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Research article

Solving the sequential and non- sequential Caputo fractional-order differential equations using the Laplace transform method

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Abstract: The Laplace transform method is nevertheless useful in spite of these drawbacks, especially when dealing with linear equations, but researchers should exercise caution while using it and interpreting the findings. In this paper, the Laplace transform approach is presented a potent tool for solving the non-sequential and Sequential Caputo fractional-order differential equations (SCFODEs) and a claim about the limitations of this method is rejected and updated. Several test cases are used to validate the approach, and the outcomes demonstrate that the Laplace transform method is a dependable and effective way to solve SCFODEs.

Keywords: Caputo fractional-order differential equations; Caputo fractional derivatives; Laplace transform; Mittag-Leffler function; Sequential Caputo fractional derivatives.

1- Introduction

The Caputo fractional-order differential equations have drawn a lot of interest lately because of their precision in simulating intricate flows and the behavior of viscous materials. More accurate predictions of material behavior under varied settings can be made thanks to these equations' strong foundation for encapsulating the non-local characteristics and memory effects present in such systems.

These formulas offer a strong foundation for examining events that conventional integer-order models might not be able to effectively capture.

Caputo fractional derivatives work especially well for simulating non-Newtonian fluids, which have intricated flow patterns under a variety of circumstances [1]. These derivatives can be used to account for heat transport and viscoelastic damping in industrial processes, as shown by the fractional Nadeem trigonometric model. However, when it comes to solving Caputo fractional-order differential equations, the Laplace transform method has a number of advantages, especially when it comes to modeling complex flows and viscous materials. It simplifies difficult equations for the solution process by offering a methodical way to convert differential equations into algebraic equations. It does, however, have drawbacks, particularly when working with non-invertible solutions in the Laplace domain, which calls for the use of numerical inversion techniques.

Numerous research examines the method's limitations and efficacy, emphasizing its application in viscoelastic fluid modeling and fractional calculus [2, 3]. When Laplace domain solutions cannot be inverted analytically, numerical inversion techniques like Talbot's and Euler's methods can be employed to obtain real-domain solutions, enhancing the applicability of the Laplace transform in practical scenarios [4]. Additionally, the approach is able to simulate various fractional differential equations, such as those with variable order or dispersed derivatives, which are essential for precisely simulating viscoelastic fluid flows [5]. But, analytical solutions in the Laplace domain can be difficult, time-consuming, and sometimes unavailable, prompting the development of numerical methods to approximate solutions [6]. Reliance on numerical inversion techniques might result in inaccuracies, therefore choosing the right approaches carefully is necessary to guarantee correct findings [4]. It can be difficult to solve non-homogeneous linear differential equations directly with the Laplace transform; handling initial conditions and non-linear components necessitates extra steps [3].

The order of the Caputo fractional derivative significantly influences the accuracy of solutions obtained through the Laplace transform method. This relationship is particularly evident in the context of fractional differential equations, where the choice of derivative order can affect convergence and the fidelity of the solution to real-world data.

Solutions involving higher-order Caputo derivatives (e.g., order 2q) tend to yield more accurate results when appropriate fractional initial and boundary conditions are applied, as demonstrated in sequential boundary value problems [7].

The Caputo derivative's convolution form allows for effective application of the Laplace transform, enhancing solution accuracy, especially in systems like the predator-prey model [8]. Studies show that methods combining Laplace transforms with decomposition techniques yield solutions that closely match known results, indicating that the order of the derivative can optimize numerical methods [2, 9]. Advanced numerical methods, such as Jacobi spectral collocation and q-homotopy techniques, have been developed to solve systems of Caputo fractional differential equations efficiently [10,11]. These methods yield high accuracy and fast convergence, making them suitable for real-world applications in fluid dynamics and wave propagation [12]. The versatility of Caputo fractional equations extends to fields like physics, control theory, and electrostatics, where they model complex interactions and dynamics [13]. Their ability to incorporate fractional parameters allows for a more nuanced understanding of diffusion and wave phenomena, enhancing predictive capabilities in engineering and scientific research [10].

2- Preliminaries

One of the basic functions of fractional calculus is the gamma function, introduced by Euler in 1729, which generalizes to integers and n!, allowing n to take on even complex values. The gamma function is defined as follows:

$$\Gamma(x)=\int_0^\infty e^{-z}z^{x-1}dz \tag{1}$$

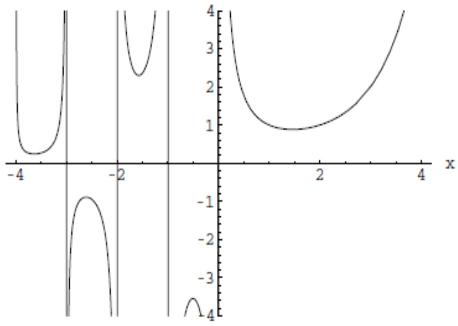

Fig. 1 Graph of gamma function

Defination 1. The Mittag-Leffler function is an important function that is widely used in the field of fractional calculus. In fact, this function is a generalization of the exponential function. Just as the exponential function plays an important role in solving differential equations, the Mittag-Leffler function also plays a similar role in solving differential equations of non-integer order. A parametric generalization of the exponential function is called the Mittag-Leffler function, which is represented as follows:

$$E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\alpha k+1)} \tag{2}$$

The two-parameter Mittag-Leffler function, which is of particular importance in fractional calculus, was first presented by Agarwal [14] as follows:

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\alpha k+\beta)} \tag{3}$$

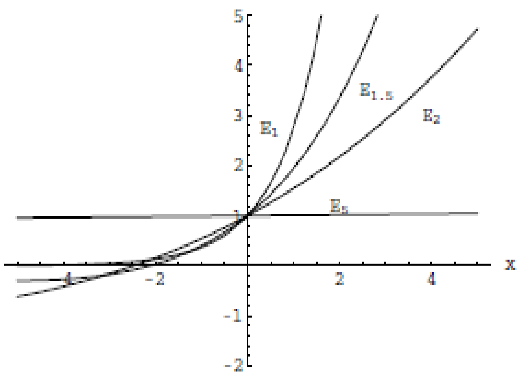

Fig. 2 Graph of ML function

For example:

$$E_{1,1}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(k+1)}=E_{1}(t)=e^{t}$$

$$E_{2,1}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(2k+1)}=E_{2}(t)=\cosh\left(\sqrt{t}\right)$$

Now, a few important functions of fractional calculus are defined here.

Defination 2. The fractional trigonometric functions are given by

$$\sin_{q,1}\left(\lambda t^{q}\right)=\frac{1}{2i}\left[E_{q,1}\left(i\lambda t^{q}\right)-E_{q,1}\left(-i\lambda t^{q}\right)\right] \tag{4}$$

and

$$\cos_{q,1}\left(\lambda t^{q}\right)=\frac{1}{2}\left[E_{q,1}\left(i\lambda t^{q}\right)+E_{q,1}\left(-i\lambda t^{q}\right)\right] \tag{5}$$

Figs. 3 and 4 are fractional trigonometric functions $\sin_{q,1}(t^{q})$ and $\cos_{q,1}(t^{q})$ when the real part of the complex roots are zero and imaginary part as one.

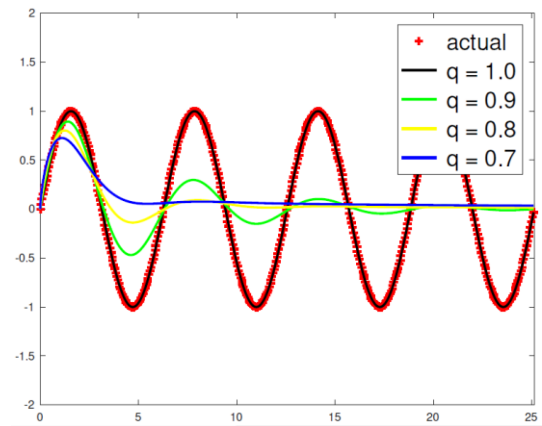

Fig. 3 Graph of $\sin_{q,1}(t^{q})$

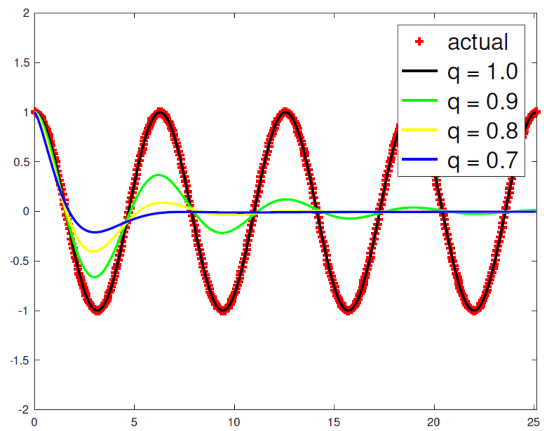

Fig. 4 Graph of $\cos_{q,1}(t^{q})$

Definition 3. The generalized fractional trigonometric functions are given by

$$G\sin_{q,1}\left((\lambda+i\mu)t^{q}\right)=\frac{1}{2i}\left[E_{q,1}\left((\lambda+i\mu)t^{q}\right)-E_{q,1}\left((\lambda-i\mu)t^{q}\right)\right] \quad (6)$$

and

$$G\cos_{q,1}\left((\lambda+i\mu)t^{q}\right)=\frac{1}{2}\left[E_{q,1}\left((\lambda+i\mu)t^{q}\right)+E_{q,1}\left((\lambda-i\mu)t^{q}\right)\right] \quad (7)$$

Remark: If q = 1 in the above definition, the functions $G\sin_{q,1}\left((\lambda+i\mu)t^{q}\right)$ and $G\sin_{q,q}\left((\lambda+i\mu)t^{q}\right)$ reduce to $e^{\lambda t}\sin\left(\mu t\right)$.

2.1. Caputo's Fractional Derivative

It is possible to identify certain flaws in the structure of the Riemann-Liouville fractional derivative by carefully defining it. Thus, by offering the following definition, renowned Italian scientist Caputo was able to significantly enhance the Riemann-Liouville definition. We observe that the ability to consider the problem's boundary and initial conditions in the formulation of the problems is arguably the greatest benefit of Caputo's definition over the Riemann-Liouville definition.

Definition 4. Assume $f\in C^{n}\left[a,b\right]$ and $n-1<\alpha<n$ the Caputo fractional derivative of α order is defined by the following expressions:

$${}_{a}D_{*x}^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{a}^{t}\left(t-s\right)^{n-\alpha-1}f^{(n)}\left(s\right)ds \quad (8)$$

Definition 5. The Caputo fractional derivative of order nq , for $n-1<\alpha<n$ is said to be sequential Caputo fractional derivative of order q , if the relation

$$\left({}^{c}D_{0+}^{nq}\right)u(t)={}^{c}D_{0+}^{q}\left({}^{c}D_{0+}^{(n-1)q}\right)u(t) \quad (9)$$

holds for $n=2,3,...,etc$. From now on we denote the sequential Caputo derivative of order q as $\left({}^{sc}D_{0+}^{nq}\right)u(t)$ where $n>2$ is an integer [16].

Note that the basis solution for $\left({}^{c}D_{0+}^{nq}\right)u(t)=0$ is

$$1,t,t^{2},...,t^{n-1}$$

whereas the basis solution of $\left({}^{sc}D_{0+}^{nq}\right)u(t)=0$; is given by

$$1,t^{q},t^{2q},...,t^{(n-1)q}$$

reasons for this difference is that the fractional derivative is not continuous, while the integer derivative is. This property makes the behavior and solution of differential equations with fractional derivatives more complicated than equations with integer derivatives. Non-sequentiality means that the order in which fractional derivatives are applied affects the final result, unlike integer derivatives. This difference is particularly important in applications where the order of derivatives is important, such as modeling systems with memory or history.

Sequential Caputo Fractional Differential in fractional calculus is described in terms of progressively applied Caputo fractional derivatives. The standard Caputo Fractional Differential, on the other hand, just requires one fractional derivative operation. Stated differently, Sequential Caputo involves the combination of multiple fractional Caputo derivatives [15]. In short,

$${}^{c}D_{0+}^{2q}\left(E_{q,1}(\lambda t^{q})\right)\neq\lambda^{2}E_{q,1}(\lambda t^{q}) \quad (10)$$

when ${}^{c}D_{0+}^{2q}\left(f\right)$ is not sequential. If ${}^{c}D_{0+}^{2q}\left(f\right)={}^{sc}D_{0+}^{2q}\left(f\right)$ is sequential, then we have

$${}^{c}D_{0+}^{2q}\left(E_{q,1}(\lambda t^{q})\right)=\lambda^{2}E_{q,1}(\lambda t^{q}) \quad (11)$$

Remark. If $n-1<Q=nq<n$ the sequential Caputo (left-sided) fractional derivative of Q order follow as:

$${}^{sc}D_{0^{+}}^{nq}f\left(t\right)=\frac{1}{\Gamma\left(n-nq\right)}\int_{0}^{t}\left(t-s\right)^{n-nq-1}f^{(n)}\left(s\right)ds \quad (12)$$

Applications with intricate time histories and dependencies benefit greatly from the usage of the sequential Caputo fractional differential. For instance, Sequential Caputo can be used to more correctly simulate reactions in the modeling of complicated dynamic systems, such engineering systems, whose behavior depends on several temporal elements.

3- Laplace transform Methodology

This section will outline a broad approach to using the Laplace transform method to solve fractional-order differential equations.

Definition 5. The Laplace transform $F(s)$ of a function $f(t)$ is

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{13}$$

defined for all s such that the integral converges. Since Laplace transform of Caputo's Fractional Derivative is defined as follow:

$$L\left[{}^{c}D_{0^+}^{q} f(t)\right] = s^q F(s) - s^{q-1} f(0) \tag{14}$$

Since, the Laplace transform of the Caputo fractional derivative of $f(t)$ of order nq for $n-1 < Q = nq < n$ order of the sequential Caputo fractional derivative of order q is

$$\begin{aligned} L\left[{}^{c}D_{0^+}^{nq} f(t)\right] &= L\left[{}^{c}D_{0^+}^{q}\left({}^{c}D_{0^+}^{(n-1)q}\right) f(t)\right] \\ &= s^{nq}F(s) - s^{nq-1}f(0) - s^{(n-1)q-1}\left({}^{c}D_{0^+}^{q} f(0)\right) \\ &\quad - s^{(n-2)q-1}\left({}^{c}D_{0^+}^{2q} f(0)\right) \cdots - s^{q-1}\left({}^{c}D_{0^+}^{(n-1)q} f(0)\right) \end{aligned} \tag{15}$$

For several fundamental functions that are relevant to our primary findings, we create a Laplace Transform Table 1 below formulas.

Table 1: Laplace transforms

$f(t)$	$L[f(t)] = F(s)$	conditions
$E_{q,1}(\pm\lambda t^q)$	$\frac{s^{q-1}}{s^q \mp \lambda}$	$s^q > \lambda, q > -1$
t^q	$\frac{\Gamma(q+1)}{s^{q+1}}$	$s > 0, q > -1$
$t^{q-1}E_{q,q}(\pm\lambda t^q)$	$\frac{1}{s^q \mp \lambda}$	$s^q > \lambda, q > -1$
$\frac{t^q}{q}E_{q,q}(\pm\lambda t^q)$	$\frac{s^{q-1}}{(s^q \mp \lambda)^2}$	$s^q > \lambda, q > -1$
$\sin_{q,1}(\lambda t^q)$	$\frac{\lambda s^{q-1}}{s^{2q} + \lambda^2}$	$s > 0$
$\cos_{q,1}(\lambda t^q)$	$\frac{s^{2q-1}}{s^{2q} + \lambda^2}$	$s > 0$
$G\sin_{q,1}((\lambda + i\mu)t^q)$	$\frac{\mu s^{q-1}}{(s^q - \lambda)^2 + \mu^2}$	-
$G\cos_{q,1}((\lambda + i\mu)t^q)$	$\frac{s^{q-1}(s^q - \lambda)}{(s^q - \lambda)^2 + \mu^2}$	-

4- Application

In this section, we present the use of the proposed techniques on the fractional-order differential equations.

4.1. Solution of a non-sequential Caputo linear fractional differential equations with initial conditions:

Problem let us assume fractional equations as

$$\begin{aligned} a_n\,{}^{c}D^{nq}u(t) + a_{n-1}\,{}^{c}D^{(n-1)q}u(t) + \cdots \\ + a_1\,{}^{c}D^{q}u(t) + a_0 u(t) = f(t) \end{aligned} \tag{16}$$

with the initial conditions

$$u^{(kq)}(0) = b_k \ , \quad k = 0,1,\ldots,n-1 \ , \ b_k \in \mathbb{R} \tag{17}$$

Initially, we recall the known method to solve the linear non-homogeneous Caputo fractional differential equations. For that purpose, consider the Caputo fractional differential equation of order $n-1<q<n \ (n\in\mathbb{N})$

$$ {}^{c}D_{0^+}^{q}u(t)=\lambda u(t)+f(t)\ ,\ t>0 \tag{18}$$

with the initial conditions

$$u^{(k)}(0)=b_k\ ,\quad k=0,1,...,n-1\ ,\ b_k\in\mathbb{R} \tag{19}$$

At first, by taking the Laplace transform of (13), we get

$$s^qU(s)-s^{n-1}u(0)-s^{n-2}\left({}^{c}D_{0^+}^{1}u(0)\right) -s^{n-3}\left({}^{c}D_{0^+}^{2}u(0)\right)\cdots-\left({}^{c}D_{0^+}^{(n-1)}u(0)\right) =\lambda U(s)+F(s) \tag{20}$$

Now solving for $U(s)$ from equation (20) and substituting the initial conditions from equation (14), we get

$$U(s)=\frac{s^{n-1}}{(s^q-\lambda)}b_0+\frac{s^{n-2}}{(s^q-\lambda)}b_1+\cdots+\frac{1}{(s^q-\lambda)}b_{n-1} +\frac{F(s)}{(s^q-\lambda)} \tag{21}$$

In order to obtain u(t), we have to take the inverse Laplace transform of (21) on both sides.

$$u(t)=b_0E_{nq,1}\left(\lambda t^{nq}\right)+b_1tE_{nq,2}\left(\lambda t^{nq}\right)+\cdots +b_{n-1}t^{n-1}E_{nq,n}\left(\lambda t^{nq}\right)+L^{-1}\left[\frac{F(s)}{(s^q-\lambda)}\right]$$

Now using the convolution, we get solution

$$u(t)=\sum_{j=0}^{n-1}b_jt^jE_{q,j+1}\left(\lambda t^q\right) +\int_0^t(t-s)^{q-1}E_{q,q}\left(\lambda(t-s)^q\right)f(s)ds \tag{22}$$

In particular, for $0<q<1$ or $n-1<nq<n$, with an initial condition $u(0)=b_0$, the solution is

$$u(t)=b_0E_{q,1}\left(\lambda t^q\right)+\int_0^t(t-s)^{q-1}E_{q,q}\left(\lambda(t-s)^q\right)f(s)ds$$

4.2. Solution of the Caputo sequential linear fractional differential equations with initial conditions:

In this section, we will provide a methodology to solve a Caputo sequential linear initial value problem with the initial conditions having fractional derivatives of lower order.

In [17], they claim that Laplace transform method encounters great difficulties in solving the equation

$${}^{c}D_{0^+}^{Q}u(t)+\lambda\,{}^{c}D_{0^+}^{q}u(t)=f(t) \tag{23}$$

when $Q-q$ is not an integer or half integer . But this claim is not complete, but it must be said, if we assume Q is an integer multiple of q and the Caputo fractional differential equation (23) is sequential, the Laplace transform method is very useful.
Consider the linear Caputo fractional sequential differential equation of order nq; which is sequential of order q; with initial condition of the form,

$$a_n\,{}^{sc}D_{0^+}^{nq}u(t)+a_{n-1}\,{}^{sc}D_{0^+}^{(n-1)q}u(t)+\cdots +a_1\,{}^{sc}D_{0^+}^{q}u(t)+a_0u(t)=f(t) \tag{24}$$

with the initial conditions

$${}^{sc}D_{0^+}^{kq}u(t)\Big|_{t=0}=b_k\ ,\quad k=0,1,...,n-1\ ,\ b_k\in\mathbb{R} \tag{25}$$

The above initial value problem can be solved using (15), just as in the integer case except that while taking the inverse Laplace transform, one needs to use the Laplace transform table developed for fractional differential equations.

Example 1. we consider the $Q=2q$ order linear non-homogeneous Caputo sequential fractional differential equation with constant coefficients of the form

$$ {}^{sc}D_{0^+}^{2q}u(t)+\lambda^2 u(t)=0\ ,\ t>0 \tag{26}$$

$1<2q<2$ or $0.5<q<1$ with initial conditions

$$u(0)=A,\quad {}^{sc}D_{0^+}^{q}u(t)\Big|_{t=0}=B \tag{27}$$

Now applying Laplace transform on (26) and (27), we get

$$s^{2q}U(s)-s^{2q-1}u(0)-s^{(q-1)}\ {}^{sc}D_{0^+}^{q}u(t)\Big|_{t=0}+\lambda^2U(s)=0 \tag{28}$$

After replacing the initial conditions from equation (27), and solving for U(s) from (28) equation, we obtain

$$U(s)=A\frac{s^{2q-1}}{s^{2q}+\lambda^2}+B\frac{s^{(q-1)}}{s^{2q}+\lambda^2} \tag{29}$$

Then the solution for Table.1 is given by the equation

$$U(s)=c_1A\cos_{q,1}\left(\lambda t^q\right)+c_2B\cos_{q,1}\left(\lambda t^q\right) \tag{30}$$

Example 2. we consider the $Q=2q$ order linear non-homogeneous Caputo sequential fractional differential equation with constant coefficients of the form

$$a\,{}^{sc}D_{0^+}^{2q}u(t)+b\,{}^{sc}D_{0^+}^{q}u(t)+cu(t)=f(t)\ ,\ t>0 \tag{31}$$

with initial conditions

$$u(0)=A,\quad {}^{sc}D_{0^+}^{q}u(t)\Big|_{t=0}=B \tag{32}$$

We assume that 0.5 < q < 1 and f(t) is a continuous function on [0, ∞), which is bounded by an exponential function. Now applying Laplace transform on (31) and (32), we get

$$\begin{aligned}&a\left(s^{2q}U(s)-s^{2q-1}u(0)-s^{(q-1)}\ {}^{sc}D_{0^+}^{q}u(t)\Big|_{t=0}\right)\\&+b\left(s^{q}U(s)-s^{(q-1)}u(0)\right)+cU(s)=F(s)\end{aligned} \tag{33}$$

After replacing the initial conditions from equation (29), and solving for U(s) from (30) equation, we obtain

$$U(s)=\frac{Aas^{2q-1}+(aB+bA)s^{(q-1)}}{a\,s^{2q}+bs^q+c}+\frac{F(s)}{a\,s^{2q}+bs^q+c} \tag{34}$$

For convenience, we can write

$$U(s)=\frac{s^{q-1}G(s)}{a\,s^{2q}+bs^q+c}+\frac{F(s)}{a\,s^{2q}+bs^q+c} \tag{35}$$

The solution depends on the roots of the quadratic equations $a\,s^{2q}+bs^q+c=0$ in terms of $t=s^q$. We then have the following four possibilities:

Case 1. If $\left(\Delta=b^2-4ac>0\right)$ the roots are real and distinct (say $r_1\neq r_2\in\mathbb{R}$), then using partial fraction method, we see that

$$u(t)=L^{-1}\left[\frac{c_1s^{q-1}}{\left(s^q-r_1\right)}+\frac{c_1s^{q-1}}{\left(s^q-r_2\right)}+\frac{c_3}{\left(s^q-r_1\right)}F(s)+\frac{c_4}{\left(s^q-r_2\right)}F(s)\right]$$

Now, we can write the solution of (28) as

$$\begin{aligned}u(t)&=c_1E_{q,1}\left(r_1t^q\right)+c_2E_{q,1}\left(r_2t^q\right)\\&+c_3\int_0^t(t-s)^{q-1}E_{q,q}\left(r_1(t-s)^q\right)f(s)ds\\&+c_4\int_0^t(t-s)^{q-1}E_{q,q}\left(r_2(t-s)^q\right)f(s)ds\end{aligned} \tag{36}$$

Case 2. If $(\Delta=0)$ the quadratic equation has real and coincident (say $r=r_1=r_2\in\mathbb{R}$), then

$$u(t)=L^{-1}\left[\frac{As^{q-1}}{\left(s^q-r\right)}+\frac{\left(r+aB+bA\right)s^{q-1}}{\left(s^q-r\right)^2}+\frac{1}{\left(s^q-r\right)^2}F(s)\right]$$

Now, we can write the solution of (28) as

$$\begin{aligned}u(t)&=AE_{q,1}\left(rt^q\right)+\frac{(r+aB+bA)}{q}E_{q,q}\left(rt^q\right)\\&+\int_0^t(t-s)^{2q-1}\sum_{k=0}^{\infty}\frac{(k+1)r^k(t-s)^{qk}}{\Gamma\left((k+2)q\right)}f(s)ds\end{aligned} \tag{37}$$

Case 3. If $(\Delta<0)$ the roots are complex (say $r=\alpha\mp i\beta$), then the solution involves the generalized fractional trigonometric functions, then

$$u(t)=L^{-1}\left[\frac{g_1 s^{q-1}\left(s^q-\alpha\right)}{\left(s^q-\alpha\right)^2+\beta^2}+\frac{\beta g_2}{\left(s^q-\alpha\right)^2+\beta^2}+\frac{1}{\beta}\frac{\beta}{\left(s^q-\alpha\right)^2+\beta^2}F(s)\right] \quad (38)$$

Now, we can write the solution of (28) as

$$u(t)=g_1 G\cos_{q,1}\left\{\left(\alpha+i\beta\right)t^q\right\}+g_2 G\sin_{q,1}\left\{\left(\alpha+i\beta\right)t^q\right\}+\frac{1}{\beta}\int_0^t (t-s)^{q-1} G\sin_{q,q}\left\{\left(\alpha+i\beta\right)\left(t-s\right)^q\right\} f(s)ds$$

In particular, If the roots are purely imaginary (say $r=\mp i\beta$), then the solution involves the fractional trigonometric functions $\sin_{q,1}\left(\beta t^q\right)$ and $\cos_{q,1}\left(\beta t^q\right)$ for the homogeneous part and $\sin_{q,q}\left(\beta t^q\right)$ in convolution with $f\left(t\right)$ for the non-homogeneous part. Then

$$u(t)=g_1 G\cos_{q,1}\left(\beta t^q\right)+g_2 G\sin_{q,1}\left(\beta t^q\right)+\frac{1}{\beta}\int_0^t (t-s)^{q-1}\sin_{q,q}\left\{\beta\left(t-s\right)^q\right\} f(s)ds \quad (39)$$

5- Conclusion

In conclusion, the Laplace transform method has proven to be an efficient technique for solving linear non-homogeneous Caputo fractional differential equation. Compared to traditional numerical methods, this approach offers a simpler and more direct approach, which is particularly useful for sequential and non- sequential equations. Furthermore, the Laplace transform method can be easily implemented using standard software packages, making it accessible to a wide range of researchers and practitioners.

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