5654	1.5653	1.5653	1.5860	1.5860	1.5859

Table 3: Non-dimensional critical buckling load ($\bar{N}$) of the nanobeam ($N_{el} = 32$)

L/h	p	Nonlocal parameter, e_0a (nm): 0			0.5			1			1.5		
		TBT [13]	SBT [13]	Present (FE-HSDT)	TBT [13]	SBT [13]	Present (FE-HSDT)	TBT [13]	SBT [13]	Present (FE-HSDT)	TBT [13]	SBT [13]	Present (FE-HSDT)
10	0	2.4056	2.4052	2.4057	2.3477	2.3473	2.3477	2.1895	2.1892	2.1896	1.9685	1.9682	1.9686
	0.3	3.9921	3.9906	3.9904	3.8959	3.8945	3.8931	3.6335	3.6322	3.6308	3.2667	3.2655	3.2643
	1	5.3084	5.3086	5.3120	5.1805	5.1808	5.1820	4.8315	4.8317	4.8329	4.3437	4.3440	4.3450
	3	6.6720	6.6780	6.6784	6.5113	6.5172	6.5179	6.0727	6.0781	6.0787	5.4596	5.4645	5.4651
	10	8.1289	8.1338	8.1338	7.9332	7.9379	7.9380	7.3987	7.4031	7.4032	6.6518	6.6558	6.6559
30	0	2.4603	2.4604	2.4603	2.4536	2.4537	2.4536	2.4336	2.4337	2.4337	2.4011	2.4011	2.4011
	0.3	4.0811	4.0826	4.0808	4.0699	4.0714	4.0702	4.0368	4.0383	4.0369	3.9828	3.9843	3.9829
	1	5.4146	5.4147	5.4155	5.3998	5.3999	5.4017	5.3559	5.3560	5.3574	5.2843	5.2843	5.2858
	3	6.8011	6.8018	6.8023	6.7825	6.7832	6.7841	6.7273	6.7280	6.7290	6.6373	6.6380	6.6390

	10	8.2962	8.2968	8.2971	8.2735	8.2741	8.2743	8.2062	8.2068	8.2070	8.0964	8.0970	8.0972
100	0	2.4667	2.4668	2.4667	2.4661	2.4662	2.4661	2.4643	2.4643	2.4643	2.4613	2.4613	2.4613
	0.3	4.0915	4.0933	4.0924	4.0905	4.0923	4.0911	4.0874	4.0893	4.0879	4.0824	4.0842	4.0828
	1	5.4270	5.4271	5.4294	5.4257	5.4257	5.4275	5.4217	5.4217	5.4231	5.4150	5.4150	5.4165
	3	6.8161	6.8162	6.8174	6.8144	6.8145	6.8157	6.8094	6.8095	6.8104	6.8010	6.8011	6.8020
	10	8.3157	8.3158	8.3160	8.3136	8.3137	8.3140	8.3075	8.3076	8.3077	8.2972	8.2973	8.2975

The dimensionless critical buckling load of the sinusoidal nanobeam is presented in Table 3. These results are compared with the findings of high-quality scientific publications, which indicate the appropriate accuracy of the proposed nonlocal finite element method. The details in this table show that the buckling load decreases with an increase in the nonlocal parameter. However, an increase in the strength of the FG material increases the critical buckling load. In general, an increase in the strength of the FG material reduces the dimensionless displacement of the nanobeam and increases the critical buckling load. The reason for these changes is the increase in the stiffness of the FG nanobeam with an increase in the strength of the FG material.

To show the effects of the nonlocal parameter on the FG nanobeam behavior, the displacement component and dimensionless critical buckling load are calculated using the nonlocal finite element method for different aspect ratios as shown in Figs. 4 and 5. These figures show the nonlinear behavior with respect to the nonlocal parameter. For the strength of the FG material, we set *p=1*. The effect of nonlocal parameter on the displacement and critical buckling load of the nanobeam is greater and more obvious, especially at smaller aspect ratios.

The effects of the FG material strength on the displacement and critical buckling load of the FG nanobeam are illustrated in Figs. 6 and 7 for $L/h=10$ and different values of nonlocal parameter. According to these results, with increasing material strength, the dimensionless displacement of the nanobeam decreases, while the critical buckling load increases. The reason for these changes is the increase in the stiffness of the FG nanobeam due to the increase in the FG material strength.

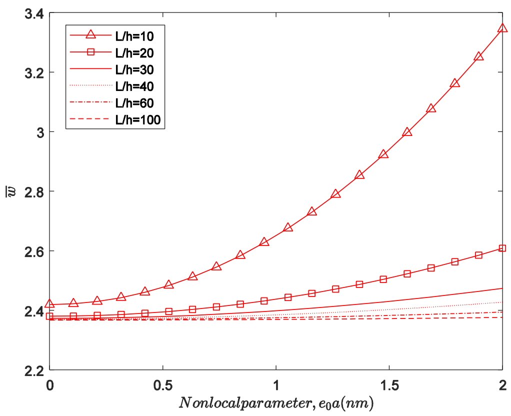

Fig. 4 Effect of nonlocal parameter on dimensionless deflection ($p=1$)

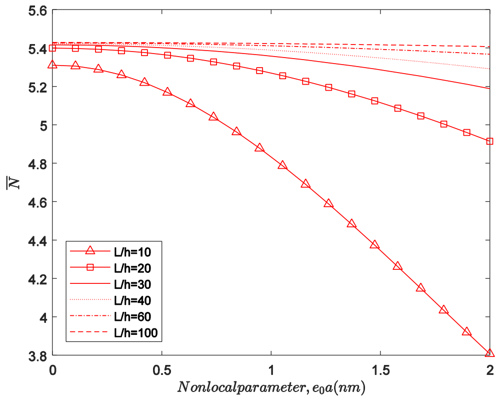

Fig. 5 Effect of nonlocal parameter on dimensionless buckling load ($p=1$)

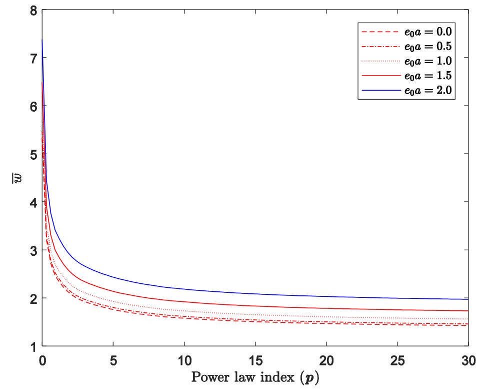

Fig. 6 Non-dimensional deflection of the nanobeam in terms of power law index

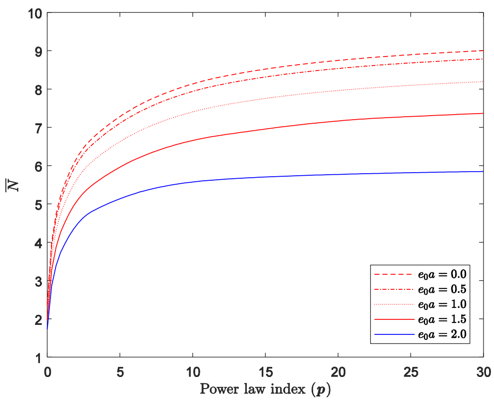

Fig. 7 Non-dimensional buckling load of the nanobeam in terms of power law index

5- Conclusion

In the present study, the size-dependent bending and buckling behavior of an FG nanobeam was investigated using the finite element method and nonlocal elasticity theory. The sinusoidal shear deformation theory was used such that the transverse shear stress at the top and bottom surfaces of the nanobeam was negligible. The mechanical properties of the FG nanobeam vary as a power function in the thickness direction. The size effects on the behavior of the nanobeam were investigated by introducing a nonlocal parameter. The capability and reliability of the proposed finite element model were evaluated by comparing our findings with the results of the existing analytical methods. The results indicated that the proposed finite element model is shear-locking-free and accurate with appropriate convergence speed for thick and thin nanobeams. Furthermore, a parametric investigation was carried out to examine the effects of various parameters such as aspect ratio, dimensionless parameter, and FG material power law index.

6- Nomenclature

P	Nanobeam material property
p	FG material power law index
C_{ij}	Stiffness coeficients
E	Young's modulus (N/m^2)
u	Axial displacement of nanobeam
w	Transverse displacement of nanobeam
	Greek symbols
μ	Non-local parameter $(nm)^2$
ϕ	Rotation of the cross section perpendicular to the neutral axis
ν	Poisson's ratio
ξ	Element local coordinate
S_{ij}	Stress components

ε_{ij} Strain components

Subscripts

b Bottom surface of nanobeam

t Top surface of nanobeam

Superscripts

i Index of nonlocal finite element

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