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Fixed point theorems in generalized C^*-valued metric spaces

H. Massit[a], M. Rossafi[b,*]

[a] *Laboratory Analysis, Geometry and Applications, University of Ibn Tofail, Kenitra, Morocco.*

[b] *Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, Kenitra, Morocco.*

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Abstract. Based on the notion and properties of C^*-algebras, this paper aims to collect important results of fixed point theorems in generalized C^*-valued metric spaces. We also prove some new notions and establish an existence result for an integral equation in C^*-valued $b-$metric spaces. Moreover, we give some fixed point theorems in different types of spaces such as C^*-valued (extended $b-$metric, $b-$rectangular metric, extended hexagonal $b-$asymmetric, $S-$metric, $G-$metric and partial metric) spaces.

Keywords: Fixed point, C^*-valued metric space, C^*-valued $b-$metric space.

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1. Introduction and preliminaries

Fixed point theory is an important tool for solving existence of solutions of many nonlinear problems in various branches of science and has been studied in different spaces. Ma et al. [13] have introduced the notion of C^*-algebra-valued metric spaces by giving the definition of C^*-algebra-valued contractive mapping analogous to Banach contraction. Many generalizations of the concept of metric spaces have been defined and some fixed point theorems have been proved in these spaces. In particular, as a generalization of metric spaces, C^*-algebra-valued metric spaces were introduced by Ma et al. [13]. They proved certain fixed point theorems, by giving the definition of C^*-algebra-valued contractive mapping analogous to Banach contraction principle. Many mathematicians also worked on this interesting space and proved various fixed point results on such

*Corresponding author.
E-mail address: massithafida@yahoo.fr (H. Massit); rossafimohamed@gmail.com, mohamed.rossafi1@uit.ac.ma (M. Rossafi).

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spaces, see [8, 10, 16] and references therein. Combining conditions used for definitions of C^*-algebra-valued metric and generalized metric spaces, Piri et al. [15] announced the notions of C^*-algebra-valued metric space and establish nice results of fixed point on such space.

Throughout this paper, we denote $\mathbb{A}$ by an unital C^*-algebra. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of x. Using positive element, we can define a partial ordering "$\preceq$" on $\mathbb{A}_h$ as follows:

$$x \preceq y \text{ if and only if } y - x \succeq \theta,$$

where θ means the zero element in $\mathbb{A}$. We denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ by $\mathbb{A}_+$ and $|x| = (x^*x)^{\frac{1}{2}}$. In 2015, Ma and Jiang [20] introduced a concept of C^*-algebra-valued $b-$metric spaces which generalize an ordinary C^*-algebra-valued space and give some fixed point theorems.

2. C^*-valued metric spaces

Definition 2.1 [13] Let X be a non-empty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:

1) $d(x, y) \succeq \theta$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;
2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

Definition 2.2 [13] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. A mapping $T : X \to X$ is said to be a C^*-valued contractive mapping on X if there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ such that $d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda$.

Theorem 2.3 [13] If $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space and T is a contractive mapping, there exists a unique fixed point in X.

Theorem 2.4 [6] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose the mapping $T : X \to X$ satisfies $d(Tx, Ty) \preceq A(d(Tx, y) + d(Ty, x))$ for all $x, y \in X$, where $A \in \mathbb{A}'_+$ and $\|A\| < \dfrac{1}{2}$. Then there exists a unique fixed point in X.

Definition 2.5 Let $F : \mathbb{A}_+ \to \mathbb{A}_+$ be a function satisfying

(i) F is continuous and nondecreasing;
(ii) $F(t) = \theta$ if and only if $t = \theta$.

A mapping $T : X \to X$ is said to be a (ϕ, F)-C^*-valued contraction of type (I) if there exists $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ an $*-$homomorphism such that

$$d(Tx, Ty) \succeq \theta \Rightarrow F(d(Tx, Ty)) + \phi(d(x, y)) \preceq F(d(x, y)) \tag{1}$$

for all $x, y \in X$.

Theorem 2.6 Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \to X$ be a (ϕ, F)-contraction mapping of type (I). Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$, $\{T^n x_0\}_{n \in \mathbb{N}}$ is convergent to x^*.

Proof. First, let us observe that T has at most one fixed point. Indeed, if x_1^*, $x_2^* \in X$, then $Tx_1^* = x_1^* \neq x_2^* = Tx_2^*$ and $\phi(d(x,y)) \preceq F(d(x_1^*, x_2^*) - F(d(Tx_1^*, Tx_2^*)) = \theta$, which is a contradiction. In order to show that it has a fixed point, let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ by $x_{n+1} = Tx_n$ for $n = 0, 1, ...$ Denote $d_n = d(x_{n+1}, x_n)$ for $n = 0, 1, ...$ If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and the proof is finished. Suppose now that $x_{n+1} \neq x_n$ for every $n \in X$. Then $d_n \succ \theta$ for all $n \in \mathbb{N}$ and using (1), $F(d_n) \preceq F(d_{n-1}) - \phi(d_{n-1}) \prec F(d_{n-1})$ for every $n \in \mathbb{N}$. Hence, F is nondecreasing and the sequence (d_n) is monotonically decreasing in $\mathbb{A}_+$. So there exists $\theta \preceq t \in \mathbb{A}_+$ such that

$$d(x_n, x_{n+1}) \to t \ \text{ as } n \to \infty. \tag{2}$$

Letting $n \to \infty$, we obtain $F(t) \preceq F(t) - \phi(t)$ and $\phi(t) \preceq \theta \Rightarrow t = \theta$. Then $\lim_{n\to\infty} d_n = \theta$. Now, we shall show that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$. Assume that $\{x_n\}$ is not a Cauchy sequence in $(X, \mathbb{A}, d)$. Then there exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k > k$ such that $\|d(x_{m_k}, x_{n_k})\| \geqslant \varepsilon$. Now, corresponding to m_k, we can choose n_k such that it is the smallest integer with $n_k > m_k$ and satisfying above inequality. Hence, $\|d(x_{m_k}, x_{n_{k-1}})\| < \varepsilon$. So we have $\varepsilon \leqslant \|d(x_{m_k}, x_{n_k})\| \leqslant \|d(x_{m_k}, x_{n_{k-1}})\| + \|d(x_{n_{k-1}}, x_{n_k})\| \leqslant \varepsilon + \|d(x_{n_{k-1}}, x_{n_k})\|$. Using (2), we have $\varepsilon \leqslant \lim_{k\to\infty} \|d(x_{m_k}, x_{n_k})\| < \varepsilon + \theta$ implying

$$\lim_{k\to\infty} \|d(x_{m_k}, x_{n_k})\| = \varepsilon. \tag{3}$$

Again,

$$\begin{aligned} \|d(x_{n_k}, x_{m_k})\| &\leqslant \|d(x_{n_k}, x_{n_{k-1}})\| + \|d(x_{n_{k-1}}, x_{m_k})\| \\ &\leqslant \|d(x_{n_k}, x_{m_{k-1}})\| + \|d(x_{n_{k-1}}, x_{m_{k-1}})\| + \|d(x_{m_{k-1}}, x_{m_k})\|. \end{aligned} \tag{4}$$

Also,

$$\begin{aligned} \|d(x_{n_{k-1}}, x_{m_{k-1}})\| &\leqslant \|d(x_{n_{k-1}}, x_{n_k})\| + \|d(x_{n_k}, x_{m_{k-1}})\| \\ &\leqslant \|d(x_{n_{k-1}}, x_{n_k})\| + \|d(x_{n_k}, x_{m_k})\| + \|d(x_{m_k}, x_{m_{k-1}})\|. \end{aligned} \tag{5}$$

Letting $k \to \infty$ in (4) and (5) and using (3), we have $\lim_{k\to\infty} \|d(x_{n_{k-1}}, x_{m_{k-1}})\| = \varepsilon$. Since $d(x_{n_{k-1}}, x_{m_{k-1}})$, $d(x_{n_k}, x_{m_k}) \in \mathbb{A}_+$ and $\lim_{k\to\infty} \|d(x_{n_{k-1}}, x_{m_{k-1}})\| = \lim_{k\to\infty} \|d(x_{n_k}, x_{m_k})\| = \varepsilon$, there exists $s \in \mathbb{A}_+$ with $\|s\| = \varepsilon$ such that

$$\lim_{k\to\infty} \|d(x_{n_{k-1}}, x_{m_{k-1}})\| = \lim_{k\to\infty} \|d(x_{n_k}, x_{m_k})\| = \|s\|, \tag{6}$$

which implies that $F(s) = \lim_{k\to\infty} F(d(x_{n_k}, x_{m_k})) \preceq \lim_{k\to\infty} F(d(x_{n_{k-1}}, x_{m_{k-1}}))$. Therefore, $F(s) + \phi(s) \preceq F(s)$. Hence, $\phi(s) = \theta$ and $s = \theta$, which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$. Hence, there exist $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = \theta$. Now, we shall show that z is fixed point of T. Using (6), we get

$$d(Tx_{n-1}, Tz) \succeq \theta \Rightarrow F(d(x_n, Tz)) + \phi(d(x_{n-1}, z)) \preceq F(d(x_{n-1}, z))$$

Letting $n \to \infty$ and using the concept of continuity of the function of T, we have $d(z, Tz) = \theta$ and so $Tz = z$. ■

Definition 2.7 A mapping $T : X \to X$ is said to be a (ϕ, F) C^*-valued contraction of type (II) if there exists an $*-$homomorphism $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ satisfying

(a) $\phi(a) \prec a$ for $a \in \mathbb{A}_+$;
(b) Either $\phi(a) \preceq d(x, y)$ or $d(x, y) \preceq \phi(a)$, where $a \in \mathbb{A}_+$ and $x, y \in X$;
(c) $F(a) \prec \phi(a)$ such that

$$d(Tx, Ty) \succeq \theta \Rightarrow F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(M(x, y)),$$

where $M(x, y) = a_1 d(x, y) + a_2[d(Tx, y) + d(Ty, x)] + a_3[d(Tx, x) + d(Ty, y)]$ with $a_1, a_2, a_3 \geqslant 0$ and $a_1 + 2a_2 + 2a_3 \leqslant 1$.

Theorem 2.8 Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \to X$ be a (ϕ, F) C^*-valued contraction of type (II). Then T has a fixed point.

Proof. Let $x_0 \in X$ and define $x_1 = Tx_0$, $x_2 = Tx_1$, ... , $x_n = Tx_{n-1}$. We have

$$\begin{aligned} F(d(x_{n+2}, x_{n+1})) &= F(d(Tx_{n+1}, Tx_n)) \\ &\preceq F(M(x_{n+1}, x_n)) + \phi(d(x_{n+1}, x_n)) \\ &= F(a_1 d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] \\ &\quad + a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]) - \phi(d(x_{n+1}, x_n)). \end{aligned}$$

Then

$$\begin{aligned} F(d(x_{n+2}, x_{n+1})) \preceq F(&a_1 d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] \\ &+ a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]). \end{aligned}$$

Using the strongly monotone property of F, we have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) \preceq\ &a_1 d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] \\ &+ a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]; \end{aligned}$$

that is,

$$(1 - a_2 - a_3) d(Tx_{n+1}, Tx_n) \preceq (a_1 + a_2 + a_3) d(x_{n+1}, x_n).$$

So $d(x_{n+2}, x_{n+1}) \preceq \dfrac{a_1 + a_2 + a_3}{1 - a_2 - a_3} d(x_{n+1}, x_n)$, which implies $d(x_{n+2}, x_{n+1}) \preceq d(x_{n+1}, x_n)$. Since $\dfrac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1$, $\{d(x_{n+1}, x_n)\}$ is monotone decreasing sequence. There exists $u \in \mathbb{A}_+$ such that $d(x_{n+1}, x_n) \to u$ as $n \to \infty$. Taking $n \to \infty$ in

$$\begin{aligned} F(d(x_{n+2}, x_{n+1})) \preceq F(&a_1 d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] \\ &+ a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]), \end{aligned}$$

and using the continuities of F and ϕ, we have

$$F(u) \preceq F((a_1 + 2a_2 + 2a_3)u) - \phi(u) \Rightarrow F(u) \preceq F(u) - \phi(u) \Rightarrow \phi(u) \preceq \theta \Rightarrow u = \theta.$$

Hence,

$$d(x_{n+1}, x_n) \to \theta \text{ as } n \to \infty. \tag{7}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence then there exists $c \in \mathbb{A}$ such that $F(c) \preceq d(x_n, x_m)$ for all $n, m \in \mathbb{N}$ with $n > m \geqslant n_0$, $n_0 \in \mathbb{N}$. Thus, there exist sequences $\{m_k\}$ and $\{n_k\}$ in $\mathbb{N}$ so that for all positive integers k, $n_k > m_k > k$ and $d(x_{n_{(k)}}, x_{m_{(k)}}) \succeq \phi(c)$ and $d(x_{n_{(k)-1}}, x_{m_{(k)}}) \preceq \phi(c)$. Then

$$\phi(c) \preceq d(x_{n_{(k)}}, x_{m_{(k)}}) \preceq d(x_{n_{(k)}}, x_{n_{(k)-1}}) + d(x_{n_{(k)-1}}, x_{m_{(k)}});$$

that is, $\phi(c) \preceq d(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)}}, x_{n_{(k)-1}}) + \phi(c)]$. Letting $k \to \infty$, we have

$$\lim_{k\to\infty} d(x_{n_{(k)}}, x_{m_{(k)}}) = \phi(c) \tag{8}$$

Again $d(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)}}, x_{n_{(k)+1}}) + d(x_{n_{(k)+1}}, x_{m_{(k)+1}}) + d(x_{m_{(k)+1}}, x_{m_{(k)}})]$ and $d(x_{n_{(k)+1}}, x_{m_{(k)+1}}) \preceq [d(x_{n_{(k)+1}}, x_{n_{(k)}}) + d(x_{n_{(k)}}, x_{m_{(k)}}) + d(x_{m_{(k)}}, x_{m_{(k)+1}})]$. Letting $k \to \infty$ in previous inequalities, we have

$$\lim_{k\to\infty} d(x_{n_{(k)+1}}, x_{m_{(k)+1}}) = \phi(c) \tag{9}$$

Again $d(x_{n_{(k)}}, x_{m_{(k)+1}}) \preceq [d(x_{n_{(k)}}, x_{m_{(k)}}) + d(x_{m_{(k)}}, x_{m_{(k)+1}})]$ and

d($x_{n_{(k)+1}}, x_{m_{(k)}}$) $\preceq [d(x_{n_{(k)+1}}, x_{n_{(k)}}) + d(x_{n_{(k)}}, x_{m_{(k)+1}}) + d(x_{m_{(k)+1}}, x_{m_{(k)}})]$. Further, $d(x_{n_{(k)+1}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)+1}}, x_{n_{(k)}}) + d(x_{n_{(k)}}, x_{m_{(k)}})]$ and $d(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)}}, x_{n_{(k)+1}}) + d(x_{n_{(k)+1}}, x_{m_{(k)}})]$. Letting $k \to \infty$ in above four inequalities, we have

$$\lim_{k\to\infty} d(x_{n_{(k)}}, x_{m_{(k)+1}}) = \phi(c) \text{ and } \lim_{k\to\infty} d(x_{n_{(k)+1}}, x_{m_{(k)}}) = \phi(c) \tag{10}$$

Using (7), (8), (9) and (10), we have

$$\begin{aligned}\lim_{k\to\infty} M(x_{n_{(k)}}, x_{m_{(k)}}) &= \lim_{k\to\infty} a_1 d(x_{n_{(k)}}, x_{m_{(k)}}) + a_2[d(x_{n_{(k)}}, x_{m_{(k)}}) + d(x_{m_{(k)}}, x_{m_{(k)+1}})] \\ &\quad + a_3[d(x_{n_{(k)}}, x_{m_{(k)+1}}) + d(x_{m_{(k)}}, x_{n_{(k)+1}})] \\ &= (a_1 + 2a_2)\phi(c) \end{aligned} \tag{11}$$

Clearly, $x_{m_k} \preceq x_{n_k}$. Putting $x = x_{n_{(k)}}$ and $y = x_{m_{(k)}}$, we have

$$F(d(x_{n_{(k)+1}}, x_{m_{(k)+1}}) = F(d(Tx_{n_{(k)}}, Tx_{m_{(k)}}) \preceq F(M(x_{n_{(k)}}, x_{m_{(k)}})) - \phi(x_{n_{(k)}}, x_{m_{(k)}}).$$

Letting $k \to \infty$ in the above inequality and using (7), (8) and (11), and the continuity of F and ϕ, we have by virtue of a property of ϕ that

$$\begin{aligned}&F(\phi(c)) \preceq F((a_1 + 2a_2)\phi(c)) - \phi(\phi(c)) \\ &\quad \Rightarrow F(\phi(c)) \preceq F(\phi(c)) - \phi(\phi(c)) \Rightarrow \phi(\phi(c)) \preceq \theta \Rightarrow \phi(c) = \theta,\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is continuous and $Tx_n \to Tz$ as $n \to \infty$, $\lim_{n\to\infty} x_{n+1} = Tz$, that is $z = Tz$. Hence z is a fixed point of T. ■

Corollary 2.9 Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \to X$ be a (ϕ, F) C^*-valued contraction of type (II). If $a_1 = a_2 = 0$ and $a_3 = \frac{1}{2}$, a mapping T is said to be $(\phi, F)-$Kannan-type C^*-valued contraction. Then T has a fixed point.

Corollary 2.10 Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space and $T : X \to X$ be a (ϕ, F) C^*-valued contraction of type (II). If $a_1 = a_3 = \frac{1}{2}$ and $a_2 = 0$, a mapping T is said to be $(\phi, F)-$Reiche-type C^*-valued contraction. Then T has a fixed point.

2.1 *Application to Fredholm's integral equations*

Here, we apply Theorem 2.6 to prove the existence and uniqueness of the solution to Fredholm's integral equations $x(t) = \lambda \int_E K(t, s, x(s))ds$.

Theorem 2.11 Let $E = [0, 1]$ be a measurable Lebesgue set of finite measure and $X = L^\infty(E)$. Consider the Hilbert space $L^2(E)$. We denote $B(L^2(E))$ the set of all linear operators bounded in $L^2(E)$, it is a unitary C^*-algebra with the usual norm of operators. For $x, y \in X$, we define the metric $d : X \times X \to B(L^2(E))^+$ by $d(x, y) = \pi_{(x-y)^2}$, where $\pi_h(f) = hf$ for $f \in L^2(E)$. It is clear that $(X, B(L^2(E)), d)$ is a complete C^*-valued metric space. Suppose that there exists $1 > \lambda > 0$ and $K : E \times E \times \mathbb{R} \to \mathbb{R}$ is a continuous application. For all $x, y \in X$ and $t, s \in E$, consider Fredholm integral equation:

$$x(t) = \int_E K(t, s, x(s))ds, \tag{12}$$

and suppose the kernel function K satisfies $|K(t, s, x(s)) - K(t, s, y(s))| \leqslant \lambda e^{-\frac{1}{|x(t)-y(t)|}} |x(t) - y(t)|$. Then, (12) admits a unique solution.

Proof. Let $T : X \to X$ be defined by $T(x)(t) = \int_E K(t, s, x(s))ds$ for all $x \in X$. Suppose that $x, y \in X$ and $t, s \in E$. Then

$$\begin{aligned}
\|d(Tx, Ty)\| = \|\pi_{(Tx-Ty)^2}\| &= \sup_{\|g\|_2=1} \langle \pi_{(Tx-Ty)^2} g, g\rangle \ \ \forall g \in L^2(E) \\
&= \sup_{\|g\|_2=1} \int_E (Tx - Ty)^2(t) g(t)\overline{g(t)} dt \\
&= \sup_{\|g\|_2=1} \int_E |\int_E K(t, s, x(s)) - K(t, s, y(s))ds|^2 g(t)\overline{g(t)} dt \\
&= \sup_{\|g\|_2=1} \int_E |\int_E K(t, s, x(s)) - K(t, s, y(s))ds|^2 |g(t)|^2 dt \\
&\leqslant \sup_{\|g\|_2=1} \int_E \lambda^2 (\int_E e^{-\frac{1}{|x(s)-y(s)|}} |x(s) - y(s)|ds)^2 |g(t)|^2 dt \\
&\leqslant \lambda^2 \sup_{\|g\|_2=1} \int_E [\int_E e^{-\frac{1}{|x(s)-y(s)|}} ds]^2 |g(t)|^2 \|(x - y)^2\|_\infty dt \\
&\leqslant \lambda^2 |\int_E e^{-\frac{1}{|x(s)-y(s)|}} ds|^2 . \sup_{\|g\|_2=1} \int_E |g(t)|^2 dt . \|(x - y)^2\|_\infty .
\end{aligned}$$

Note that the set $\{s \in E / x(s) = y(s)\}$ is negligible. In addition, we have

$$|x(s) - y(s)| \leqslant \sup_{s \in E} |x(s) - y(s)| = \|x - y\|_\infty \Leftrightarrow \frac{1}{|x(s) - y(s)|} \geqslant \frac{1}{\|x - y\|_\infty}$$
$$\Leftrightarrow \frac{-1}{|x(s) - y(s)|} \leqslant \frac{-1}{\|x - y\|_\infty} \Leftrightarrow \int_E e^{\frac{-1}{|x(s)-y(s)|}} ds \leqslant \int_E e^{\frac{-1}{\|x(s)-y(s)\|_\infty}} ds.$$

and as $\ln \int_E f \leqslant \int_E \ln(f)$ where $(E = [0, 1])$,

$$\begin{aligned}\tilde{F}(\|d(Tx, Ty)\|) &\leqslant 2\ln(\lambda) + 2\ln(\|d(x, y)\|) + 2\int_E \frac{-1}{\|x - y\|_\infty} ds \\ &\leqslant 2(\ln(\lambda) + \ln(\|d(x, y)\|) - \frac{m(E)}{\|d(x, y)\|}) \\ &\leqslant 2[\tilde{F}(\|d(x, y)\|) - m(E)\tilde{\phi}(\|d(x, y)\|))].\end{aligned}$$

If we take $F(d(x, y)) = \tilde{F}(\|d(x, y)\|)I$ and $\phi(d(x, y)) = \tilde{\phi}(\|d(x, y)\|)I$, then

$$F((d(Tx, Ty)) + \phi(d(x, y)) \leqslant F\left(d(x, y)\right).$$

Then T satisfies the condition (1), and the equation (12) has a unique solution. ■

3. C^*-valued $b-$metric spaces

In this section, we introduce the notion of C^*-valued $b-$metric space.

Definition 3.1 [3] Let X be a non-empty set and $\mathbb{A}$ be a unital C^*-algebra. Let $b \in \mathbb{A}$ such that $\|b\| \geqslant 1$. A C^*-valued $b-$metric on X is a mapping $T : X \times X \to \mathbb{A}$ satisfying the following conditions:

(1) $d(x, y) \succeq \theta$, $\forall x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \preceq b[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

$(X, \mathbb{A}, d)$ is called a C^*-valued $b-$metric space with the coefficient b.

Definition 3.2 [3] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space. A mapping $T : X \to X$ is said to be a contraction if there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ such that $d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda$ for all $x, y \in X$.

Theorem 3.3 [3] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued $b-$metric space. Let $T : X \to X$ be a contraction with the contraction constant $\lambda \in \mathbb{A}$ such that $\|b\|\|\lambda\|^2 < 1$. Then T has a unique fixed point in X.

Lemma 3.4 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space with $b \succeq I$. Suppose that $\{x_n\}$ is a sequence in X such that $d(x_{n+1}, x_n) \preceq \delta d(x_n, x_{n-1})$ for all $n \in \mathbb{N}$ and $\delta \in [0, 1)$ with $\|b\| < \dfrac{1}{\delta}$. Then $\{x_n\}$ is a Cauchy sequence.

Definition 3.5 [7] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space and $\{x_n\}$ a sequence in X.

(1) $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to \theta$ as $n \to \infty$.

(2) $\{x_n\}$ is a Cauchy sequence if $d(x_m, x_n) \to \theta$ as $m, n \to \infty$
(3) $(X, \mathbb{A}, d)$ is complete if very Cauchy sequence in X is convergent.

Definition 3.6 [7] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{A}'_+$ be two mappings. T is said to be $\alpha-$admissible if $\alpha(x, y) \succeq I$ implies $\alpha(Tx, Ty) \succeq I$.

Definition 3.7 [7] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{A}'_+$ be two mappings such that T is $\alpha-$admissible. T is said to be triangular $\alpha-$admissible if $\alpha(x, y) \succeq I$ and $\alpha(y, z) \succeq I$ imply $\alpha(x, z) \succeq I$.

Definition 3.8 [7] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{A}'_+$ be two mappings. T is said to be $\alpha-$orbital admissible if $\alpha(x, Tx) \succeq I$ implies $\alpha(Tx, T^2x) \succeq I$.

Definition 3.9 [7] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{A}'_+$ be two mappings such that T is $\alpha-$orbital admissible. T is said to be triangular $\alpha-$orbital admissible if $\alpha(x, y) \succeq I$ and $\alpha(y, Ty) \succeq I$ imply $\alpha(x, Ty) \succeq I$.

4. Fixed point theorems for C^*-multivalued contractions in $b-$metric space

The concept of multivalued contraction mappings was introduced by Nadler [14]. He established that a multivalued contraction mapping has a fixed point in a complete metric space. In 2017, Amer [1] introduced a new concept known as generalized $\alpha_* - \psi$-Geraghty contraction type for multivalued mappings. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space. We will denote by $\mathcal{CB}(X)$ the set of non-empty bounded closed subsets of X. For $M, N \in \mathcal{CB}(X)$ and $x \in X$, we define $d(x, M) = \inf_{a\in M} d(x, a)$ and $d(M, N) = \sup_{a\in M} d(a, N)$. The mapping $h : \mathcal{CB}(X)\times\mathcal{CB}(X) \to \mathbb{A}_+$ given by $h(M, N) = \max\{\sup_{a\in M} d(a, N), \sup_{b\in N} d(b, M)\}$ is the Hausdorff distance between M and N in $\mathcal{CB}(X)$. A point x is said to be a fixed point of multivalued mapping $T : X \to \mathcal{CB}(X)$ provided $x \in T(x)$.

Definition 4.1 [1] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space and $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping. The space X is said to be $\alpha-$complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \succeq I$ for all $n \in \mathbb{N}$ converges in X.

Definition 4.2 [1] Let $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping satisfying the property that if $\alpha(x, y) \succeq I$ implies $\alpha_*(Tx, Ty) \succeq I$, where $\alpha_*(M, N) = \inf\{\alpha(x, y) : x \in M, y \in N\}$, then T is said to be α_*-admissible.

Definition 4.3 [1] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space and $\alpha, \eta : X \times X \to \mathbb{A}_+$ be two mappings. T is said to be $\alpha-\eta-$continuous on $(X, \mathbb{A}, d)$ if for given $x \in X$ and a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \succeq I$ for all $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$ imply that $Tx_n \to Tx$ as $n \to \infty$. If $\eta(x_n, x_{n+1}) = I$, then T is an $\alpha-$continuous mapping.

Definition 4.4 [1] Let $T, S : X \to \mathcal{CB}(X)$ be two multivalued mappings and $\alpha : X \times X \to \mathbb{A}'_+$ be a function. Then the pair (T, S) is said to be triangular α_*-admissible if the following conditions hold:

(i) $\alpha(x, y) \succeq I \Rightarrow \alpha_*(Tx, Sy) \succeq I$ and $\alpha_*(Sx, Ty) \succeq I$;
(ii) $\alpha(x, y) \succeq I$ and $\alpha(y, z) \succeq I \Rightarrow \alpha(x, z) \succeq I$.

Definition 4.5 [1] Let $T, S : X \to \mathcal{CB}(X)$ be two multivalued mappings and $\alpha : X \times X \to \mathbb{A}'_+$ be a function. Then the pair (T, S) is said to be triangular α_*-orbital

admissible if the following condition holds:
$\alpha(x,Tx) \succeq I$ and $\alpha_*(x,Sx) \succeq I \Rightarrow \alpha_*(Tx,S^2x) \succeq I$ and $\alpha_*(Sx,T^2x) \succeq I$.

Definition 4.6 [1] Let $T,S : X \to \mathcal{CB}(X)$ be two multivalued mappings and $\alpha : X \times X \to \mathbb{A}'_+$ be a function. Then the pair (T,S) is said to be triangular α_*-orbital admissible if the following conditions hold:

- (i) (T,S) is α_*-orbital admissible.
- (ii) $\alpha(x,y) \succeq I$, $\alpha(y,Ty) \succeq I$ and $\alpha_*(y,Sy) \succeq I$ imply $\alpha_*(x,Ty) \succeq I$ and $\alpha_*(x,Sy) \succeq I$.

Lemma 4.7 [1] Let $T,S : X \to \mathcal{B}(X)$ be two multivalued mappings such that the pair (T,S) is triangular α_*-orbital admissible. Assume that there exists $x_0 \in X$ such that $\alpha_*(x_0,Tx_0) \succeq I$. Define a sequence $\{x_n\} \in X$ by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in S(x_{2n+1})$, where $n = 0,1,....$ Then, for all $n,m \in \mathbb{N}$ with $m > n$, we have $\alpha(x_n,x_m) \succeq I$.

Using C^*-Hausdorff metric on $\mathcal{CB}(X)$, we give a generalization of some common fixed point results for rational contraction of multivalued mappings defined on a C^*-algebra-valued $b-$metric space.

Lemma 4.8 Let $M,N \in \mathcal{CB}(X)$ such that $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space. Suppose that the range of the metric d is a totally ordered subset of $\mathbb{A}^+$. For all $a \in M$, we have $d(a,N) \preceq h(M,N)$.

Lemma 4.9 Let $M,N \in \mathcal{CB}(X)$ such that $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space. Suppose that the range of the metric d is a totally ordered subset of $\mathbb{A}^+$. For all $a \in M$, if $r \succeq \theta$, there exists $u \in N$ such that $d(a,u) \preceq h(M,N) + r$.

Lemma 4.10 Let $M,N \in \mathcal{CB}(X)$ such that $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space. Then for all $a \in M$ and $q < 1$, there exists $u \in N$ such that $qd(a,u) \preceq h(M,N)$.

Definition 4.11 Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space. Let $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping. Then T is said an $\alpha-$continuous multivalued mapping on $(\mathcal{CB}(X),h)$.

Definition 4.12 Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space with a coefficient $b \succeq I$. A mapping $T : X \to \mathcal{CB}(X)$ is called a C^*-multivalued contraction if there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ and $\|b\|\|\lambda\|^2 < 1$ such that $h(Tx,Ty) \preceq \lambda^* d(x,y)\lambda$ for all $x,y \in X$.

Theorem 4.13 Let $(X,\mathbb{A},d)$ be a complete C^*-algebra-valued $b-$metric space with a coefficient $b \succeq I$ and $T : X \to \mathcal{CB}(X)$ be a C^*-multivalued contraction. That is, there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ and $\|b\|\|\lambda\|^2 < 1$ such that $h(Tx,Ty) \preceq \lambda^* d(x,y)\lambda$ for all $x,y \in X$. Then T has a fixed point.

Proof. Let $x_0 \in X$. Consider $x_1 \in Tx_0$ and $x_2 \in Tx_1$ such that $d(x_1,x_2) \preceq h(Tx_0,Tx_1) + \lambda^*\lambda$. Again, since Tx_1 and Tx_2 are closed and bounded subsets of X and x_2 lies in Tx_1, there will be a point $x_3 \in Tx_2$, which satisfies $d(x_2,x_3) \preceq h(Tx_1,Tx_2) + (\lambda^*\lambda)^2$. Proceeding in this way, we obtain a sequence $\{x_n\}_{n\in\{1,2,..\}}$ of points of X such that $x_{n+1} \in Tx_n$ and $d(x_n,x_{n+1}) \preceq h(Tx_{n-1},Tx_n) + (\lambda^*\lambda)^n$ for all $n \geqslant 1$. We note that

$$\begin{aligned} d(x_n,x_{n+1}) &\preceq h(Tx_{n-1},Tx_n) + (\lambda^*\lambda)^n \preceq \lambda^* d(x_{n-1},x_n)\lambda) + (\lambda^*\lambda)^n \\ &\preceq \lambda^*[h(Tx_{n-2},Tx_{n-1}) + (\lambda^*\lambda)^{n-1}]\lambda + (\lambda^*\lambda)^n = \lambda^*[h(Tx_{n-2},Tx_{n-1})]\lambda + 2(\lambda^*\lambda)^n \\ &\preceq \lambda^{*n} d(x_0,x_1)\lambda^n + n(\lambda^*\lambda)^n. \end{aligned}$$

for all $n \geqslant 1$. Hence, for all $n, m \geqslant 1$,

$$\begin{aligned}
d(x_m, x_n) &\preceq b[d(x_m, x_{m+1}) + d(x_{m+1}, x_n)] \\
&\preceq bd(x_m, x_{m+1}) + b^2 d(x_{m+1}, x_{m+2}) + ... + b^{n-m} d(x_{n-1}, x_n) \\
&\preceq b[\lambda^{*m} d(x_0, x_1)\lambda^m + m(\lambda^*\lambda)^m] + b^2[\lambda^{*(m+1)} d(x_0, x_1)\lambda^{m+1} + (m+1)(\lambda^*\lambda)^{m+1}] \\
&\quad + ... + b^{n-m}[\lambda^{*(n-1)} d(x_0, x_1)\lambda^{n-1} + (n-1)(\lambda^*\lambda)^{n-1}] \\
&\preceq \|b\|^{-m+1}[\sum_{k=m}^{n-1} \|b^k\| \|(\lambda^*)^k\|^2 \|(d(x_0, x_1))^{\frac{1}{2}}\|^2 I_{\mathbb{A}} + \sum_{k=m}^{n-1} \|b^k\| k \|\lambda^k\|^2 I_{\mathbb{A}}] \\
&\preceq \|b\|^{-m+1} \|(d(x_0, x_1))^{\frac{1}{2}}\|^2 \sum_{k=m}^{n-1} \|b^k\| \|\lambda^k\|^2 I_{\mathbb{A}} + \|b\|^{-m+1} \sum_{k=m}^{n-1} k \|b^k\| \, \|\lambda^k\|^2 I_{\mathbb{A}} \\
&\preceq \|b\|^{-m+1} \|(d(x_0, x_1))^{\frac{1}{2}}\|^2 \sum_{k=m}^{n-1} (\|b\| \|\lambda^2\|)^k I_{\mathbb{A}} + \|b\|^{-m+1} \sum_{k=m}^{n-1} k(\|b\| \, \|\lambda^2\|)^k I_{\mathbb{A}} \to \theta,
\end{aligned}$$

as $m \to \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, the sequence $\{x_n\}$ will converge to some $x_0 \in X$. Also, $h(Tx_n, Tx_0) \preceq \lambda^* d(x_n, x_0)\lambda$. Therefore, the sequence $\{Tx_n\}$ converges to Tx_0. Also, $x_n \in Tx_{n-1}$ for all $n \in \{1, ...\}$ and $d(x_n, Tx_0) \to \theta$ as $n \to \infty$. We obtain that $x_0 \in Tx_0$. ■

If whenever $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \succeq I$ for all $n \in \mathbb{N}$ and $x \in X$ such that $\lim_{n\to+\infty} d(x_n, x) = \theta$, then $\lim_{n\to+\infty} h(Tx_n, Tx) = \theta$.

Definition 4.14 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space. Let $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping and $T, S : X \to \mathcal{CB}(X)$ two multivalued mappings said to be a pair of generalized rational α_*-contraction type for multivalued mappings if $\alpha(x, y) \succeq I$ and $h(Tx, Sy) \preceq \lambda^* M(x, y)\lambda$ for all $x, y \in X$, where $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ and $\|b\|\|\lambda\|^2 < 1$, and $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy)\}$.

Theorem 4.15 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space with $b \succeq I$ and $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping. Let $T, S : X \to \mathcal{CB}(X)$ be a pair of generalized rational α_*-contraction type for multivalued mappings and

(i) $(X, \mathbb{A}, d)$ is an $\alpha-$complete;
(ii) (T, S) is triangular α_*-orbital admissible;
(iii) $\alpha_*(x_0, Tx_0) \succeq I$ for $x_0 \in X$;
(iv) T and S are $\alpha-$continuous.

Then there exists a common fixed point of T and S in X.

Proof. Let $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \succeq I$. Let $x_1 \in Tx_0$ so that $\alpha(x_0, x_1) \succeq I$ and $x_1 \neq x_0$. We have $0 < d(x_1, Sx_1) \preceq h(Tx_0, Sx_1) \preceq \lambda^* M(x_0, x_1)\lambda$. There exists $x_2 \in Sx_1$ such that $d(x_1, x_2) \preceq h(Tx_0, Sx_1) \preceq \lambda^* M(x_0, x_1)\lambda$ with

$$\begin{aligned}
M(x_0, x_1) &= \max\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Sx_1)\} \\
&= \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, Sx_1)\} = \max\{d(x_0, x_1), d(x_1, Sx_1)\}.
\end{aligned}$$

If $\max\{d(x_0, x_1), d(x_1, Sx_1)\} = d(x_1, Sx_1)$, we get $d(x_1, Sx_1) \preceq \lambda^* d(x_1, Sx_1)\lambda$, implying that $\|d(x_1, Sx_1)\| \leqslant \|\lambda\|^2 \|d(x_1, Sx_1)\| < \|d(x_1, Sx_1)\|$, which is a contradiction. Hence,

$\max\{d(x_0, x_1), d(x_1, Sx_1)\} = d(x_0, x_1)$, by Lemma 4.10 with $q = 1/\|b\|$, there is $x_2 \in Sx_1$ so that $\|d(x_1, x_2)\| \leqslant \|b\| \, \|\lambda\|^2 \, \|d(x_0, x_1)\| < \|d(x_0, x_1)\|$. Similarly, for $x_3 \in Tx_2$, $\frac{1}{\|b\|} d(x_2, x_3) \preceq h(Sx_1, Tx_2) \preceq \lambda^* M(x_1, x_2)\lambda$, where

$$M(x_1, x_2) = \max\{d(x_1, x_2), d(x_1, Sx_1), d(x_2, Tx_2)\} = \max\{d(x_1, x_2), d(x_2, Tx_2)\}.$$

We obtain $d(x_2, x_3) \preceq h(Sx_1, Tx_2) \preceq \lambda^* M(x_1, x_2)\lambda$ where

$$M(x_1, x_2) = \max\{d(x_1, x_2), d(x_1, Sx_1), d(x_2, Tx_2),\} = \max\{d(x_1, x_2), d(x_2, Tx_2)\}.$$

If $M(x_1, x_2) = d(x_2, Tx_2)$, by $0 \prec d(x_2, Tx_2) \preceq h(Sx_1, Tx_2) \preceq \lambda^* d(x_2, Tx_2)\lambda$, we have

$$\frac{1}{\|b\|} d(x_{2n+1}, x_{2n+2}) \preceq h(Tx_{2n}, Sx_{2n+1}) \preceq \lambda^* M(x_{2n}, x_{2n+1})\lambda,$$

and $\|d(x_2, Tx_2)\| < \|\lambda\| \|d(x_2, Tx_2)\|$, a contradiction. So $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_1, x_2)$ and $\|d(x_2, x_3)\| \leqslant \|b\| \, \|\lambda\|^2 \, \|d(x_1, x_2)\|$. We define a sequence $\{x_n\}$ by $x_{2n+1} \in Tx_{2n}$ and $x_{2n} \in Sx_{2n+1}$ for $n = 0, 1, ...$ Thus, $\alpha(x_n, x_{n+1}) \succeq I$ for all $n \in \mathbb{N}$ and

$$0 \prec d(x_{2n+1}, Sx_{2n+1}) \preceq h(Tx_{2n}, Sx_{2n+1}) \preceq \lambda^* M(x_{2n}, x_{2n+1})\lambda, \tag{13}$$

and

$$\frac{1}{\|b\|} d(x_{2n+1}, x_{2n+2}) \preceq h(Tx_{2n}, Sx_{2n+1}) \preceq \lambda^* M(x_{2n}, x_{2n+1})\lambda,$$

we have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1})\} \\ &= \max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Sx_{2n+1})\}. \end{aligned}$$

If $\max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Sx_{2n+1})\} = d(x_{2n+1}, Sx_{2n+1})$, then we have by (13) that

$$d(x_{2n+1}, Sx_{2n+1}) \preceq \lambda^* d(x_{2n+1}, Sx_{2n+1})\lambda \Rightarrow \|d(x_{2n+1}, Sx_{2n+1})\| < \|\lambda\| \|d(x_{2n+1}, Sx_{2n+1})\|,$$

which is a contradiction. Hence, $\max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Sx_{2n+1})\} = d(x_{2n}, x_{2n+1})$ and $d(x_{2n+1}, Sx_{2n+1}) \preceq \lambda^* d(x_{2n}, x_{2n+1})\lambda$, which implies that $\|d(x_{2n+1}, x_{2n+2})\| \leqslant \|b\| \, \|\lambda\|^2 \, \|d(x_{2n+1}, x_{2n})\|$ and by Lemma 3.4, $\{x_n\}$ is a Cauchy sequence. By completeness of $(X, \mathbb{A}, d)$, there is $z \in X$ so that for all $n \in \mathbb{N} \cup \{0\}$, $\lim_{n\to+\infty} d(x_n, z) = \theta$ implying $\lim_{n\to+\infty} d(x_{2n+1}, z) = \lim_{n\to+\infty} d(x_{2n+2}, z) = \theta$. As S is $\alpha-$continous, $\lim_{n\to+\infty} h(Sx_{2n+2}, Sz) = \theta$. Therefore, $d(z, Sz) \preceq b[d(z, x_{2n+1}) + d(x_{2n+1}, Sz)] \to \theta$as $n \to \infty$ then, $z \in Sz$. Similarly, we obtain $z \in Tz$. Thus, z is a common fixed point of T and S. ■

In the following, the $\alpha-$continuity proprety is replaced by a new condition.

Theorem 4.16 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space with $b \succeq I$ and $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping and $T, S : X \to \mathcal{CB}(X)$ be a pair of generalized rational α_*-contraction type, where

(i) $(X, \mathbb{A}, d)$ is an $\alpha-$complete;

(ii) (T,S) is triangular α_*-orbital admissible;
(iii) $\alpha_*(x_0,Tx_0)\succeq I$ for $x_0\in X$;
(iv) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\succeq I$ for all $n\in\mathbb{N}\cup\{0\}$ and $\lim\limits_{n\to\infty} d(x_n,z)=\theta$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},z)\succeq I$ for all $k\in\mathbb{N}\cup\{0\}$.

Then, T and S have a common fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X such that $x_{2n+1}\in Tx_{2n}$ and $x_{2n+2}\in Sx_{2n+1}$ for $n=0,1,...$, with $\alpha(x_n,x_{n+1})\succeq I$ and $x_n\to z\in X$. By (iv), we have for all $k\in\mathbb{N}$ that

$$\begin{aligned} d(z,Tz) &\preceq b[d(z,x_{2n(k)+1})+d(x_{2n(k)+1},Tz)]\preceq bd(z,x_{2n(k)+1})+bh(Sx_{2n(k)},Tz)\\ &\preceq bd(z,x_{2n(k)+1})+b\lambda^*M(x_{2n(k)},z)\lambda, \end{aligned} \tag{14}$$

where $M(x_{2n(k)},z)=\max\{d(x_{2n(k)},z),d(x_{2n(k)},Sx_{2n(k)}),d(z,Tz)\}$. Letting $k\to\infty$, we get $M(x_{2n(k)},z)\to d(z,Tz)$, and by (14), we have

$$d(z,Tz)\preceq b\lambda^*d(z,Tz)\lambda\Rightarrow \|d(z,Tz)\|<\|b\|\|\lambda\|^2\|d(z,Tz)\|,$$

which is a contradiction. Then $z\in Tz$ i.e, z is a fixed point of T. Proceeding in this manner we prove that $z\in Sz$, i.e, z is the common fixed point of T and S. ■

We denote Φ the class of all functions $\phi:\mathbb{A}_+\to\mathbb{A}_+$ such that for any bounded sequence $\{t_n\}$ of positive real numbers, $\lim\limits_{n\to\infty}\phi(t_n)=I$ implies $\lim\limits_{n\to\infty}t_n=\theta$ and $\|\phi\|<1$, and Ψ the class of the functions $\psi:\mathbb{A}_+\to\mathbb{A}_+$ satisfying the conditions:

(i) ψ is nondecreasing and continuous;
(ii) $\psi(t)=\theta\Leftrightarrow t=\theta$.

Definition 4.17 Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space with $b\succ I$ and $\alpha:X\times X\to\mathbb{A}'_+$ be a mapping. Let $T,S:X\to\mathcal{CB}(X)$ be a pair of generalized rational $\alpha_*-\psi-$ Geraghty contraction type for multivalued mappings if there is $\phi\in\Phi$ and $\psi\in\Psi$ such that for $x,y\in X$ with $\alpha(x,y)\succeq I$, the pair (T,S) satisfies the following inequality:

$$\alpha(x,y)\psi(h(Tx,Sy))\preceq\phi(\psi(M(x,y)))\psi(M(x,y)), \tag{15}$$

where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Sy)\}$.

Theorem 4.18 Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space with $b\succeq I$, $\alpha:X\times X\to\mathbb{A}'_+$ be a mapping and $T,S:X\to\mathcal{CB}(X)$ be a pair of generalized rational $\alpha_*-\psi-$Geraghty contraction type, where

(i) $(X,\mathbb{A},d)$ is an $\alpha-$complete;
(ii) (T,S) is triangular α_*-orbital admissible;
(iii) $\alpha_*(x_0,Tx_0)\succeq 1$ for $x_0\in X$;
(iv) T and S are $\alpha-$continuous.

Then there exists a common fixed point of T and S in X.

Proof. Let $x_0\in X$. Construct the sequence $\{x_n\}$ such that $x_{2n+1}\in Tx_{2n}$ and $x_{2n+2}\in Sx_{2n+1}$ for $n=0,1,...$ with $\alpha(x_n,x_{n+1})\succeq I$. By (15), we have

$$\begin{aligned} 0<\psi(d(x_1,Sx_1))&\preceq\psi(h(Tx_0,Sx_1))\preceq\alpha(x_0,x_1)\psi(h(Tx_0,Sx_1))\\ &\preceq\phi(\psi(M(x_0,x_1)))\psi(M(x_0,x_1)). \end{aligned}$$

There exists $x_2 \in Sx_1$ such that

$$\psi(d(x_1,x_2)) \preceq \alpha(x_0,x_1)\psi(h(Tx_0,Sx_1)) \preceq \phi(\psi(M(x_0,x_1)))\psi(M(x_0,x_1))$$

with

$$\begin{aligned} M(x_0,x_1) &= \max\{d(x_0,x_1),d(x_0,Tx_0),d(x_1,Sx_1)\} = \max\{d(x_0,x_1),d(x_0,x_1),d(x_1,Sx_1)\} \\ &= \max\{d(x_0,x_1),d(x_1,Sx_1)\}. \end{aligned}$$

If $\max\{d(x_0,x_1),d(x_1,Sx_1)\} = d(x_1,Sx_1)$, we get

$$\begin{aligned} &\psi(d(x_1,Sx_1)) \preceq \phi(\psi(d(x_1,Sx_1)))\psi(d(x_1,Sx_1)) \\ &\qquad \Rightarrow \|\psi(d(x_1,Sx_1))\| \leqslant \|\phi(\psi(d(x_1,Sx_1)))\|\|\psi(d(x_1,Sx_1))\|, \end{aligned}$$

which is a contradiction. Hence, $\max\{d(x_0,x_1),d(x_1,Sx_1)\} = d(x_0,x_1)$. Then

$$\psi(d(x_1,x_2)) \preceq \phi(\psi(d(x_0,x_1)))\psi(d(x_0,x_1)).$$

In the same way, for $x_2 \in Sx_1$ and $x_3 \in Tx_2$, we obtain

$$\psi(d(x_2,x_3)) \preceq \alpha(x_1,x_2)\psi(h(Sx_1,Tx_2)) \preceq \phi(\psi(M(x_1,x_2)))\psi(M(x_1,x_2))$$

where $M(x_1,x_2) = \max\{d(x_1,x_2),d(x_1,Sx_1),d(x_2,Tx_2)\} = \max\{d(x_1,x_2),d(x_2,Tx_2)\}$. If $M(x_1,x_2) = d(x_2,Tx_2)$, we obtain

$$\begin{aligned} &\psi(d(x_2,x_3)) \preceq \phi(\psi(d(x_2,Tx_2)))\psi(d(x_2,Tx_2)) \\ &\Rightarrow \|\psi(d(x_2,Tx_2))\| \leqslant \|\phi(\psi(d(x_2,Tx_2)))\|\|\psi(d(x_2,Tx_2))\|, \end{aligned}$$

which is a contradiction. Hence, $\max\{d(x_1,x_2),d(x_2,Tx_2)\} = d(x_1,x_2)$ and we have $\psi(d(x_2,x_3)) \preceq \phi(\psi(d(x_1,x_2)))\psi(d(x_1,x_2))$. We define a sequence $\{x_n\}$ by $x_{2n+1} \in Tx_{2n}$ and $x_{2n} \in Sx_{2n+1}$ for $n = 0,1,....$ So $\alpha(x_n,x_{n+1}) \succeq I$ for all $n \in \mathbb{N}\cup\{0\}$ and

$$\psi(d(x_{2n+1},Sx_{2n+1})) \preceq \psi(h(Tx_{2n},Sx_{2n+1})) \preceq \phi(\psi(M(x_{2n},x_{2n+1})))\psi(M(x_{2n},x_{2n+1})),$$

and

$$\psi(d(x_{2n+1},x_{2n+2})) \preceq \psi(h(Tx_{2n},Sx_{2n+1})) \preceq \phi(\psi(M(x_{2n},x_{2n+1})))\psi(M(x_{2n},x_{2n+1})),$$

where

$$\begin{aligned} M(x_{2n},x_{2n+1}) &= \max\{d(x_{2n},x_{2n+1}),d(x_{2n},Tx_{2n}),d(x_{2n+1},Sx_{2n+1})\} \\ &= \max\{d(x_{2n+1},x_{2n}),d(x_{2n+1},Sx_{2n+1})\}. \end{aligned}$$

If $\max\{d(x_{2n+1},x_{2n}),d(x_{2n+1},Sx_{2n+1})\} = d(x_{2n+1},Sx_{2n+1})$, then

$$\begin{aligned} &\psi(d(x_{2n+1},Sx_{2n+1})) \preceq \phi(\psi(d(x_{2n+1},Sx_{2n+1})))\psi(d(x_{2n+1},Sx_{2n+1})) \\ &\Rightarrow \|\psi(d(x_{2n+1},Sx_{2n+1}))\| \leqslant \|\phi(\psi(d(x_{2n+1},Sx_{2n+1})))\|\|\psi(d(x_{2n+1},Sx_{2n+1}))\|, \end{aligned}$$

which is a contradiction. Hence, $\max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Sx_{2n+1})\} = d(x_{2n+1}, x_{2n})$ and we have $\psi(d(x_{2n+1}, Sx_{2n+1})) \preceq \phi(\psi(d(x_{2n+1}, x_{2n})))\psi(d(x_{2n+1}, x_{2n}))$. Using properties of ψ and ϕ, we conclude that $\{x_n\}$ is a Cauchy sequence. By completeness of $(X, \mathbb{A}, d)$, there exists $z \in X$ such that

$$\lim_{n\to+\infty} d(x_n, z) = \theta \Rightarrow \lim_{n\to+\infty} d(x_{2n+1}, z) = \lim_{n\to+\infty} d(x_{2n+2}, z) = \theta.$$

for all $n \in \mathbb{N} \cup \{0\}$. Since S is $\alpha-$continuous, $\lim\limits_{n\to+\infty} h(Sx_{2n+2}, Sz) = \theta$. Therefore, $d(z, Sz) \preceq b[d(z, x_{2n+1}) + d(x_{2n+1}, Sz)] \to \theta$. So, $z \in Sz$. Similarly, we show that $z \in Tz$. Then T and S have a common fixed point in X. ■

5. C^*-valued extended $b-$metric spaces

Definition 5.1 [2] Let X be a non-empty set and $E : X \times X \to \mathbb{A}'$. A function $d : X \times X \to \mathbb{A}$ is called a C^*-algebra-valued extended $b-$metric spaces on X if

1) $d(x, y) = \theta \Leftrightarrow x = y$ for all $x, y \in X$ and $d(x, y) \succeq \theta$;
2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
3) $d(x, y) \preceq E(x, y)[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

$(X, \mathbb{A}, d)$ is called a C^*-algebra-valued extended $b-$metric space.

Theorem 5.2 [2] Let $(X, \mathbb{A}, d)$ be complete C^*-algebra-valued extended $b-$metric space and $T : X \to X$ satisfies $d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda$ for all $x, y \in X$, where $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ and $\lim\limits_{n,m\to\infty} E(x_n, x_m)\|\lambda\| \prec I$. Then T has a unique fixed point $x \in X$.

6. C^*-valued rectangular metric spaces

In 2000, Branciari [4] introduced the notion of a generalized (rectangular) metric space. Rectangular metric space is different with metric space and $b-$metric space. The difference is located in the last properties, that is triangle inequality, which is in this space we used rectangular inequality.

Definition 6.1 Assume X is a non-empty set and the mapping $d : X \times X \to \mathbb{A}_+$ satisfies:

(i) $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
(iii) $d(x, y) \preceq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and for all distinct points $u, v \in X - \{x, y\}$.

Then $(X, \mathbb{A}, d)$ is called a C^*-valued rectangular metric space.

Definition 6.2 Let $(X, \mathbb{A}, d)$ be a C^*-valued rectangular metric space. A mapping $T : X \to X$ is said to be a contraction if there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ such that $d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda$ for all $x, y \in X$.

Theorem 6.3 Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued rectangular metric space. Let $T : X \to X$ be a contraction with the contraction constant $\lambda \in \mathbb{A}$ such that $\|\lambda\| < 1$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$ and define $x_1 = Tx_0$, $x_2 = Tx_1$, ... , $x_n = Tx_{n-1}$. Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \preceq \lambda^* d(x_n, x_{n-1})\lambda \\ &\preceq (\lambda^*)^2 d(x_{n-1}, x_{n-2})\lambda^2 \preceq (\lambda^*)^n d(x_1, x_0)\lambda^n = (\lambda^*)^n B\lambda^n, \end{aligned}$$

where $B = d(x_1, x_0)$. For any $m \geqslant 1$ and $p \geqslant 1$, we have

$$\begin{aligned} d(x_{m+p}, x_m) &\preceq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m) \\ &\preceq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_{m+p-3}) + ... \\ &\quad + d(x_{m+3}, x_{m+2}) + d(x_{m+2}, x_{m+1}) + d(x_{m+1}, x_m) \\ &\preceq (\lambda^*)^{m+p-1} B\lambda^{m+p-1} + (\lambda^*)^{m+p-2} B\lambda^{m+p-2} + (\lambda^*)^{m+p-3} B\lambda^{m+p-3} \\ &\quad + (\lambda^*)^{m+p-4} B\lambda^{m+p-4} + ... + (\lambda^*)^{m+1} B\lambda^{m+1} + (\lambda^*)^m B\lambda^m \\ &\preceq \sum_{k=1}^{\frac{P-1}{2}} \|B^{\frac{1}{2}}\lambda^{m+p-(2k-1)}\|^2 I + \sum_{k=1}^{\frac{P-1}{2}} \|B^{\frac{1}{2}}\lambda^{m+p-(2k)}\|^2 + \|B^{\frac{1}{2}}\lambda^m\|^2 I \\ &\preceq \|B\| \sum_{k=1}^{\frac{P-1}{2}} \lambda\|^{2(m+p-(2k-1)}I + \|B\| \sum_{k=1}^{\frac{P-1}{2}} \|\lambda\|^{2(m+p-(2k)}I + \|B\|\|\lambda\|^{2m} I \to \theta \end{aligned}$$

as $m, p \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence in X with respect to $\mathbb{A}$. By completeness of $(X, \mathbb{A}, d)$, $x_n \to x \in X$, i.e., $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} Tx_{n-1} = x$. We have

$$\begin{aligned} d(Tx, x) &\preceq d(Tx, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x) \\ &\preceq \lambda^* d(x, x_n)\lambda + \lambda^* d(x_n, x_{n+1})\lambda + d(x_{n+2}, x), \end{aligned}$$

which implies that $d(Tx, x) \to \theta$ as $n \to \infty$. Hence, $Tx = x$. To prove the uniqueness of the fixed point x, suppose that u is another fixed point of T. We have $d(x, u) = d(Tx, Tu) \preceq \lambda^* d(x, u)\lambda$. Using the norm of $\mathbb{A}$, we have

$$\begin{aligned} &\|d(x, y)\| = \|d(Tx, Ty)\| \leqslant \|\lambda^* d(x, y)\lambda\| \leqslant \|\lambda^*\|\|d(x, y)\|\|\lambda\| = \|\lambda\|^2 \|d(x, y)\| \\ &\quad \Rightarrow d(x, y) = \theta \Rightarrow x = y. \end{aligned}$$

■

Theorem 6.4 Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued rectangular metric space. Assume the mapping $T : X \to X$ satisfies $d(Tx, Ty) \preceq \lambda(d(Tx, x) + d(Ty, y))$ for all $x, y \in X$, where $\lambda \in \mathbb{A}_+$ such that $\|\lambda\| < \frac{1}{2}$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$ and define $x_1 = Tx_0$, $x_2 = Tx_1$, ... , $x_n = Tx_{n-1}$. We have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \preceq \lambda(d(Tx_n, x_n) + d(Tx_{n-1}, x_{n-1})) \\ &= \lambda(d(x_{n+1}, x_n) + d(x_n, x_{n-1})). \end{aligned}$$

As $\|B\| < \dfrac{1}{2}$ and $I - \lambda$ is invertible, $d(x_{n+1}, x_n) \preceq (I - \lambda)^{-1}\lambda d(x_n, x_{n-1}) = \alpha d(x_n, x_{n-1})$, where $\alpha = (I - \lambda)^{-1}\lambda$. So $d(x_{n+1}, x_n) \preceq \alpha d(x_n, x_{n-1}) \preceq \alpha^2 d(x_{n-1}, x_{n-2}) \preceq \alpha^n d(x_1, x_0)$.

For any $m \geqslant 1$ and $p \geqslant 1$, we have

$$\begin{aligned} d(x_{m+p}, x_m) &\preceq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m) \\ &\preceq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_{m+p-3})... \\ &+ d(x_{m+3}, x_{m+2}) + d(x_{m+2}, x_{m+1}) + d(x_{m+1}, x_m) \\ &\preceq \alpha^{m+p-1}B + \alpha^{m+p-2}B + \alpha^{m+p-3}B + ... + \alpha^{m+2}B + \alpha^{m+1}B + \alpha^m B \\ &\preceq \|B\| \sum_{k=1}^{\frac{P-1}{2}} \|\|\alpha\|^{m+p-(2k-1)}\|I + \|B\| \sum_{k=1}^{\frac{P-1}{2}} \|\alpha\|^{m+p-2k} I + \|B\|\|\alpha\|^m I \\ &\preceq \|B\|\|\alpha\|^{m+p-1}[\frac{\|\alpha\|^{-p+1}-1}{\|\alpha\|^{-2}-1}]I + \|B\|\|\alpha\|^{m+p-1}\|[\frac{\|\alpha\|^{-p+1}-1}{\|\alpha\|^{-2}-1}]I \\ &\quad + \|B\|\|\alpha\|^m I \to \theta \text{ as } m, p \to \infty. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} Tx_{n-1} = x$. Since

$$\begin{aligned} \theta \preceq d(Tx, x) &\preceq d(Tx, x_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x) \\ &\preceq \lambda d(Tx, x) + \lambda d(Tx_n, x_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x), \end{aligned}$$

we have

$$d(Tx, x) \preceq (I-\lambda)^{-1}\lambda d(Tx_n, Tx_{n-1}) + (I-\lambda)^{-1} d(Tx_n, Tx_{n+1})(I-\lambda)^{-1} d(Tx_{n+1}, x).$$

Then,

$$\begin{aligned} \|d(Tx, x)\| &\leqslant \|(I-\lambda)^{-1}\lambda\|\|d(Tx_n, Tx_{n-1})\| + \|(I-\lambda)^{-1}\|\|d(Tx_n, Tx_{n+1})\| \\ &+ \|(I-\lambda)^{-1}\|\|d(Tx_{n+1}, x)\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that x is a fixed point of T. Now if $y \neq x$ is another fixed point of T, then $\theta \preceq d(x, y) = d(Tx, Ty) \preceq \lambda(d(Tx, y) + d(Ty, y)) = \theta$. Hence, $x = y$. Therefore, the fixed point is unique. ■

7. C^*-valued $b-$rectangular metric spaces

Definition 7.1 [11] Let X be a non-empty set and $b \succeq I$. Assume $d : X \times X \to \mathbb{A}_+$ satisfies

- (i) $d(x, y) = \theta$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leqslant b[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and for all $u, v \in X - \{x, y\}$.

Then $(X, \mathbb{A}, d)$ is called a C^*-valued rectangular $b-$metric space.

Definition 7.2 [11] Let $(X, \mathbb{A}, d)$ be a C^*-valued $b-$rectangular metric space. A mapping $T : X \to X$ is said to be a contraction if there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ such that $d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda$ for all $x, y \in X$.

Theorem 7.3 [11] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued $b-$rectangular metric space and $T : X \to X$ be a contraction with the contraction constant $\lambda \in \mathbb{A}$ such that $\|\lambda\| < 1$. Then T has a unique fixed point in X.

Theorem 7.4 [11] (Kannan type) Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued $b-$rectangular metric space. Suppose the mapping $T : X \to X$ satisfies $d(Tx, Ty) \preceq \lambda(d(Tx, x) + d(Ty, y))$ for all $x, y \in X$, where $\lambda \in \mathbb{A}_+$ such that $\|\lambda\| < \frac{1}{2}$. Then T has a unique fixed point in X.

We give some fixed point theorems in C^*-algebra-valued rectangular $b-$metric space using a positive function.

Theorem 7.5 Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued rectangular $b-$metric space. Assume $T : X \to X$ satisfies $d(Tx, Ty) \preceq a^*d(x, y)a - \psi(d(x, y))$, where ψ is $*-$homomorphism and $\lim_{a\to\infty} \psi(a) = \infty$ and $\|b\|\|a\|^2 < 1$. Then T has a unique fixed point.

Proof. Choose $x_0 \in X$ and define $x_1 = Tx_0$, $x_2 = Tx_1$, ... , $x_n = Tx_{n-1}$. We have

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\preceq a^*d(x_n, x_{n-1})a - \psi(d(x_n, x_{n-1})) \\ &\preceq (a^*)^n d(x_1, x_0)(a)^n - \psi^n(d(x_1, x_0)). \end{aligned}$$

Then, for $m \geqslant 1$ and $p \geqslant 1$, we have

$$\begin{aligned} d(x_{m+p}, x_m) &\preceq b[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)] \\ &\preceq bd(x_{m+p}, x_{m+p-1}) + bd(x_{m+p-1}, x_{m+p-2}) + b[b[d(x_{m+p-2}, x_{m+p-3}) \\ &\quad + d(x_{m+p-3}, x_{m+p-4}) + d(x_{m+p-4}, x_m)]] \\ &\preceq \sum_{k=1}^{\frac{p-1}{2}} \|b^{\frac{k}{2}}\|[\|(a^*)^{m+p-(2k-1)} d(x_1, x_0)^{\frac{1}{2}}\|^2 - \|\psi^{m+p-(2k-1)}(d(x_1, x_0))\|]I \\ &\quad + \sum_{k=1}^{\frac{p-1}{2}} \|b^{\frac{k}{2}}\|[\|(a^*)^{m+p-2k} d(x_1, x_0)^{\frac{1}{2}}\|^2 - \|\psi^{m+p-2k}(d(x_1, x_0)\|]I \\ &\quad + \|b^{\frac{p-1}{2}}\|[\|a^m(d(x_1, x_0)^{\frac{1}{2}}\|^2 - \|\psi^m(d(x_1, x_0)\|]I \to \theta \quad (m \to \infty). \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an $x \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x = Tx$. Let y be another fixed point of T, where

$$d(x, y) = d(Tx_n, Ty_n) \preceq (a^*)^n d(x, y)a^n - \psi^n(d(x, y)).$$

We have $\|d(x, y)\| \leqslant \|a\|^{2n}\|d(x, y)\| - \|\psi^n(d(x, y))\| \to \theta$ as $n \to \infty$, which implies that the fixed point is unique. ■

8. C^*-algebra-valued extended hexagonal $b-$asymmetric metric spaces

The notion of extended hexagonal $b-$metric spaces was introduced by Kalpana et al. [12].

Definition 8.1 Let X be a non-empty set and $b \in \mathbb{A}^{'}$ such that $b \succeq I$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies

1) $d(x, y) \succeq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y$ for all $x, y \in X$;
2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
3) $d(x, y) \preceq b[d(x, u) + d(u, v) + d(v, w) + d(w, z) + d(z, y)]$ for all $x, y, u, v, w, z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

d is called a C^*-algebra-valued hexagonal $b-$metric and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued hexagonal $b-$metric space.

The definition of C^*-algebra-valued extended hexagonal $b-$metric space was defined in the following way in [12].

Definition 8.2 Let X be a non-empty set and $E : X \times X \to \mathbb{A}^{'}$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies

1) $d(x, y) \succeq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y$ for all $x, y \in X$;
2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
3) $d(x, y) \preceq E(x, y)[d(x, u)+d(u, v)+d(v, w)+d(w, z)+d(z, y)]$ for all $x, y, u, v, w, z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

$(X, \mathbb{A}, d)$ is called a C^*-algebra-valued extended hexagonal $b-$metric space.

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, asymmetric metric space were introduce by Wilson [19] as metric spaces, but without the requirement that the asymmetric metric d has to satisfy $d(x, y) = d(y, x)$.

Definition 8.3 Let X be a non-empty set and $b \in \mathbb{A}^{'}$ such that $b \succeq I$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies

1) $d(x, y) \succeq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y$ for all $x, y \in X$;
2) $d(x, y) \preceq b[d(x, u) + d(u, v) + d(v, w) + d(w, z) + d(z, y)]$ for all $x, y, u, v, w, z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

d is called a C^*-algebra-valued hexagonal $b-$asymmetric metric and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued hexagonal $b-$asymmetric metric space.

Definition 8.4 Let X be a non-empty set and $E : X \times X \to \mathbb{A}^{'}$. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies

1) $d(x, y) \succeq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y$ for all $x, y \in X$;
2) $d(x, y) \preceq E(x, y)[d(x, u)+d(u, v)+d(v, w)+d(w, z)+d(z, y)]$ for all $x, y, u, v, w, z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

$(X, \mathbb{A}, d)$ is called a C^*-algebra-valued extended hexagonal $b-$asymmetric metric space.

Definition 8.5 Assume that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued extended hexagonal $b-$asymmetric metric space. A sequence $\{x_n\}$ in X is said to be

(i) $\{x_n\}$ $b-$forward (respectively, $b-$backward) converges to $x \in X$ with respect to $\mathbb{A}$ iff for all $\varepsilon \succ \theta$, there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x, x_n) \preceq \varepsilon$ (respectively, $d(x_n, x) \preceq \varepsilon$);
(ii) $\{x_n\}$ converges to x if $\lim_{n\to\infty} d(x, x_n) = \lim_{n\to\infty} d(x_n, x) = \theta$;
(iii) $\{x_n\}$ is $b-$forward Cauchy sequence respect with $\mathbb{A}$ if for all $\varepsilon \succ \theta$, there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x_m, x_n) \preceq \varepsilon$ for all $m > n \geqslant N_\varepsilon$;

(iv) $\{x_n\}$ is $b-$backward Cauchy sequence respect with $\mathbb{A}$ if for all $\varepsilon \succ \theta$, there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x_m, x_n) \preceq \varepsilon$ for all $n > m \geqslant N_\varepsilon$.

Definition 8.6 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued extended hexagonal $b-$asymmetric metric space. X is said to be $b-$forward (respectively, $b-$backward) complete if every $b-$forward (respectively, $b-$backward) Cauchy sequence $\{x_n\}$ converges to x in X.

Definition 8.7 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued extended hexagonal $b-$asymmetric metric space. X is said to be complete if X is $b-$forward and $b-$backward complete.

Lemma 8.8 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued extended hexagonal $b-$asymmetric metric space and $\{x_n\}_n$ be a forward (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.

Theorem 8.9 Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued hexagonal $b-$asymmetric metric space and $T : X \to X$ be a mapping satisfying $d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda$ for all $x, y \in X$ with $\lambda \in \mathbb{A}$ and $\|\lambda\| < 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \preceq \lambda^* d(x_n, x_{n-1})\lambda \\ &\preceq (\lambda^*)^2 d(x_{n-1}, x_{n-2})\lambda^2 \preceq (\lambda^*)^n d(x_1, x_0)\lambda^n. \end{aligned}$$

Thus, $d(x_{n+1}, x_n) \to \theta$ as $n \to \infty$. For $m \geqslant 1$ and $r \geqslant 1$, it follows that

$$\begin{aligned} d(x_{m+r}, x_m) &\preceq b[d(x_{m+r}, x_{m+r-1}) + d(x_{m+r-1}, x_{m+r-2}) + ... + d(x_{m+r-4}, x_m)] \\ &\preceq b[d(x_{m+r}, x_{m+r-1}) + ... + d(x_{m+r-3}, x_{m+r-4})] \\ &\quad + b^2[d(x_{m+r-4}, x_{m+r-5}) ... + d(x_{m+r-7}, x_{m+r-8})] \\ &\quad + ... + b^{r-1}[d(x_{m+5}, x_{m+4}) + ... + d(x_{m+1}, x_m)] \\ &\preceq b\sum_{k=1}^{4}(\lambda^*)^{m+r-k} d(x_1, x_0)\lambda^{m+r-k} + ... + b^{r-1}\sum_{k=1}^{4}(\lambda^*)^{m+k} d(x_1, x_0)\lambda^{m+k} \\ &\quad + b^{r-1}(\lambda^*)^m d(x_1, x_0)\lambda^m \\ &\preceq (\|b\|\sum_{k=1}^{4}\|\lambda\|^{2(m+r-k)} + \|d(x_1, x_0)\| + ... + \|b\|^{r-1}\sum_{k=1}^{4}\|\lambda\|^{2(m+k)})\|d(x_1, x_0) \\ &\quad + \|b^{r-1}\|\|\lambda\|^{2m}\|d(x_1, x_0)\|)I \to \theta \;\; as \;\; m \to \infty. \end{aligned}$$

Similarly, we obtain $d(x_m, x_{m+r}) \to \theta$ as $m \to \infty$. Consequently, $\{x_n\}$ is $b-$forward and $b-$backward Cauchy sequence. By completeness of X, there exists $z \in X$ such that

$\lim_{n\to\infty} x_n = z$. Now, we show that $d(z, Tz) = d(Tz, z) = \theta$.

$$\begin{aligned}
d(Tz, z) &\preceq b[d(Tz, Tx_n) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z)] \\
&\preceq b[\lambda^* d(z, x_n)\lambda + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z)] \\
&\Leftrightarrow \|d(z, Tz)\| \leqslant \|b\|[\|\lambda\|^2\|d(z, x_n)\| + \|d(x_{n+1}, x_{n+2})\| + \|d(x_{n+2}, x_{n+3})\| \\
&\quad + \|d(x_{n+3}, x_{n+4})\| + \|d(x_{n+4}, z)\|] \to \theta \ (n \to \infty).
\end{aligned}$$

Hence, $Tz = z$ and z is a fixed point of T.

Unicity: Let $z' \neq z$ be another fixed point of T. We have $0 \leqslant \|d(z, z')\| \leqslant \|\lambda^* d(z, z')\lambda\| \leqslant \|\lambda\|^2\|d(z, z')\|$, which is a contradiction ($\|\lambda\|^2 \geqslant 1$). Hence, the fixed point z is unique. ■

Theorem 8.10 Let $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued hexagonal $b-$asymmetric metric space and $T : X \to X$ be a mapping satisfying $d(Tx, Ty) \preceq \lambda[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$ with $\lambda \in \mathbb{A}$ and $\|\lambda\| < \frac{1}{2}$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\preceq \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&\Rightarrow (I - \lambda)d(x_n, x_{n+1}) \preceq \lambda d(x_{n-1}, x_n) \preceq \beta^n d(x_0, x_1).
\end{aligned}$$

Let $\beta = (I - \lambda)^{-1}(\lambda)$. Since $\|\lambda\| < \frac{1}{2}$, we have $\|\beta\| < 1$. Then

$$(I - \lambda)d(x_n, x_{n+1}) \preceq \lambda d(x_{n-1}, x_n) \preceq (\beta)^n d(x_0, x_1)$$

and $d(x_n, x_{n+1}) \to \theta$ as $n \to \infty$. For $m \geqslant 1$ and $r \geqslant 1$, it follows that

$$\begin{aligned}
d(x_m, x_{m+r}) &\preceq b[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + ... + d(x_{m+4}, x_m)] \\
&\preceq b[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + d(x_{m+3}, x_{m+4})] \\
&b^2[d(x_{m+4}, x_{m+5}) + d(x_{m+5}, x_{m+6}) + d(x_{m+6}, x_{m+7}) + d(x_{m+7}, x_{m+8})]+ \\
&... + b^{r-1}[d(x_{m+r-5}, x_{m+r-4}) + d(x_{m+r-4}, x_{m+r-3})+ \\
&d(x_{m+r-3}, x_{m+r-2}) + d(x_{m+r-2}, x_{m+r-1}) + d(x_{m+r-1}, x_{m+r})] \\
&\preceq b\sum_{k=1}^{4}(\beta)^{m+r-k}d(x_0, x_1) + ... + b^{r-1}\sum_{k=1}^{4}(\beta)^{m+k}d(x_0, x_1) + b^{r-1}\beta^m d(x_0, x_1) \\
&\preceq (\|b\|\sum_{k=1}^{4}\|\beta\|^{2(m+r-k)} + \|d(x_0, x_1)\| + ... + \|b\|^{r-1}\sum_{k=1}^{4}\|\beta\|^{2(m+k)})\|d(x_0, x_1)+ \\
&\|b^{r-1}\|\|\beta\|^{2m}\|d(x_0, x_1)\|)I \to \theta \ \ as \ \ m \to \infty.
\end{aligned}$$

Similarly, we obtain $d(x_{m+r}, x_m) \to \theta$ as $m \to \infty$. Consequently, $\{x_n\}$ is $b-$forward and $b-$backward Cauchy sequence. By completeness of X, there exists $z \in X$ such that

$\lim_{n\to\infty} x_n = z$. Now, we show that $d(z, Tz) = d(Tz, z) = \theta$.

$$\begin{aligned} d(Tz, z) &\preceq b[d(Tz, Tx_n) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z)] \\ &\preceq b[\lambda(d(z, Tz) + d(x_n, Tx_n)) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+4}, z)] \\ &\Leftrightarrow \|d(z, Tz)\| \leqslant \|b\|[\|\lambda\| \, \|d(z, x_n)\| + \|\lambda\| \, \|d(x_n, x_{n+1}) + \|d(x_{n+1}, x_{n+2})\| \\ &+ \|d(x_{n+2}, x_{n+3})\| + \|d(x_{n+3}, x_{n+4})\| + \|d(x_{n+4}, z)\|] \to \theta \ (n \to \infty). \end{aligned}$$

Hence, $Tz = z$ and z is a fixed point of T.

Unicity: Let $z' \neq z$ be another fixed point of T. We have $d(z, z') \leqslant \lambda(d(z, Tz) + d(z', Tz')) = \lambda(d(z, z) + d(z', z')) = \theta$, which is a contradiction ($d(z, z') = \theta \Rightarrow z = z'$), hence the fixed point z is unique. ■

9. C^*-algebra-valued S-metric spaces

Definition 9.1 [9] Let X be non-empty set and $S : X \times X \times X \to \mathbb{A}^+$ be a function satisfying the following properties:

1) $S(x, y, , z) \succeq \theta$ for all $x, y, z \in X$;
2) $S(x, y, z) = \theta$ if and only if $x = y = z$;
3) $S(x, y, z) \preceq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

Then S is said to be C^*-algebra-valued $S-$metric on X and $(X, \mathbb{A}, S)$ is said to be a C^*-algebra-valued $S-$metric space.

Definition 9.2 Suppose that $(X, \mathbb{A}, S)$ be a C^*-algebra-valued $S-$metric space. Let $\{x_n\}_n$ be a sequence in X. If $\|S(x_n, x_n, x)\| \to 0$ as $(n \to \infty)$, then it is said that $\{x_n\}_n$ converges to x and we denote it by $\lim_{n\to+\infty} x_n = x$. If for any $p \in \mathbb{N}$ $\|S(x_{n+p}, x_{n+p}, x_n)\| \to \theta$ as $(n \to \infty)$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then $(X, \mathbb{A}, S)$ is called a complete C^*-algebra-valued $S-$metric space.

Lemma 9.3 [9]

1) If $\{b_n\}_n \subset \mathbb{A}$ and $\lim_{n\to\infty} b_n = \theta$, then $\lim_{n\to\infty} a^* b_n a = \theta$ for any $a \in \mathbb{A}$.
2) If $a, b \in \mathbb{A}_h$ and $c \in \mathbb{A}$ $a \preceq b \Rightarrow ca \preceq cb$.

Lemma 9.4 [9] Let (X, A, S) be a complete C^*-algebra-valued $S-$metric space. Then, $S(x, x, y) = S(y, y, x)$.

Lemma 9.5 [9] Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converge to x and y, then $x = y$.

Theorem 9.6 [9] Let (X, S, d) a complete C^*-algebra-valued $S-$metric space. Suppose that the mapping $f : X \to X$ satisfies $S(fx, fx, fy) \preceq a^* S(x, x, y)a$ for all $x, y \in X$ with $\|a\| < 1$. Then there exists a unique fixed point in X.

10. C^*-algebra-valued G-metric spaces

Definition 10.1 [17] Let X be non-empty set and S_3 be the permutation group on $\{1, 2, 3\}$. $G : X \times X \times X \to \mathbb{A}^+$ be a function satisfying the following properties:

1) $G(x_1, x_2, x_3) = \theta \Leftrightarrow x_1 = x_2 = x_3$;

2) $G(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = G(x_1, x_2, x_3)$ for all $x_1, x_2, x_3 \in X$ and $\sigma \in S_3$;
3) $G(x_1, x_1, x_3) \preceq G(x_1, x_2, x_3)$ for all $x_1, x_2, x_3 \in X$ with $x_2 \neq x_3$;
4) $G(x_1, x_1, x_3) \preceq G(x_1, a, a) + G(a, x_2, x_3)$ for all $x_1, x_2, x_3, a \in X$.

Then G is said to be C^*-algebra-valued $G-$metric on X and $(X, \mathbb{A}, G)$ is said to be a C^*-algebra-valued $G-$metric space.

Definition 10.2 [17] Let $(X, \mathbb{A}, G)$ be a C^*-algebra-valued $G-$metric space and $\{x_n\}_n$ be a sequence in X. If $\|G(x_n, x_n, x)\| \to 0$ as $(n \to \infty)$, then it is said that $\{x_n\}_n$ converges to x and we denote it by $\lim_{n\to+\infty} x_n = x$. If $\|G(x_{n+p}, x_{n+p}, x_n)\| \to \theta$ for any $p \in \mathbb{N}$ as $(n \to \infty)$, then $\{x_n\}$ is called a $G-$ Cauchy sequence in X. If every $G-$Cauchy sequence is convergent in X, then $(X, \mathbb{A}, G)$ is called a complete C^*-algebra-valued $G-$metric space.

Proposition 10.3 [17] Let $(X, \mathbb{A}, G)$ be a C^*-algebra-valued $G-$metric space and $\{x_n\} \subset X$. Then $\{x_n\}$ is a $G-$Cauchy sequence if and only if for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ so that $\|G(x_m, x_n, x_n)\| < \varepsilon$ for all $m, n > N$.

Definition 10.4 [17] Let $(X, \mathbb{A}, G)$ be a C^*-algebra-valued $G-$metric space and $T : X \to X$ is a mapping. If there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ such that

$$G(Tx, Ty, Tz) \preceq \lambda^* G(x, y, z)\lambda$$

for all $x, y, z \in X$, then T is called a contractive mapping on $(X, \mathbb{A}, G)$.

Theorem 10.5 [17] Let $(X, \mathbb{A}, G)$ be a C^*-algebra-valued $G-$metric space. If $T : X \to X$ is a contractive mapping on $(X, \mathbb{A}, G)$, then there is a unique fixed point of T on X.

11. C^*-algebra-valued partial metric space

Definition 11.1 [5] Let X be a non-empty set. A mapping $p : X \times X \to \mathbb{A}$ is called a C^*-algebra-valued metric on X if the following conditions are satisfied:

(i) $\theta \preceq p(x, y)$ for all $x, y \in X$ and $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$;
(ii) $p(x, y) = p(y, x)$ for all $x, y \in X$;
(iii) $p(x, x) \preceq p(x, y)$ for all $x, y \in X$;
(iv) $p(x, y) \preceq p(x, z) + p(z, y) - p(z, z)$ for all $x, y, z \in X$.

Then $(X, \mathbb{A}, p)$ is called a C^*-algebra-valued partial metric space.

If we take $\mathbb{A} = \mathbb{R}$, then the new notion of C^*-algebra-valued partial metric space becomes equivalent to the definition of the real partial metric space.

Definition 11.2 [5] Let $(X, \mathbb{A}, p)$ be a C^*-algebra-valued partial metric space.

(1) A sequence $\{x_n\} \subset X$ converges to $x \in X$, whenever for every $\varepsilon > 0$, there is a natural number N such that for all $n > N$, $\|p(x_n, x) + p(x, x)\| \leqslant \varepsilon$. We denote it by $\lim_{n\longrightarrow\infty} p(x_n, x) - p(x, x) = \theta$.
(2) $\{x_n\}$ is a partial Cauchy sequence respect to $\mathbb{A}$, whenever $\varepsilon > 0$ there is a natural number N such that
$(p(x_n, x_m) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x_m, x_m))((p(x_n, x_m) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x_m, x_m))^* \preceq \varepsilon^2$
for all $n, m > N$.

(3) $(X, \mathbb{A}_+, p)$ is said to be complete with respect to $\mathbb{A}$ if every partial Cauchy sequence with respect to $\mathbb{A}$ converges to $x \in X$ such that $\lim_{n\to\infty}(p(x_n, x) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x, x)) = \theta$.

If we take $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, then p^s is a C^*-algebra-valued metric.

Lemma 11.3 [5] Let $(X, \mathbb{A}, p)$ be a C^*-algebra-valued partial metric space.

(1) $\{x_n\}$ is a partial Cauchy sequence in $(X, \mathbb{A}, p)$ if and only if it is Cauchy sequence in the C^*-algebra-valued metric $(X, \mathbb{A}, p^s)$.
(2) A C^*-algebra-valued partial metric space $(U, \mathbb{A}, p)$ is complete if and only if C^*-algebra-valued metric space $(X, \mathbb{A}, p^s)$ is complete. Furthermore,

$$\lim_{n\to\infty} p^s(x_n, x) = \theta \Leftrightarrow \lim_{n\to\infty}(2p(x_n, x) - p(x_n, x_n) - p(x, x)) = \theta$$

or

$$\lim_{n\to\infty} p^s(x_n, x) = \theta \Leftrightarrow \lim_{n\to\infty}(p(x_n, x) - p(x_n, x_n)) = \theta \text{ and } \lim_{n\to\infty}(p(x_n, x) - p(x, x)) = \theta.$$

Lemma 11.4 [5] If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a C^*-algebra-valued partial metric space $(X, \mathbb{A}, p)$, then $\lim_{n\to\infty}(p(x_n, y_n) - p(x_n, x_n)) = p(x, y) - p(x, x)$ and $\lim_{n\to\infty}(p(x_n, y_n) - p(y_n, y_n)) = p(x, y) - p(y, y)$.

Definition 11.5 [18] Let $(X, \mathbb{A}, p)$ be a C^*-algebra-valued partial metric space. A mapping $T : X \to X$ is said to be a C^*-valued contractive mapping on X if there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ such that $p(Tx, Ty) \preceq \lambda^* p(x, y)\lambda$.

Theorem 11.6 [18] If $(X, \mathbb{A}, p)$ is a complete C^*-algebra-valued partial metric space and T is a contractive mapping, then T has a unique fixed point.

Definition 11.7 Let $F : \mathbb{A}_+ \to \mathbb{A}_+$ be a function satisfying

(i) F is continuous and nondecreasing;
(ii) $F(T) = \theta$ if and only if $T = \theta$.

A mapping $T : X \to X$ is said to be a (ϕ, F) C^*-valued partial contraction of type (I) if there exists $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ an $*-$homomorphism such that

$$p(Tx, Ty) \succeq \theta \Rightarrow F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(p(x, y)) \tag{16}$$

for all $x, y \in X$.

Theorem 11.8 Let $(X, \mathbb{A}, p)$ be a complete C^*-algebra-valued partial metric space and $T : X \to X$ be a (ϕ, F) C^*-valued partial contraction mapping of type (I). Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. First, let us observe that T has at most one fixed point. Indeed if $x_1^*, x_2^* \in X$: $Tx_1^* = x_1^* \neq x_2^* = Tx_2^*$, then we get $\phi(p(x, y)) \preceq F(p(x_1^*, x_2^*) - F(p(Tx_1^*, Tx_2^*)) = \theta$, which is a contradiction. In order to show that it has a fixed point, let $x_0 \in X$ be arbitrary and fixed. We define a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ by $x_{n+1} = Tx_n$ for $n = 0, 1, ...$ Denote $p_n = p(x_{n+1}, x_n)$ for $n = 0, 1, ...$ If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and the proof is finished. Suppose now that $x_{n+1} \neq x_n$ for every $n \in X$, Then $p_n \succ \theta$ for all $n \in \mathbb{N}$. Using (16), the following holds

$$F(p_n) \preceq F(p_{n-1}) - \phi(p_{n-1}) \prec F(p_{n-1}) \tag{17}$$

for every $n \in \mathbb{N}$. Hence, F is non decreasing and so the sequence (p_n) is monotonically decreasing in $\mathbb{A}_+$. So there exists $\theta \preceq t \in \mathbb{A}_+$ such that $p(x_n, x_{n+1}) \to t$ as $n \to \infty$. From (17), we obtain $\lim_{n\to\infty} F(p_n) = \theta$ that together with (ii) gives

$$\lim_{n\to\infty} p_n = \theta. \tag{18}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p)$. By Lemma 11.4, it is sufficient to prove that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p^s)$. We have proved $\lim_{n\to\infty} p_n = \theta$. Keeping in mind that $\theta \preceq p(x_n, x_n) \preceq p(x_n, x_{n+1})$, we get

$$\lim_{n\to\infty} p(x_n, x_n) = \theta. \tag{19}$$

Also, $\theta \preceq p(x_{n+1}, x_{n+1}) \preceq p(x_n, x_{n+1})$ implies $\lim_{n\to\infty} p(x_{n+1}, x_{n+1}) = \theta$. Assume that $\{x_n\}$ is not a Cauchy sequence in $(X, \mathbb{A}, p^s)$. Then there exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k > k$ such that $\|p^s(x_{m_k}, x_{n_k})\| > \varepsilon$. Now, corresponding to m_k, we can choose n_k such that it is the smallest integer with $n_k > m_k$ satisfying above inequality. Hence, $\|p^s(x_{m_k}, x_{n_{k-1}})\| \leqslant \varepsilon$. So, we have

$$\begin{aligned}
\varepsilon &\leqslant \|p^s(x_{m_k}, x_{n_k})\| \\
&\leqslant \|p^s(x_{m_k}, x_{n_{k-1}}) + p^s(x_{n_{k-1}}, x_{n_k}) - p^s(x_{n_{k-1}}, x_{n_{k-1}})\| \\
&\leqslant \|p^s(x_{m_k}, x_{n_{k-1}})\| + \|p^s(x_{n_{k-1}}, x_{n_k})\| \\
&\leqslant \varepsilon + \|p^s(x_{n_{k-1}}, x_{n_k})\|.
\end{aligned} \tag{20}$$

We know that

$$p^s(x_{n_{k-1}}, x_{n_k}) = 2p(x_{n_{k-1}}, x_{n_k}) - p^s(x_{n_{k-1}}, x_{n_{k-1}}) - p^s(x_{n_k}, x_{n_k}). \tag{21}$$

Using (18), (19), (20), and (21), we have $\varepsilon \leqslant \lim_{k\to\infty} \|p^s(x_{n_{k-1}}, x_{n_k})\| < \varepsilon + \theta$. This implies

$$\lim_{k\to\infty} \|p^s(x_{m_k}, x_{n_k})\| = \varepsilon. \tag{22}$$

Again,

$$\begin{aligned}
\|p^s(x_{n_k}, x_{m_k})\| &\leqslant \|p^s(x_{n_k}, x_{n_k-1}) + p^s(x_{n_k-1}, x_{m_k}) - p^s(x_{n_k-1}, x_{n_k-1})\| \\
&\leqslant \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k})\| \\
&\leqslant \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k-1}) + p^s(x_{m_k-1}, x_{m_k}) \\
&\quad - p^s(x_{m_k-1}, x_{m_k-1})\| \\
&\leqslant \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k-1})\| + \|p^s(x_{m_k-1}, x_{m_k})\|.
\end{aligned} \tag{23}$$

Also,

$$\begin{aligned}\|p^s(x_{n_k-1}, x_{m_k-1})\| &\leqslant \|p^s(x_{n_k-1}, x_{n_k}) + p^s(x_{n_k}, x_{m_k-1}) - p^s(x_{n_k}, x_{m_k})\| \\ &\leqslant \|p^s(x_{n_k-1}, x_{n_k})\| + \|p^s(x_{n_k}, x_{m_k-1})\| \\ &\leqslant \|p^s(x_{n_k-1}, x_{nk})\| + \|p^s(x_{n_k}, x_{m_k}) + p^s(x_{m_k}, x_{mk-1}) - p^s(x_{m_k}, x_{m_k})\| \\ &\leqslant \|p^s(x_{n_k-1}, x_{nk})\| + \|p^s(x_{n_k}, x_{m_k})\| + \|p^s(x_{m_k}, x_{m_k-1})\|. \quad (24)\end{aligned}$$

Letting $k \to \infty$ in (23) and (24) and using (19) and (22), $\lim_{k\to\infty} \|p^s(x_{n_{k-1}}, x_{m_{k-1}})\| = \varepsilon$. Thus,

$$\begin{aligned}\lim_{k\to\infty} \|p(x_{n_{k-1}}, x_{m_{k-1}})\| &= \frac{1}{2}\lim_{k\to\infty} \|2p^s(x_{n_{k-1}}, x_{m_{k-1}}) - p^s(x_{n_{k-1}}, x_{n_{k-1}}) - p^s(x_{m_{k-1}}, x_{m_{k-1}})\| \\ &= \frac{1}{2}\lim_{k\to\infty} \|p^s(x_{n_{k-1}}, x_{m_{k-1}})\| \\ &= \frac{\varepsilon}{2}.\end{aligned}$$

As $p(x_{n_{k-1}}, x_{m_{k-1}}), p(x_{n_k}, x_{m_k}) \in \mathbb{A}_+$ and $\lim_{k\to\infty} \|p(x_{n_{k-1}}, x_{m_{k-1}})\| = \lim_{k\to\infty} \|p(x_{n_k}, x_{m_k})\| = \frac{\varepsilon}{2}$, there is $s \in \mathbb{A}_+$ with $\|s\| = \varepsilon$ such that

$$\lim_{k\to\infty} \|p(x_{n_{k-1}}, x_{m_{k-1}})\| = \lim_{k\to\infty} \|p(x_{n_k}, x_{m_k})\| = s \quad (25)$$

By (25), we have $F(s) \prec F(s) - \phi(s)$. Thus, $\phi(s) = \theta$ and so $s = \theta$ which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p^s)$ and $\{x_n\}$ is partially Cauchy in the complete C^*-algebra-valued partial metric space $(X, \mathbb{A}, p)$. Hence, there exist $z \in X$ such that $\lim_{n\to\infty} p(x_n, z) - p(x_n, x_n) = \theta$. Using (19), we get $\lim_{n\to\infty} p(x_n, z) = \theta$ and thus, $p(z, z) = \theta$. Now, we shall show that z is fixed point of T. Using (16), we get $\theta \preceq F(p(Tz, Tz)) \prec F(p(z, z)) = F(\theta) = \theta$. Thus, $F(p(Tz, Tz)) = \theta$, which implies $p(Tv, Tv) = \theta$. On the other hand, $F(p(x_n, Tz)) \prec F(p(x_{n-1}, z))$. Letting $n \to \infty$ and using the concept of continuity of the function of T, we have $p(z, Tz) = \theta$. Hence, by Definition 11.1, we have $p(z, z) = p(Tz, Tz) = p(z, Tz) = \theta$ and then $Tz = z$, which completes the proof. ∎

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