A framework for the numerical solution of Klein–Gordon equations using three new collocation methods

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Abstract

The Klein−Gordon equation is a relativistic version of the Schrödinger equation and has a large range of applications in contemporary physics. In this paper, we present a unified framework for the numerical solution of nonlinear Klein−Gordon equations using three new collocation schemes. The solutions are approximated by the two−dimensional Legendre−Gauss−Radau interpolation directly. Moreover, using the properties of Jacobi polynomials, the partial derivatives of the solutions are expressed in terms of Jacobi polynomials that makes the approximations numerically more stable. We first derive a single−domain collocation method (SDC) that is suited for problems in small and moderate domains. Next, a multi−domain collocation method (MDC) is presented for large domains. The proposed MDC method solves the problem step by step in subdomains whilst is of the BN−stability. We then construct a single−domain iterative collocation method (SDIC). The proposed SDIC method is based on the quasilinearization (QL) technique and is suited for highly nonlinear problems. The key properties of the SDC, MDC and SDIC schemes are explained and the convergence of the QL approach applied to PDEs with the differential operator of the wave equation is assessed. Numerical examples are included to assess the accuracy and features of each collocation scheme.

Keywords: Klein−Gordon equations; Legendre interpolation; Legendre−Gauss−Radau collocation; Domain decomposition; Quasilinearization.

1 Introduction

One of the most efficient type of numerical methods for solving differential equations are spectral methods [1, 2]. Spectral method employs global orthogonal polynomials as trial functions. Orthogonal collocation methods are a class of spectral methods where the differential equation is transcribed to system of algebraic equations by parameterizing the unknown solution using global polynomials and collocating the differential equation using nodes obtained from a Gaussian quadrature [3]. They are easy for implementation and enjoy the so-called "spectral accuracy" for smooth solutions. For problems where the solution is nonsmooth or not well approximated by global polynomials of a reasonably low degree, it is preferable to use a finite-element approach where the domain of the underlying problem is partitioned into subdomains and a different polynomial is used over each subdomain [3]. Moreover, for highly nonlinear problems where the resulting nonlinear system of algebraic equations is complicated, implementing the quasilinearization (QL) technique is a suitable alternative [4]. In the QL technique, a nonlinear differential equation is replaced with a sequence of linear differential equations which simplifies calculations and saves work.

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Consider the nonlinear Klein−Gordon equation

$$Lu + H(u) = f(x,t), \qquad (x,t) \in \Omega = [0,X] \times [0,T], \tag{1}$$

subject to initial conditions

$$u(x,0) = g(x), \qquad \partial_t u(x,0) = h(x). \tag{2}$$

where L is the differential operator of the wave equation defined by $L = \partial_{tt} - c^2 \partial_{xx}$, u is a function of x and t, H is a generally nonlinear function and f, g and h are known analytic functions. In the special mode when $H(u) = \alpha \sin(u)$ we have the Sine−Gordon equation. The nonlinear Klein−Gordon equation appears in many types of nonlinearities and plays a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory [5]. During the past decade, several different numerical methods have been developed for solving Klein−Gordon and Sine−Gordon equations [6]−[31].

In this paper, we examine the properties of global and multi−domain Legendre−Gauss−Radau (LGR) collocation methods for the numerical solution of Klein−Gordon equations. This work is for three new collocation methods. In the next section, we explain approximating the exact solutions directly by the two−dimensional direct LGR interpolation. Some interpolation error estimates in the norm of Sobolev spaces are also investigated. In Section 3, we first derive a single−domain collocation method which possesses the spectral accuracy and is easy to be implemented. A multi−domain version of collocation method is then proposed. By this approach with moderate number of collocation points, we could evaluate numerical solutions step by step as time increases. Therefore, it is more appropriate for long−time calculation. Next, by employing the QL technique, we propose an iterative version of the single−domain collocation method. In Section 4, convergence of the QL technique applied to a general PDE with the differential operator of the wave equation, is investigated. In Section 5, we present some numerical results which demonstrate the spectral accuracy of proposed schemes. We also compare the accuracy and features of the suggested new methods. This paper provides the first comparison that identifies the mathematical properties of different types of collocation methods, enabling a researcher or end-user to see clearly the accuracy and convergence (or nonconvergence) that can be expected when applying a particular collocation method on a problem of interest. The final section is for some concluding remarks.

2 Direct Legendre interpolation

Let $\alpha, \beta > -1$ and $P_n^{(\alpha,\beta)}(x)$ be the standard Jacobi polynomial of degree n. We have

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)}, \qquad P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}. \tag{3}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{4}$$

The nth-degree standard Legendre polynomial is represented in terms of Jacobi polynomials as $P_n(x) = P_n^{(0,0)}(x)$. We denote by $P_{I,n}(x)$ the nth-degree shifted Legendre polynomial on the arbitrary interval $I = [a,b]$ and, according to (3) and (4), we get

$$P_{I,n}(a) = (-1)^n, \qquad P_{I,n}(b) = 1, \tag{5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{I,n}(x) = \frac{1}{b-a}(n+1)P^{(1,1)}_{I,n-1}(x). \tag{6}$$

We denote by x_j, $0 \le j \le N$ the nodes of the standard LGR interpolation on the half open interval $[-1, 1)$. The corresponding Christoffel numbers are w_j, $0 \le j \le N$. Then the nodes of the shifted LGR interpolation on the half open interval $[a, b)$ and their corresponding Christoffel numbers are denoted by $x_{I,j}$ and $w_{I,j}$, respectively. Let $\mathcal{P}_N(a, b)$ be the set of polynomials of degree at most N on the interval I. Thanks to the property of the standard LGR quadrature, it follows that for any $\psi \in \mathcal{P}_{2N}$ on I,

$$\int_I \psi(x)dx = \sum_{j=0}^{N} w_{I,j}\ \psi(x_{I,j}). \tag{7}$$

Let $\langle u, v\rangle_I$ and $\|v\|_I$ be the inner product and the norm of space $L^2(I)$, respectively. We also define the following discrete inner product and norm,

$$\langle u, v\rangle_{I,N} = \sum_{j=0}^{N} w_{I,j}u(x_{I,j})v(x_{I,j}), \qquad \|v\|_{I,N} = \sqrt{\langle v, v\rangle_{I,N}}. \tag{8}$$

Due to Eq. (7), for any $\varphi\psi \in \mathcal{P}_{2N}$ and $\psi \in \mathcal{P}_N$,

$$\langle \varphi, \psi\rangle_I = \langle \varphi, \psi\rangle_{I,N}, \qquad \|\varphi\|_I = \|\varphi\|_{I,N}. \tag{9}$$

For any $U \in \mathcal{C}(I)$, the direct LGR interpolation, $\mathcal{I}^N U(x)$, in the interval I can be expanded as

$$\mathcal{I}^N U(x) = \sum_{j=0}^{N} \widetilde{U}_{I,j}P_{I,j}(x), \tag{10}$$

where

$$\widetilde{U}_{I,j} = \frac{2j+1}{b-a}\langle \mathcal{I}^N U, P_{I,j}\rangle_I = \frac{2j+1}{b-a}\langle U, P_{I,j}\rangle_{I,N}, \quad 0 \le j \le N. \tag{11}$$

2.1 Two-dimensional direct LGR interpolation

Let $I = [0, X]$, $J = [0, T]$, $\Omega = I \times J$ and $t_0 \in J$ be fixed. For any $u \in \mathcal{C}(\Omega)$, Eqs. (10)–(11) imply

$$\mathcal{I}^N u(x, t_0) = \sum_{j=0}^{N} \widetilde{u}^N_j P_{I,j}(x), \tag{12}$$

where

$$\widetilde{u}^N_j = \sum_{p=0}^{N} \frac{2j+1}{X} w_p \mathcal{I}^N u(x_{I,p}, t_0)P_{I,j}(x_{I,p}).$$

Similarly, for any given shifted LGR point $x_{I,p}$, we get

$$\mathcal{I}^M u(x_{I,p}, t) = \sum_{i=0}^{M} \widehat{u}^M_i P_{J,i}(t), \tag{13}$$

where

$$\widehat{u}_i^M = \sum_{q=0}^{M} \frac{2i+1}{T} w_q \mathcal{I}^M u(x_{I,p}, t_{J,q}) P_{J,i}(t_{J,q}).$$

Let $\mathcal{I}^{N,M}u(x,t)$ denotes the two dimensional direct Legendre interpolation. Combining Eqs. (12)–(13) we deduce that

$$\begin{aligned}
\mathcal{I}^{N,M}u(x,t) &= \sum_{j=0}^{N} \Big(\frac{2j+1}{X} \sum_{p=0}^{N} w_p \mathcal{I}^N u(x_{I,p}, t) P_{I,j}(x_{I,p}) \Big) P_{I,j}(x) \\
&= \sum_{j=0}^{N} \Big(\frac{2j+1}{X} \sum_{p=0}^{N} w_p \Big(\sum_{i=0}^{M} \widehat{u}_i^M P_{J,i}(t) \Big) P_{I,j}(x_{I,p}) \Big) P_{I,j}(x) \\
&= \frac{1}{XT} \sum_{j=0}^{N} \sum_{p=0}^{N} \sum_{i=0}^{M} \sum_{q=0}^{M} (2j+1)(2i+1) w_p w_q u(x_{I,p}, t_{J,q}) P_{J,i}(t_{J,q}) P_{I,j}(x_{I,p}) P_{I,j}(x) P_{J,i}(t) \\
&:= \sum_{j=0}^{N} \sum_{i=0}^{M} \widetilde{u}_{j,i}^{N,M} P_{I,j}(x) P_{J,i}(t), \qquad (14)
\end{aligned}$$

where

$$\widetilde{u}_{j,i}^{N,M} = \frac{1}{XT} \sum_{p=0}^{N} \sum_{q=0}^{M} (2j+1)(2i+1) w_p w_q u(x_{I,p}, t_{J,q}) P_{I,j}(x_{I,p}) P_{J,i}(t_{J,q}).$$

Furthermore, the first and second order partial derivatives of $\mathcal{I}^{N,M}u(x,t)$ in terms of Jacobi polynomials can be computed using Eq. (6) as

$$\partial_x \mathcal{I}^{N,M}u(x,t) = \frac{1}{X} \sum_{j=1}^{N} \sum_{i=0}^{M} (j+1) \widetilde{u}_{j,i}^{N,M} P_{I,j-1}^{(1,1)}(x) P_{J,i}(t), \qquad (15)$$

$$\partial_t \mathcal{I}^{N,M}u(x,t) = \frac{1}{T} \sum_{j=0}^{N} \sum_{i=1}^{M} (i+1) \widetilde{u}_{j,i}^{N,M} P_{I,j}(x) P_{J,i-1}^{(1,1)}(t), \qquad (16)$$

and

$$\partial_{xx} \mathcal{I}^{N,M}u(x,t) = \frac{1}{X^2} \sum_{j=2}^{N} \sum_{i=0}^{M} (j+1)(j+2) \widetilde{u}_{j,i}^{N,M} P_{I,j-2}^{(2,2)}(x) P_{J,i}(t), \qquad (17)$$

$$\partial_{tt} \mathcal{I}^{N,M}u(x,t) = \frac{1}{T^2} \sum_{j=0}^{N} \sum_{i=2}^{M} (i+1)(i+2) \widetilde{u}_{j,i}^{N,M} P_{I,j}(x) P_{J,i-2}^{(2,2)}(t). \qquad (18)$$

Note that, in the representations (14)–(18), instead of Lagrange interpolating polynomials, a double summation of Jacobi polynomials is utilized. This makes the computations more stable specially for large mode M and N [3].

2.2 Error estimates

Here, we investigate the error estimate for the two-dimensional direct Legendre interpolation and its partial derivatives. We suppose that $N = \max\{M, N\}$ and for brevity we set $\mathcal{I}^{N,M}u := \mathcal{I}^N u$. Let $\widetilde{\Omega} = [-1,1]^2$ and H^m be the Sobolev space of integer order m.

Theorem 1. *Suppose that $u \in H^m(\Omega)$ for $m \geqslant 2$ and $h = \max\{X, T\}$. Then, for $0 \leqslant l \leqslant m$ we have*

$$\left\|u - \mathcal{I}^N u\right\|_{H^l(\Omega)} \leqslant \begin{cases} C_l h^m N^{2l+1-m} \left\|u\right\|_{H^m(\Omega)} & \text{if} \quad 0 < h < 1, \\ C_l h^l N^{2l+1-m} \left\|u\right\|_{H^m(\Omega)} & \text{if} \quad h \geqslant 1, \end{cases} \tag{19}$$

where C_l is a positive constant that depends on l and m, but which is independent of N and the function u.

Proof. Using the definition of the H^m–norm, for $u \in H^m(\Omega)$ we have

$$\left\|u - \mathcal{I}^N u\right\|^2_{H^l(\Omega)} = \sum_{r=0}^{l} \left\|D^r(u - \mathcal{I}^N u)\right\|^2_{L^2(\Omega)}, \tag{20}$$

where $D^r v$ denotes the r^{th} weak derivative of v [2]. On the other hand,

$$\begin{aligned} \left\|D^r(u - \mathcal{I}^N u)\right\|^2_{L^2(\Omega)} &= \int\limits_{\Omega}\int \left|D^r\big(u(x,t) - \mathcal{I}^N u(x,t)\big)\right|^2 \mathrm{d}x\mathrm{d}t \\ &= \frac{XT}{4}\int\limits_{\widetilde{\Omega}}\int \left|D^r\Big(u(\frac{X}{2}(z+1), \frac{T}{2}(w+1)) - \mathcal{I}^N u(\frac{X}{2}(z+1), \frac{T}{2}(w+1))\Big)\right|^2 \mathrm{d}z\mathrm{d}w. \end{aligned}$$

By introducing the multi-index $r = (r_1, r_2)$ in the weak derivative D^r, we obtain

$$\begin{aligned} \left\|D^r(u - \mathcal{I}^N u)\right\|^2_{L^2(\Omega)} &= \frac{XT}{4}\int\limits_{\widetilde{\Omega}}\int \left(\frac{X}{2}\right)^{2r_1}\left(\frac{T}{2}\right)^{2(r-r_1)} \left|D^{(r_1, r-r_1)}\big(v(z,w) - \mathcal{I}^N v(z,w)\big)\right|^2 \mathrm{d}z\mathrm{d}w. \\ &= 2^{-2r-2} X^{2r_1+1} T^{2r-2r_1+1} \left\|D^r(v - \mathcal{I}^N v)\right\|^2_{L^2(\widetilde{\Omega})}. \end{aligned} \tag{21}$$

Furthermore, according to equation (9.7.19) in [1], one has

$$\left\|v - \mathcal{I}^N v\right\|_{H^l(\widetilde{\Omega})} \leqslant C N^{2l+1-m} \left\|v\right\|_{H^m(\widetilde{\Omega})}, \quad 0 \leqslant l \leqslant m. \tag{22}$$

Substituting Eq. (21) into Eq. (20) and employing Eq. (22), we deduce that

$$\begin{aligned} \left\|u - \mathcal{I}^N u\right\|^2_{H^l(\Omega)} &= \sum_{r=0}^{l} 2^{-2r-2} X^{2r_1+1} T^{2r-2r_1+1} \left\|D^r(v - \mathcal{I}^N v)\right\|^2_{L^2(\widetilde{\Omega})} \\ &\leqslant \sum_{r=0}^{l} 2^{-2r-2} h^{2r_1+1} h^{2r-2r_1+1} \left\|D^r(v - \mathcal{I}^N v)\right\|^2_{L^2(\widetilde{\Omega})} \\ &\leqslant 2^{-2} \sum_{r=0}^{l} h^{2r+2} \left\|D^r(v - \mathcal{I}^N v)\right\|^2_{L^2(\widetilde{\Omega})} \\ &\leqslant \begin{cases} 2^{-2} h^2 C^2 N^{4l+2-2m} \left\|v\right\|^2_{H^m(\widetilde{\Omega})}, & 0 < h < 1 \\ 2^{-2} h^{2l+2} C^2 N^{4l+2-2m} \left\|v\right\|^2_{H^m(\widetilde{\Omega})}, & h \geqslant 1. \end{cases} \end{aligned}$$

In addition, it is easy to show that

$$\left\|v\right\|^2_{H^m(\widetilde{\Omega})} \leqslant \begin{cases} h^{2m-2} \left\|u\right\|^2_{H^m(\Omega)}, & 0 < h < 1 \\ h^{-2} \left\|u\right\|^2_{H^m(\Omega)}, & h \geqslant 1, \end{cases}$$

which implies

$$\left\|u-\mathcal{I}^N u\right\|_{H^l(\Omega)} \leqslant \begin{cases} C_l h^m N^{2l+1-m} \left\|u\right\|_{H^m(\Omega)}, & 0<h<1 \\ C_l h^l N^{2l+1-m} \left\|u\right\|_{H^m(\Omega)}, & h \geqslant 1, \end{cases}$$

as desired. □

Remark 1. *According to the definition of H^l–norm, the interpolation error estimate* (19)*, also provides upper bounds for the errors in approximating partial derivatives given in Eqs.* (15)–(18) *for $l=1$ and* 2*.*

3 The proposed collocation schemes

In this section, we develop our framework for the numerical solution of Klein−Gordon equation (1) − (2) using collocation at shifted LGR points. We derive three different numerical schemes, namely, single-domain collocation (SDC), multi time-domain collocation (MDC) and single-domain iterative collocation (SDIC).

3.1 SDC method

Consider the Klein-Gordon equation (1) with initial conditions (2) defined on the rectangular domain $\Omega = I \times J$. The SDC method for solving (1) is to seek a polynomial $u^{N,M}(x,t)$ such that

$$\begin{cases} L\ u^{N,M}\big(x_{I,j},t_{J,i}\big) + H\big(u^{N,M}(x_{I,j},t_{J,i})\big) = f(x_{I,j},t_{J,i}), \\ \\ u^{N,M}(x_{I,j},0) = g(x_{I,j}), \qquad\qquad 0 \leq j \leq N, \quad 2 \leq i \leq M, \\ \\ \partial_t u^{N,M}(x_{I,j},0) = h(x_{I,j}). \end{cases} \tag{23}$$

The equation in (23) is valid only for $0 \leq j \leq N$ and $2 \leq i \leq M$. Indeed, since $u(x,0)$ and $\partial_t u(x,0)$ are given, the above treatment is more reasonable. It should be noted that, the scheme (23) is implicit and if H satisfies certain conditions, then (23) has a unique solution [1, 2].

We next describe the numerical implementation for Eq. (23). The unknown function $u(x,t)$ and its first and second order partial derivatives defined on the domain Ω are approximated by Eqs. (14)–(18). Substituting them into Eq. (1) and collocating the result at shifted LGR points $(x_{I,j},t_{J,i})$ for $0 \leq j \leq N$ and $2 \leq i \leq M$ we produce $(N+1)(M-1)$ collocation equations as

$$L\ \mathcal{I}^{N,M} u(x_{I,j},t_{J,i}) + H\big(u(x_{I,j},t_{J,i})\big) = f(x_{I,j},t_{J,i}). \tag{24}$$

Moreover, collocating the initial conditions (2) at shifted LGR points $x_{I,j}$, $0 \leq j \leq N$ we get

$$u(x_{I,j},0) = g(x_{I,j}), \tag{25}$$

$$\partial_t \mathcal{I}^{N,M} u(x_{I,j},0) = h(x_{I,j}). \tag{26}$$

Eqs. (24)–(26) give a system of nonlinear algebraic equations arising from the SDC discretization, where the variables are $u(x_{I,j},t_{J,i})$, $0 \leq j \leq N$, $0 \leq i \leq M$. By solving this system, we obtain the values $u^{N,M}(x_{I,j},t_{J,i})$ as approximations of $u(x_{I,j},t_{J,i})$.

Remark 2. *In most collocation methods for PDEs, the unknown function is approximated by the Lagrange interpolation [1]. As we know, the Lagrange interpolation is not stable for large M and N. However, we used the shifted Legendre interpolation in this work, which is stable for large M and N.*

3.2 MDC method

In the SDC method, for ensuring the convergence, the length of the time interval on which the underlying problems are defined should not be large. In addition, it is not convenient to resolve the corresponding discrete system (24)–(26) with very large mode M and N. In order to improve the performance of the SDC method, we propose a multi–domain version of the collocation method in this subsection. This technique simplifies computation and saves work, while keeping the spectral accuracy.

We first divide the time interval $J = [0, T]$ into a mesh consisting of K non-overlapping mesh intervals J_k, $k = 1, \ldots, K$. Then the original problem defined on the domain Ω is replaced with a sequence of local problems defined on the subdomains $\Omega_k = I \times J_k$, where $J_k = [(k-1)h, kh]$ and h is a real step-size. Replacing $u(x,t)$ and J by $u_k(x,t)$ and J_k in Eq. (23), the MDC method for solving Eq. (1) is to consecutively seek polynomials $u_k^{N,M}(x,t)$, $k = 1, \ldots, K$ such that

$$\begin{cases} L\, u_k^{N,M}\big(x_{I,j}, t_{J_k,i}\big) + H\big(u_k^{N,M}\big(x_{I,j}, t_{J_k,i}\big)\big) = f(x_{I,j}, t_{J_k,i}), \\ u_k^{N,M}(x_{I,j}, (k-1)h) = u_{k-1}^{N,M}(x_{I,j}, (k-1)h), \qquad 0 \le j \le N,\ 2 \le i \le M, \\ \partial_t u_k^{N,M}(x_{I,j}, (k-1)h) = \partial_t u_{k-1}^{N,M}(x_{I,j}, (k-1)h). \end{cases} \tag{27}$$

Note that, in the first step we have $k = 1$, $\Omega_1 = I \times [0, h]$ and

$$u_0^{N,M}(x_{I,j}, 0) = g(x_{I,j}), \quad \partial_t u_0^{N,M}(x_{I,j}, 0) = h(x_{I,j}).$$

Similar to the SDC method, the unknown functions $u_k(x,t)$, $1 \le k \le K$ and their first and second order partial derivatives defined on the subdomain Ω_k are approximated using Eqs. (14)–(18). Substituting them into Eq. (1) and collocating the result at shifted LGR points $(x_{I,j}, t_{J_k,i})$ for $0 \le j \le N$ and $2 \le i \le M$, in each step, we produce $(N+1)(M-1)$ collocation equations as

$$L\, \mathcal{I}^{N,M} u_k(x_{I,j}, t_{J_k,i}) + H\big(u_k(x_{I,j}, t_{J_k,i})\big) = f(x_{I,j}, t_{J_k,i}). \tag{28}$$

The continuity conditions are collocated at shifted LGR points $x_{I,j}$, $0 \le j \le N$ as

$$u_k(x_{I,j}, (k-1)h) = u_{k-1}(x_{I,j}, (k-1)h), \tag{29}$$

$$\partial_t \mathcal{I}^{N,M} u_k(x_{I,j}, (k-1)h) = \partial_t \mathcal{I}^{N,M} u_{k-1}(x_{I,j}, (k-1)h), \tag{30}$$

where we set $u_0(x_{I,j}, 0) = g(x_{I,j})$ and $\partial_t u_0(x_{I,j}, 0) = \partial_t h(x_{I,j})$. In each step, Eqs. (28)–(30) give a system of nonlinear algebraic equations arising from the MDC discretization, where the variables are $u_k(x_{I,j}, t_{J,i})$, $0 \le j \le N$, $0 \le i \le M$. Finally, we use Eq. (14) to obtain the approximation of $u_k(x,t)$ (denoted by $u_k^{N,M}(x,t)$) to be used in the next step.

In numerical tests we will demonstrate that the above algorithm is of the BN–stability [3]. BN–stability means that the propagated error arising from the starting error can be controlled effectively for long–time calculation.

3.3 SDIC method

In this subsection, we derive an iterative version of the SDC method, which replaces the original nonlinear problem with a sequence of linear problems using the QL technique. The iterations of the QL technique are constructed to yield rapid convergence and often monotonicity [4].

In the SDIC method, we assume that the function H in Eq. (1) has continuous first order derivative. The SDIC prescription determines the $(r+1)$th iterative approximation $u^{(r+1)}(x,t)$ to the solution of Eq. (1) as a solution of the linear PDE

$$Lu^{(r+1)} + H(u^{(r)}) + \partial_u H\Big|_{u^{(r)}} \cdot \big(u^{(r+1)} - u^{(r)}\big) = f(x,t), \qquad (x,t) \in \Omega, \tag{31}$$

with initial conditions

$$u^{(r+1)}(x,0) = g(x), \qquad \partial_t u^{(r+1)}(x,0) = h(x). \tag{32}$$

The initial guess $u^{(0)}(x,t)$ is chosen from mathematical or physical considerations and it would be beneficial if it satisfies the initial conditions.

Let

$$A^{(r)}(x,t) = \partial_u H\Big|_{u^{(r)}},$$

$$B^{(r)}(x,t) = f(x,t) - H(u^{(r)}(x,t)) + A^{(r)}(x,t)u^{(r)}(x,t).$$

In each iteration of the SDIC method, the unknown function $u^{(r+1)}(x,t)$ and its first and second order partial derivatives defined on the domain Ω are approximated by Eqs. (14)-(18) and the results are substituted into Eqs. (31)-(32) to obtain the following linear system of algebraic equations

$$\begin{cases} L\,\mathcal{I}^{N,M}u^{(r+1)}(x_{I,j},t_{J,i}) + A^{(r)}(x_{I,j},t_{J,i})u^{(r+1)}(x_{I,j},t_{J,i}) = B^{(r)}(x_{I,j},t_{J,i}), \quad 0 \le j \le N, \quad 2 \le i \le M, \\ \\ u^{(r+1)}(x_{I,j},0) = g(x_{I,j}), \qquad \partial_t \mathcal{I}^{N,M}u^{(r+1)}(x_{I,j},0) = h(x_{I,j}), \end{cases} \tag{33}$$

where the variables are $u^{(r+1)}(x_{I,j},t_{J,i})$, $0 \le j \le N$, $0 \le i \le M$. We substitute the obtained variables into Eq. (14) to obtain the approximation ${u^{(r+1)}}^{N,M}(x,t)$ of $u^{(r+1)}(x,t)$ to be used in the next iteration.

The procedure for solving the above sequence of systems of linear algebraic equations starts with $r = 0$ and an appropriate initial guess $u^{(0)}(x,t)$. Then, the procedure is repeated until a specified convergence criterion is fulfilled. In this paper, we consider the following convergence criterion,

$$\frac{\Big\|{u^{(r+1)}}^{N,M}(x,t) - {u^{(r)}}^{N,M}(x,t)\Big\|_{L^\infty(\Omega)}}{\Big\|{u^{(r+1)}}^{N,M}(x,t)\Big\|_{L^\infty(\Omega)}} < \epsilon, \tag{34}$$

where ϵ is a user defined error tolerance. In Section 4, we will discus more about the quadratic and often monotonic convergence of the QL approach applied to problem (1).

4 Convergence of the QL approach

Consider the following general model problem

$$Lu(x,t) = F\big(x,t,u(x,t)\big), \tag{35}$$

with initial conditions

$$u(x,0) = g(x), \qquad \partial_t u(x,0) = h(x), \tag{36}$$

where L is the differential operator of the wave equation defined by $L = \partial_{tt} - c^2\partial_{xx}$ and F is a nonlinear function of $u(x,t)$. As stated in Subsection 3.3, the QL approach replaces the nonlinear problem (35)-(36) with a sequence of linear problems as

$$\begin{cases} Lu^{(r+1)}(x,t) = \partial_u F\big(x,t,u^{(r)}(x,t)\big)u^{(r+1)}(x,t) + F\big(x,t,u^{(r)}(x,t)\big) - \partial_u F\big(x,t,u^{(r)}(x,t)\big)u^{(r)}(x,t), \\ \\ u^{(r+1)}(x,0) = g(x), \qquad\qquad \partial_t u^{(r+1)}(x,0) = h(x), \end{cases} \tag{37}$$

where $u^{(0)}(x,t)$ is an initial guess for $u(x,t)$.

Now, let $\delta u^{(r+1)}(x,t) = u^{(r+1)}(x,t) - u^{(r)}(x,t)$, which is the difference between two subsequent iterations. Then,

$$\begin{cases} L\delta u^{(r+1)}(x,t) = \partial_u F\big(x,t,u^{(r)}(x,t)\big)\delta u^{(r+1)}(x,t) - \partial_u F\big(x,t,u^{(r-1)}(x,t)\big)\delta u^{(r)}(x,t) \\ \qquad\qquad + F\big(x,t,u^{(r)}(x,t)\big) - F\big(x,t,u^{(r-1)}(x,t)\big), \\ \\ \delta u^{(r+1)}(x,0) = 0, \qquad\qquad \partial_t \delta u^{(r+1)}(x,0) = 0. \end{cases} \tag{38}$$

By the Taylor's expansion of the functional F around $u^{(r-1)}(x,t)$, one has

$$\begin{aligned} F\big(x,t,u^{(r)}(x,t)\big) &= F\big(x,t,u^{(r-1)}(x,t)\big) + \partial_u F\big(x,t,u^{(r-1)}(x,t)\big)\delta u^{(r)}(x,t) \\ &\quad + \frac{1}{2}\partial_{uu} F\big(x,t,\bar{u}^{(r-1)}(x,t)\big)\Big(\delta u^{(r)}(x,t)\Big)^2, \end{aligned} \tag{39}$$

where $\bar{u}^{(r-1)}(x,t)$ lies between $u^{(r)}(x,t)$ and $u^{(r-1)}(x,t)$. Thereby, Eq. (38) can be written as

$$L\delta u^{(r+1)}(x,t) = \partial_u F\big(x,t,u^{(r)}(x,t)\big)\delta u^{(r+1)}(x,t) + \frac{1}{2}\partial_{uu} F\big(x,t,\bar{u}^{(r-1)}(x,t)\big)\Big(\delta u^{(r)}(x,t)\Big)^2. \tag{40}$$

Next, we define the differential operator $\widetilde{L} = L - \partial_u F\big(x,t,u^{(r)}(x,t)\big)$. Denoting $G_{\widetilde{L}}(x,t)$ as the Green's function associated with the operator $\widetilde{L}$, one can express the solution for Eq. (40) as

$$\delta u^{(r+1)}(x,t) = \frac{1}{2}\int\int_{\Omega} G_{\widetilde{L}}(x,t)\ \partial_{uu} F\big(x,t,\bar{u}^{(r-1)}(x,t)\big)\Big(\delta u^{(r)}(x,t)\Big)^2 \mathrm{d}A. \tag{41}$$

Therefore from Eq. (41) follows

$$\Big|\delta u^{(r+1)}(x,t)\Big| \leq \frac{1}{2} M_{G_{\widetilde{L}}} M_{F_{uu}} H^2 \Big\|\delta u^{(r)}(x,t)\Big\|^2_{L^\infty(\Omega)}, \tag{42}$$

where $M_{G_{\widetilde{L}}} = \sup\limits_{(x,t)\in\Omega} \big|G_{\widetilde{L}}(x,t)\big|$, $M_{F_{uu}} = \big\|\partial_{uu} F\big(x,t,u(x,t)\big)\big\|_{L^\infty(\Omega)}$ and $H = XT$. In view of (42), the difference between the solution of subsequent iterations of Eq. (35) decreases quadratically with each iteration, provided that the quantity $\widetilde{M} := \frac{1}{2} M_{G_{\widetilde{L}}} M_{F_{uu}} H^2$ is less than one. Furthermore, a simple induction shows that

$$\Big\|\delta u^{(r+1)}(x,t)\Big\|_{L^\infty(\Omega)} \leq \widetilde{M}^{(r)} \Big\|u^{(1)}(x,t) - u^{(0)}(x,t)\Big\|^2_{L^\infty(\Omega)}.$$

Hence, the convergence also depends on $\big\|u^{(l)}(x,t) - u^{(l-1)}(x,t)\big\|_{L^\infty(\Omega)}$ for $1 \leq l \leq r$. This suggests that, if just one of the mentioned quantities is small enough, one can always hope that even if the first convergent coefficient $\big\|u^{(1)}(x,t) - u^{(0)}(x,t)\big\|_{L^\infty(\Omega)}$ is large, a well chosen initial guess $u^{(0)}(x,t)$ results in the smallness of at least one of the convergence coefficients for $l > 1$.

Table 1: L^∞–norm errors for Example 1 for $T = 1$ and $T = 5$.

	$T = 1$			$T = 5$		
$N(= M)$	SDC	MDC ($K = 2$)	SDIC	SDC	MDC ($K = 5$)	SDIC
6	2.5×10^{-4}	6.8×10^{-6}	2.5×10^{-4}	8.6×10^{-1}	8.5×10^{-4}	–
8	5.1×10^{-6}	8.6×10^{-9}	5.2×10^{-6}	6.7×10^{-2}	4.4×10^{-6}	–
10	4.0×10^{-9}	6.6×10^{-11}	4.5×10^{-9}	2.8×10^{-2}	7.9×10^{-7}	–
12	1.2×10^{-9}	4.1×10^{-12}	2.0×10^{-9}	8.9×10^{-3}	4.2×10^{-8}	–

5 Numerical examples

In this section, we present some numerical results to illustrate the efficiency, convergence and numerical stability of the SDC, MDC and SDIC methods. In addition, we assess the key characteristics of the proposed collocation methods by comparing them with each other and with other available methods in the literature.

Example 1. Consider the following nonlinear Klein−Gordon equation with quadratic nonlinearity [15, 17, 21, 22, 23, 24]:

$$\partial_{tt}u - \partial_{xx}u + \frac{\pi^2}{4}u + u^2 = x^2 \sin^2(\frac{\pi}{2}t), \quad 0 \le x \le 1, \quad 0 \le t \le T, \tag{43}$$

with initial conditions

$$u(x, 0) = 0, \qquad \partial_t u(x, 0) = \frac{\pi}{2}x.$$

The exact solution to this problem is $u(x, t) = x \sin(\frac{\pi}{2}t)$.

This problem was solved using the proposed collocation methods for $T = 1$ and $T = 5$. In the SDIC method, the initial guess, satisfying the initial conditions, was chosen as $u^{(0)}(x, t) = \frac{\pi}{2}xt$, and with $\epsilon = 10^{-8}$ the convergence of the QL scheme for $T = 1$ was achieved in four iterations. It is important to mention that, due to the oscillatory behavior of the exact solution on the larger domain $[0, 1] \times [0, 5]$, the SDIC method with the considered initial guess fails to converge (see Table 1).

Table 1 gives the L^∞–norm errors for $T = 1$ and $T = 5$. Fig. 1 shows the base 10 logarithm of the absolute errors for $T = 1$ and $T = 5$. It is seen that, in this example, the numerical results of the SDC method for $T = 1$ are analogous to the SDIC method. In addition, we observe that the errors using the MDC method are by far smaller than the errors of the SDC and SDIC methods and as the time domain of the problem becomes larger, the MDC method becomes more reliable. This demonstrates the BN−stability of the MDC method for this problem.

This problem has been solved in [17] using the optimal homotopy asymptotic method (OHAM) and in [21] using mixed Crank−Nicolson scheme and Tau method (CN−T). Moreover, in [15] the boundary integral equation method and the dual reciprocity boundary element method (BIE−DRM) have been employed for solving this problem. Comparison between the numerical results are given in Tables 2–3.

Example 2. Consider the following nonlinear Klein−Gordon equation with cubic nonlinearity [16, 25]:

$$\partial_{tt}u - \partial_{xx}u + u + u^3 = (-2 + x^2)\cosh(x+t) + x^6 \cosh^3(x+t) - 4x \sinh(x+t), \quad -1 \le x \le 1, \quad 0 \le t \le T, \tag{44}$$

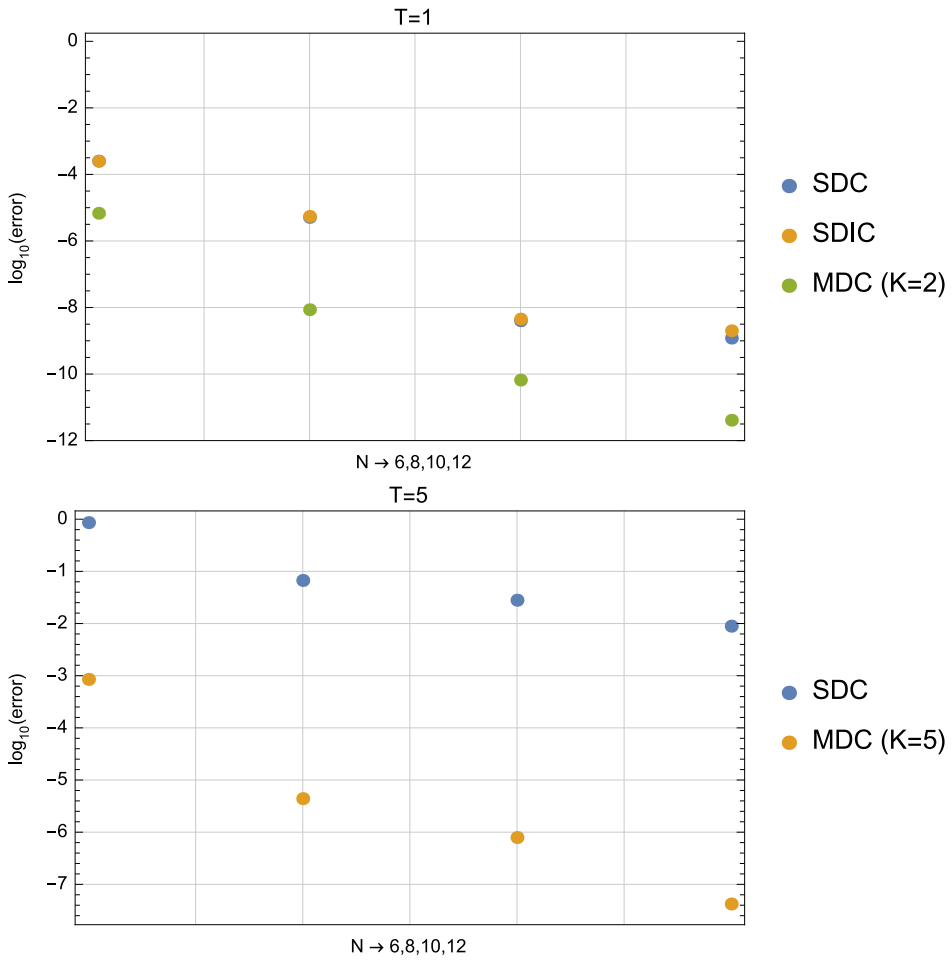

Figure 1: Comparison between errors in SDC, SDIC and MDC methods for Example 1.

Table 2: Comparison between the absolute errors at $t = 5$ for Example 1.

x	OHAM [17]	CN−T [21]	MDC ($N = 12$, $K = 5$)
0	0	0	0
0.1	3.3×10^{-3}	2.9×10^{-4}	3.3×10^{-10}
0.2	6.5×10^{-3}	4.8×10^{-4}	4.9×10^{-10}
0.3	9.3×10^{-3}	5.5×10^{-5}	6.4×10^{-10}
0.4	1.2×10^{-2}	2.6×10^{-5}	7.8×10^{-10}
0.5	1.3×10^{-2}	3.7×10^{-4}	9.1×10^{-10}
0.6	1.4×10^{-2}	1.0×10^{-4}	1.0×10^{-9}
0.7	1.4×10^{-2}	1.0×10^{-4}	1.1×10^{-9}
0.8	1.1×10^{-2}	4.5×10^{-4}	1.2×10^{-9}
0.9	7.1×10^{-3}	3.6×10^{-4}	1.3×10^{-9}
1	0	0	1.7×10^{-10}

Table 3: Comparison between the L^∞-norm errors for Example 1.

T	BIE−DRM [15]	MDC ($N=12, K=T$)
1	7.0×10^{-5}	4.1×10^{-12}
2	9.0×10^{-5}	3.2×10^{-11}
3	3.0×10^{-4}	6.0×10^{-10}
4	6.0×10^{-4}	4.6×10^{-9}
5	6.9×10^{-4}	4.2×10^{-8}

Table 4: L^∞–norm errors for Example 2 for $T=1$ and $T=5$.

	$T=1$			$T=5$	
$N(=M)$	SDC	MDC ($K=2$)	SDIC	SDC	MDC ($K=5$)
6	6.4×10^{-2}	2.4×10^{-2}	2.4×10^{-2}	3.0×10^{-1}	8.5×10^{-2}
8	6.8×10^{-4}	2.6×10^{-4}	6.8×10^{-4}	5.5×10^{-3}	9.3×10^{-4}
10	2.2×10^{-5}	1.3×10^{-5}	2.2×10^{-5}	8.9×10^{-4}	7.8×10^{-5}
12	9.1×10^{-7}	6.5×10^{-7}	6.2×10^{-7}	9.2×10^{-5}	8.6×10^{-6}

Table 5: Comparison between the L^∞-norm errors for Example 2.

T	TPS−RBF [25] ($\mathrm{d}t=0.0001,\ \mathrm{d}x=0.01$)	B−SC [16] ($h=0.02$)	MDC ($N=12, K=T$)
1	5.1×10^{-5}	1.1×10^{-5}	6.2×10^{-7}
2	5.0×10^{-4}	1.3×10^{-5}	9.4×10^{-7}
3	2.1×10^{-3}	1.6×10^{-6}	1.5×10^{-6}
4	6.6×10^{-3}	2.3×10^{-5}	5.5×10^{-6}
5	1.9×10^{-2}	3.9×10^{-5}	8.6×10^{-6}

with initial conditions

$$u(x,0)=x^2\cosh(x),\qquad \partial_t u(x,0)=x^2\sinh(x). \tag{45}$$

The exact solution to this problm is $u(x,t)=x^2\cosh(x+t)$.

Similar to the previous example, we solved this example using the proposed collocation methods for $T=1$ and $T=5$. In the SDIC method, the initial guess was the exact solution and with $\epsilon=10^{-8}$ the convergence of the QL scheme was achieved in four iterations.

Table 4 gives the L^∞–norm errors for $T=1$ and $T=5$. Moreover, Fig. 2 shows the base 10 logarithm of the absolute errors for $T=1$ and $T=5$. It is observed that, in this example, for $T=1$ the numerical results of the three proposed methods are almost the same. Therefore, in the domain $\Omega=[-1,1]\times[0,1]$, the SDC method is preferable as it has less computational complexity compared with the MDC and SDIC methods. However, we observe that in larger domains (e.g. $\Omega=[-1,1]\times[0,5]$) the MDC method may be more efficient and it provides more accurate numerical results. This also demonstrates the BN−stability of the MDC method for this problem.

In Table 5 we compare the errors obtained using the MDC method with the results in [16] and [25]. The method in [16] uses cubic B−spline collocation method (B−SC) on the uniform mesh points and both Dirichlet and Neumann boundary conditions are considered. The method in [25] uses the collocation points and approximates the solution using Thin Plate Splines (TPS) radial basis functions (RBF).

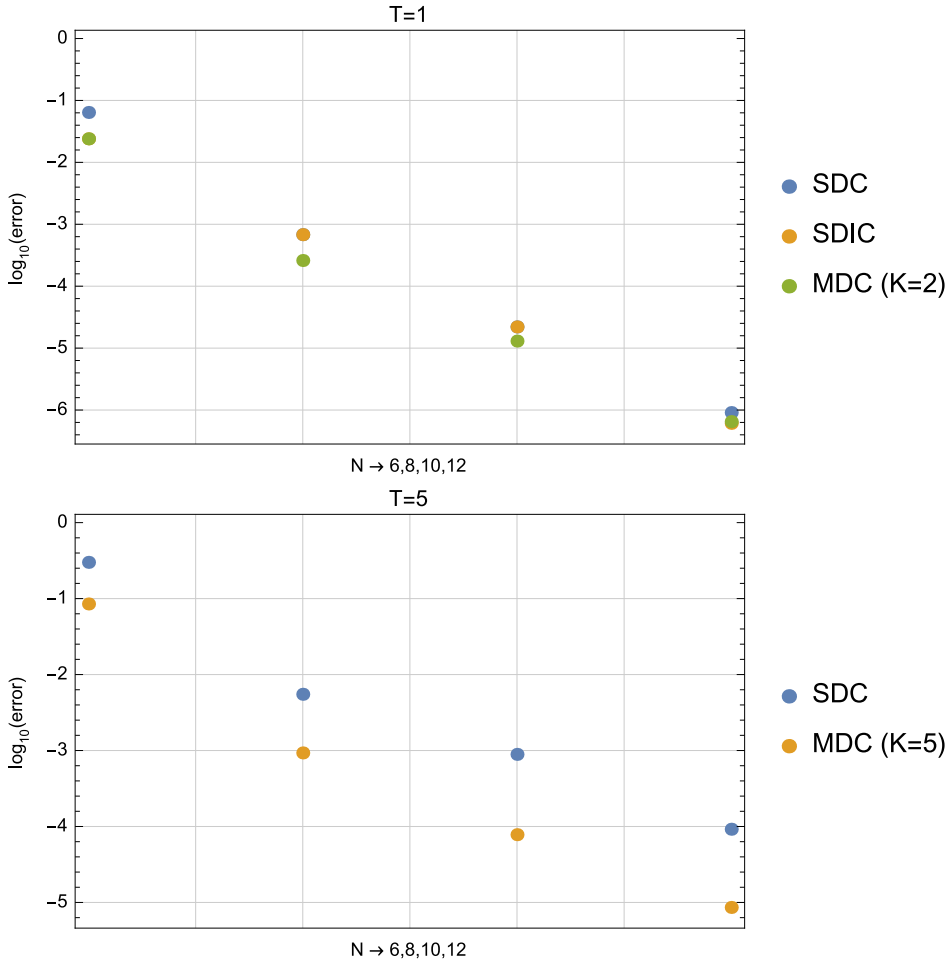

Figure 2: Comparison between errors in SDC, SDIC and MDC methods for Example 2.

Table 6: L^∞-norm errors for Example 3.

$N(=M)$	SDC ($T=0.1$)	SDC ($T=0.5$)
8	4.5×10^{-5}	1.1×10^{-2}
10	7.9×10^{-6}	5.4×10^{-3}
12	1.5×10^{-6}	2.5×10^{-3}
14	2.4×10^{-7}	8.6×10^{-4}

Table 7: Comparison between the absolute errors for Example 3 at $x=0.1$ and different values of t.

t	MADM [28]	VIM [27]	VHPM [26]	SDC ($N=14$)
0.01	1.925×10^{-4}	4.974×10^{-7}	5.007×10^{-8}	1.7×10^{-10}
0.02	3.926×10^{-4}	3.978×10^{-6}	4.003×10^{-7}	3.3×10^{-10}
0.03	6.079×10^{-4}	1.341×10^{-5}	1.350×10^{-6}	4.9×10^{-10}
0.04	8.453×10^{-4}	3.176×10^{-5}	3.195×10^{-6}	6.4×10^{-10}
0.05	1.112×10^{-3}	6.195×10^{-5}	6.229×10^{-6}	7.8×10^{-10}
0.06	1.413×10^{-3}	1.069×10^{-4}	1.074×10^{-5}	9.1×10^{-10}
0.07	1.757×10^{-3}	1.694×10^{-4}	1.701×10^{-5}	1.0×10^{-9}
0.08	2.147×10^{-3}	2.523×10^{-4}	2.531×10^{-5}	1.1×10^{-9}
0.09	2.591×10^{-3}	3.583×10^{-4}	3.592×10^{-5}	1.2×10^{-9}
0.1	3.092×10^{-3}	4.901×10^{-4}	4.909×10^{-5}	1.3×10^{-9}

Example 3. Consider the following nonlinear Sine-Gordon equation [26, 27, 28, 29]:

$$\partial_{tt}u-\partial_{xx}u+\sin(u)=0, \tag{46}$$

with initial conditions

$$u(x,0)=0,\qquad \partial_t u(x,0)=4\ \mathrm{sech}(x),\qquad -1\le x\le 1. \tag{47}$$

The exact solution to this problem is $u(x,t)=4\tan^{-1}\big(t\ \mathrm{sech}(x)\big)$.

This problem was solved for $T=0.1$ and $T=0.5$ and it was observed that the three methods provide quite similar results in this example. In the SDIC method, the initial guess of the QL scheme is considered as $u^{(0)}(x,t)=4t\ \mathrm{sech}(x)$ and for $\epsilon=10^{-10}$ the convergence is achieved in only two iterations.

In Table 6, the numerical results of the proposed SDC method are listed, which demonstrate the spectral accuracy of this method. The comparison between absolute errors of exact solution with the 5-terms of modified adomian decomposition method (MADM) [28], 2-iteration solution of the variational iteration method (VIM) [27], and 4-iteration solution of variational homotopy perturbation method (VHPM) [26] is given in Table 7.

In [29], this problem was considered with additional Dirichlet boundary condition

$$u(x,t)=h(x,t),\quad x\in\partial\Omega,\quad 0<t\leqslant T.$$

and it was solved using a TPS−RBF collocation method. The L^∞-norm errors for $t=0.25,0.5,0.75$ and 1 with considering the above Dirichlet boundary conditions are reported in Table 8. It is seen that our SDC method is more accurate than the TPS−RBF collocation method.

Table 8: L^∞-norm errors at t for Example 3.

t	TPS−RBF [29] ($\mathrm{d}t = 0.0001, \mathrm{d}x = 0.04$)	T	SDC ($N = 14$)
0.25	5.9×10^{-6}	0.25	8.1×10^{-7}
0.5	2.0×10^{-5}	0.5	1.4×10^{-6}
0.75	3.6×10^{-5}	0.75	5.4×10^{-6}
1	5.1×10^{-5}	1	9.2×10^{-6}

6 Conclusions

A unified framework has been presented for the numerical solution of Klein–Gordon equations using LGR collocation and single–domain, multi–domain and single–domain–iterative schemes. Instead of Lagrange interpolation, two–dimensional direct Legendre interpolation has been utilized which makes the proposed methods numerically more stable. It was demonstrated that, in small domains, the SDC and SDIC schemes provide almost the same numerical results for nonlinear Klein–Gordon equations. However, for highly nonlinear problems the SDIC scheme may be more efficient as it replaces the original nonlinear problem with a sequence of linear problems and with an appropriate initial guess it provides monotonic and quadratic convergence. In large domains, however, the more efficiency of the MDC scheme comes to light due to its BN−stability. Three numerical examples were studied, where it was observed that the errors in approximations tend to zero at an exponential rate for all three schemes. Moreover, comparisons with other finite–element and spectral methods available in the literature demonstrated that the present methods are more accurate. Of course, obtaining some theoretical estimates for the approximation errors would be desirable, which is currently in progress.

References

[1] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer, Verlag Berlin Heidelberg (1988).

[2] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods: Fundamentals in Single Domains, Springer, Berlin (2006).

[3] Z.h. Wang, B. Guo, Legendre-Gauss-Radau collocation method for solving initial value problems of first order ordinary differential equations, J. Sci. Comput. 52 (2012) 226–255.

[4] R.E. Bellman, R.E. Kalaba, Quasilinearization and Nonlinear Boundary-Value Problems, Elsevier Publishing, New York, NY, USA, 1965.

[5] A.M. Wazwaz, New travelling wave solutions to the Boussinesq and the Klein–Gordon equations, Commun. Nonlinear Sci. Numer. Simulat. 13 (2008) 889–901.

[6] S. Abbasbandy, Numerical solution of non-linear Klein–Gordon equations by variational iteration method, Int. J. Numer. Meth. Engng. 70 (2007) 876–881.

[7] J. Li, T. Wang, Analysis of a conservative fourth-order compact finite difference scheme for the Klein–Gordon–Dirac equation, Comput. Appl. Math. 40 (114) (2021).

[8] S.G. Venkatesh, S. Raja Balachandar, S.K. Ayyaswamy, K. Krishnaveni, An efficient approach for solving Klein–Gordon equation arising in quantum field theory using wavelets, Comput. Appl. Math. 37 (2018) 81–98.

[9] N. Hayashi, P.I. Naumkin, The initial value problem for the quadratic nonlinear Klein-Gordon equation, Adv. Math. Phys, vol. 2010, Article ID 504324, 35 pages, 2010. https://doi.org/10.1155/2010/504324.

[10] I.L. Elviyanti, B.N. Pratiwi, A. Suparmi, C. Cari, The application of minimal length in Klein-Gordon equation with hulthen potential using asymptotic iteration method, Adv. Math. Phys, vol. 2018, Article ID 9658679, 8 pages, 2018. https://doi.org/10.1155/2018/9658679.

[11] A.T. Deresse, Y.O. Mussa, A.K. Gizaw, Analytical solution of two-dimensional Sine-Gordon equation, Adv. Math. Phys, vol. 2021, Article ID 6610021, 15 pages, 2021. https://doi.org/10.1155/2021/6610021.

[12] S.A. Khuri, A. Sayfy, A spline collocation approach for the numerical solution of a generalized nonlinear Klein–Gordon equation, J. Appl. Math. Comput. 216 (4) (2010) 1047–1056.

[13] M. Lakestani, M. Dehghan, Collocation and finite difference-collocation methods for the solution of nonlinear Klein–Gordon equation, J. Computer Phys. Commun. 181 (8) (2010) 1392–1401.

[14] M. Dehghan , A. Mohebbi, Z. Asgari, Fourth-order compact solution of the nonlinear Klein–Gordon equation, J. Numer. Algor. 52 (2009) 523–540.

[15] M. Dehghan , A. Ghesmati, Application of the dual reciprocity boundary integral equation technique to solve the nonlinear Klein–Gordon equation, J. Computer Phys. Commun. 181 (2010) 1410–1418.

[16] J. Rashidinia, M. Ghasemia, R. Jalilian, Numerical solution of the nonlinear Klein–Gordon equation, J. Comput. Appl. Math. 233 (2010) 1866–1878.

[17] S. Iqbal, M. Idrees, A.M. Siddiqui, A.R. Ansari, Some solutions of the linear and nonlinear Klein–Gordon equations using the optimal homotopy asymptotic method, J. Appl. Math. Comput. 216 (10) (2010) 2898–2909.

[18] J. Chen, J. Yang, H. Yang, Single soliton solutions of the coupled nonlinear Klein–Gordon equations with power law nonlinearity, J. Appl. Math. Comput. 246 (2014) 184–191.

[19] F. Yin, T. Tian, J. Song, M. Zhu, Spectral methods using Legendre wavelets for nonlinear Klein, Sine–Gordon equations, J. Comput and Appl. Math. 275 (2015) 321–334.

[20] M.S. Semary, H.N. Hassan, A new approach for a class of nonlinear boundary value problems with multiple solutions, J. Association of Arab Universities for Basic and Applied Sciences. 17 (2015) 27–35.

[21] B.N. Saray, M. Lakestani, C. Cattani, Evaluation of mixed Crank–Nicolson scheme and Tau method for the solution of Klein–Gordon equation, J. Appl. Math. Comput. 331 (2018) 169–181.

[22] E.Y. Deeba, S.A. Khuri, A decomposition method for solving the nonlinear Klein–Gordon equation, J. Comput. Phys. 124 (4) (1996) 42–48.

[23] A.S.V. Ravi Kanth, K. Aruna, Differential transform method for solving the linear and nonlinear Klein–Gordon equation, Comput. Phys. Commun. 180 (5) (2009) 708–711.

[24] E.S. Suharto, N.I.Z. Othman, M.A.S. Nasir, Nonstandard finite difference scheme associated with harmonic mean averaging for the nonlinear Klein–Gordon equation, J. Advanced Research in Applied Sciences and Engineering Technology. 6 (1) (2017) 9–19.

[25] M. Dehghan, A. Shokri, Numerical solution of the nonlinear Klein–Gordon equation using radial basis functions, J. Comput. Appl. Math. 230 (2) (2009) 400–410.

[26] A.Y. Majeed, A.M. Bewar, Approximate solutions for solving the Klein–Gordon and sine–Gordon equations, J. Arab Universities for Basic and Applied Sciences. 22 (2017) 83–90.

[27] B. Batiha, M.S.M. Noorani, I. Hashim, Numerical solution of sine-Gordon equation by variational iteration method, Physics Letters A. 370 (5-6) (2007) 437–440.

[28] D. Kaya, A numerical solution of the sine–Gordon equation using the modified decomposition method, J. Appl. Math. Comput. 143 (23) (2003) 309-317.

[29] M. Dehghan, A. Shokri, A numerical method for one-dimensional nonlinear Sine–Gordon equation using collocation and radial basis functions, Numer. Methods Partial Diff. Eq. 24 (2)(2008) 687–698.

[30] M.O. Korpusov, A.N. Levashov, D.V. Lukyanenko, Analytical-Numerical Study of Finite-Time Blow-up of the Solution to the Initial-Boundary Value Problem for the Nonlinear Klein-Gordon Equation, Comput. Math. Math. Phys. 90 (2020) 1452–1460.

[31] E.G. Ekomasov, R.K. Salimov, Pseudo-spectral fourier method as applied to finding localized spherical soliton solutions of (3+1)-dimensional Klein-Gordon equations, Comput. Math. Math. Phys. 56 (2016) 1604–1610.