Dual Steffensen-Popoviciu measure in plane

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Dual Steffensen-Popoviciu measure is investigated in plane. Several properties and results introduce in this setting. Suitable examples are also involved.

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1. Introduction

The dual Steffensen-Popoviciu measure is an extension of the Steffensen-Popoviciu measure, which is used in convex analysis. While the original measure deals with convex functions, the dual measure focuses on concave functions. The convex and concave functions play important role in many mathematics topics, especialy in the optimization problems. The study of convex functions encompasses different types, including strictly convex, strongly convex, preinvex, proper convex, uniformly convex, and concave functions, see [3, 8–10] and the references therein.The integral of a nonnegative convex or concave functions, in terms of an arbitrary measure, may not be necessarily nonnegative, but their integral in terms of Steffensen-Popoviciu measures and dual Steffensen-Popoviciu measures are nonnegative. Characterization of Steffensen-Popoviciu measures investigated by T. Popoviciu in [7] and A. M. Fink in [2]. Then C. P. Niculescu in [6] introduced the notion of dual Steffensen-Popoviciu measure on interval and obtained sufficient condition for dual Steffensen-Popoviciu measure.

First we recall some required notions and results in the literature which we need throughout the paper. Let I be an interval of real numbers and let $C(I)$ be the set of all real-valued continuous functions on I, C be all of real-valued continuous convex functions on I, $-C$ be all of real-valued continuous concave functions on I and $A(I)$ be all of real-valued continuous affine functions on I. We recall the definition of Steffensen-Popoviciu measure and dual Steffensen-Popoviciu from [4, 6]:

Definition 1.1 *A real Borel measure μ on $I = [a, b]$ is said to be*

(1) Steffensen-Popoviciu measure provided that,

i) $\mu(I) > 0$,

ii) $\int_a^b f(x)d\mu(x) \geqslant 0$ *for every nonnegative* $f \in C$.

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(2) dual Steffensen-Popoviciu measure provided that,
i) $\mu(I) > 0$,
ii) $\int_a^b f(x)d\mu(x) \geqslant 0$ *for every nonnegative* $f \in -C$.

It is clear that every finite positive Borel measure is both Steffensen-Popoviciu measure and dual Steffensen-Popoviciu measure.

The following result from T. Popoviciu [7] and A. M. Fink [2] gives a characterization of the Steffensen-Popoviciu measures. See also [5, p177], for details.

LEMMA 1.2 *Let* μ *be a real Borel measure on interval* $[a, b]$ *such that* $\mu([a, b]) > 0$. *Then* μ *is a Steffensen-Popoviciu measure if and only if, it verifies the following conditions*

$$\int_a^t (t-x)d\mu(x) \geqslant 0 \quad \textit{and} \quad \int_t^b (x-t)d\mu(x) \geqslant 0$$

for every $t \in [a, b]$.

The measures

$$(x^2 - 1/6)^3 dx \quad on \quad [-1, 1]$$
$$\left[(\frac{2x-a-b}{b-a})^2 + \lambda\right] dx \quad on \quad [a, b] \quad (\lambda \geqslant -\frac{1}{4})$$
$$\left[(\frac{2x-a-b}{b-a})^2 - \lambda(\frac{2x-a-b}{b-a})\right] dx \quad on \quad [a, b] \quad (|\lambda| \leqslant \frac{2}{3})$$

are examples of Steffensen-Popoviciu measures, see [6]. Moreover we recall the following examples of dual Steffensen-Popoviciu measures on an interval $[a, b]$ (see [4] and [6]).

Example 1.3 The three measures

$$-\delta_a + \delta_{\frac{3a+b}{2}} + \delta_{\frac{a+b}{2}} + \delta_{\frac{a+3b}{2}} - \delta_b$$
$$\left[6\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2\right] dx$$
$$\left[\left(\frac{2x-a-b}{b-a}\right)^2 + \lambda\right] dx \quad for \lambda \geqslant -\frac{1}{6}$$

are dual Steffensen-Popoviciu measures interval $[a, b]$.

We also need the notion of concave on the co-ordinates function, see [1].

DEFINITION 1.4 *The function* $f : [a, b] \times [c, d] \to \mathbb{R}$ *is called concave on the co-ordinates if the partial mappings* $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ *and* $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ *be concave for every* $x \in [a, b]$ *and* $y \in [c, d]$ *respectively.*

It is clear that every two variable concave function on a rectangle in $\mathbb{R}^2$, is concave on the co-ordinates function. See [1]. Motivated by the above studies we made dual Steffensen-Popoviciu measures in dimension 2 and prove some inequalities by dual Steffensen-Popoviciu measures in plane.

2. Main Results

In this section we introduce generalization of dual Steffensen-Popoviciu measure in case of two dimension and obtain some new inequalities by using these measures. Now we introduce the notion of daul Steffensen-Popoviciu measure in plane. Let D be a nonempty compact convex subset of $\mathbb{R}^2$, and let $C(D)$ be the space of all two variable real-valued continuous functions on D and $-C$ be all of two variable real-valued continuous concave functions on D.

DEFINITION 2.1 *A dual Steffensen-Popoviciu measure on D is a real Borel measure μ on D such that*

i) $\mu(D) = \int\int_D d\mu(x,y) > 0$,
ii) $\int\int_D f(x,y)d\mu(x,y) \geqslant 0$ *for every nonnegative* $f \in -C$.

In what follows we introduce some results and properties of daul Steffensen-Popoviciu measures in this setting.

THEOREM 2.2 *Let $p(x)dx$ and $q(x)dx$ be two daul Steffensen-Popoviciu measure on interval $[a,b]$ and $[c,d]$ respectively. Then $\mu(x,y) = (p(x)+q(y))dxdy$ is a daul Steffensen-Popoviciu measure on $[a,b]\times[c,d]$.*

Proof *At first*

$$\begin{aligned}
\mu([a,b]\times[c,d]) &= \int_c^d \int_a^b (p(x)+q(y))dxdy \\
&= \int_c^d \int_a^b p(x)dxdy + \int_c^d \int_a^b q(y)dxdy \\
&= (d-c)\int_a^b p(x)dx + (b-a)\int_c^d q(y)dy > 0,
\end{aligned}$$

as a sum of nonnegative numbers. Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a nonnegative continuous real-valued concave function, so f is concave on coordinates. Then

$$f(tx_1 + (1-t)x_2, y) \geqslant tf(x_1,y) + (1-t)f(x_2,y),$$

for every $x_1, x_2 \in [a,b]$, $y \in [c,d]$ and $t \in [0,1]$. By integrating the above inequality we have

$$\int_c^d f(tx_1+(1-t)x_2,y)dy \geqslant t\int_c^d f(x_1,y)dy + (1-t)\int_c^d f(x_2,y)dy,$$

Thus the map

$$x \to \int_c^d f(x,y)dy,$$

is a nonnegative concave function. Similarly the map

$$y \to \int_a^b f(x,y)dx,$$

is also a nonnegative concave function. Therefore

$$\int_a^b \int_c^d f(x,y)(p(x)+q(y))dydx$$
$$= \int_a^b \left(\int_c^d f(x,y)dy \right) p(x)dx + \int_c^d \left(\int_a^b f(x,y)dx \right) q(y)dy \geqslant 0$$

as a sum of nonnegative numbers. ■

Remark 1 It is easy to see that $(\alpha p(x) + \beta q(y))dxdy$ is also a daul Steffensen-Popoviciu measure on $[a,b] \times [c,d]$ for every $\alpha, \beta \geqslant 0$.

Example 2.3 According to Example 1.3, $\left[(\dfrac{2x-a-b}{b-a})^2 + \lambda \right] dx$ is a daul Steffensen-Popoviciu measure on $[a,b]$ for every $\lambda \geqslant -\frac{1}{6}$, so by Theorem 2.2

$$\left[(\frac{2x-a-b}{b-a})^2 + (\frac{2y-c-d}{d-c})^2 + \gamma \right] dxdy,$$

is a daul Steffensen-Popoviciu measure on $[a,b] \times [c,d]$ for every $\gamma \geqslant -\frac{1}{3}$.

THEOREM 2.4 *Let $p(x)dx$ and $q(y)dy$ be daul Steffensen-Popoviciu measure on intervals $[a,b]$ and $[c,d]$ respectively. Then*

$$\mu(x,y) = p(x)q(y)dxdy$$

is a daul Steffensen-Popoviciu measure on $[a,b] \times [c,d]$ if at least one of the functions $p(x)$ or $q(y)$ is nonnegative.

Proof

$$\mu([a,b] \times [c,d]) = \int_c^d \int_a^b p(x)q(y)dxdy$$
$$= \int_c^d \left(\int_a^b p(x)dx \right) q(y)dy = \left(\int_a^b p(x)dx \right) \left(\int_c^d q(y)dy \right) > 0$$

as a product of positive numbers. Now let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a nonnegative continuous concave function, without loss of generality let p be a nonnegative function on $[a,b]$. It is clear that the map

$$y \to \int_a^b f(x,y)p(x)dx,$$

is a nonnegative concave function on interval $[c,d]$. Then

$$\int_c^d \int_a^b f(x,y)p(x)q(y)dxdy = \int_c^d \left(\int_a^b f(x,y)p(x)dx \right) q(y)dy \geqslant 0.$$

Therefore the $\mu(x,y) = p(x)q(y)dxdy$ is a daul Steffensen-Popoviciu measure on $[a,b] \times [c,d]$. ■

These kind of measures also works on triangular domains of the form $\{(x,y) : x \geqslant 0, y \geqslant 0, x+y \leqslant c\}$ for $c > 0$.

Example 2.5 According to Example 1.3 and by applying Theorem 2.4 it is easy to see that

$$\left(\frac{2x-a-b}{b-a}\right)^2\left(\left(\frac{2y-c-d}{d-c}\right)^2+\lambda\right)dxdy$$

is a daul Steffensen-Popoviciu measure on $[a,b]\times[c,d]$ for every $\lambda \geqslant -\frac{1}{6}$.

Remark 2 Let $p(x)dx$ and $q(y)dy$ be two dual Steffensen-Popoviciu measures as in Theorem 2.4 and $f:[a-d,b-c]\to\mathbb{R}$ be a nonnegative concave function of class C^2 whose second derivative is also concave. Then

$$\int_a^b\int_c^d f(x-y)p(x)q(y)dydx \geqslant 0.$$

In next theorem we obtain a daul Steffensen-Popoviciu measure under certain conditions.

THEOREM 2.6 *Let $p(x)dx$ and $q(y)dy$ be daul Steffensen-Popoviciu measures on intervals $[a,b]$ and $[c,d]$ respectively. Then $\mu(x,y) = \frac{p(x)}{q(y)}dxdy$ is a daul Steffensen-Popoviciu measure on $[a,b]\times[c,d]$ if $q(y)$ be a positive function on interval $[c,d]$.*

Proof

$$\begin{aligned}\mu([a,b]\times[c,d]) &= \int_c^d\int_a^b \frac{p(x)}{q(y)}dxdy\\ &=\int_c^d\left(\int_a^b p(x)dx\right)\frac{1}{q(y)}dy=\left(\int_a^b p(x)dx\right)\left(\int_c^d\frac{1}{q(y)}dy\right)>0,\end{aligned}$$

as a product of positive numbers. Now let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a nonnegative continuous concave function. Hence, the map

$$x\to\int_c^d\frac{f(x,y)}{q(y)}dy,$$

is a nonnegative concave function on interval $[a,b]$. Indeed,

$$\begin{aligned}\int_c^d\frac{f(tx_1+(1-t)x_2,y)}{q(y)}dy &\geqslant \int_c^d\frac{tf(x_1,y)+(1-t)f(x_2,y)}{q(y)}dy\\ &= t\int_c^d\frac{f(x_1,y)}{q(y)}dy+(1-t)\int_c^d\frac{f(x_2,y)}{q(y)}dy,\end{aligned}$$

for every $x_1,x_2\in[a,b]$ and $t\in[0,1]$.

Since $p(x)dx$ is a daul Steffensen-Popoviciu measure on $[a,b]$ we have

$$\int_c^d\int_a^b f(x,y)\frac{p(x)}{q(y)}dxdy=\int_a^b\left(\int_c^d\frac{f(x,y)}{q(y)}dy\right)p(x)dx\geqslant 0.$$

Therefore $\mu(x,y)=\frac{p(x)}{q(y)}dxdy$ is a daul Steffensen-Popoviciu measure on $[a,b]\times[c,d]$. ■

By using the Example 1.3, Remark *1*, Theorem 2.6 and choosing

$$p(x)dx := \left(\left(\frac{2x-a-b}{b-a} \right)^2 + \lambda \right) dx,$$

and

$$q(y)dy := \left(\frac{2y-c-d}{d-c} \right)^2 dy,$$

it is easy to show that

$$\left(\left(\frac{2x-a-b}{2y-c-d} \right)^2 + \lambda \left(\frac{b-a}{2y-c-d} \right)^2 \right) dxdy,$$

is a daul Steffensen-Popoviciu measure on $[a,b] \times [c,d]$ for every $\lambda \geqslant -\frac{1}{6}$.

Now we are in position to investigate dual Steffensen-Popoviciu measure on a compact disc.

PROPOSITION 2.7 *Let $p(x)dx$ be a dual Steffensen-Popoviciu measure on interval $[0,R]$, then $\frac{p(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}dxdy$ is a dual Steffensen-Popoviciu measure on the compact disc $\bar{D}_R(0) = \{(x,y) : x^2+y^2 \leqslant R\}$.*

Proof *By apply converting from rectangular to polar coordinates we have*

$$\iint_{\bar{D}_R(0)} \frac{p(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} dxdy = \int_0^{2\pi} \int_0^R \frac{p(r)}{r} r dr d\theta = 2\pi \int_0^R p(r)dr > 0.$$

Now let f be a nonnegative real-valued continuous concave function on compact disc $\bar{D}_R(0)$. In the other hand $p(r)dr$ and $1d\theta$ are two dual Steffensen-Popoviciu measures on intervals $[0,R]$ and $[0,2\pi]$ respectively wich satisfy in Theorem 2.4. Hence $p(r)drd\theta$ is a dual Steffensen-Popoviciu measure on $\bar{D}_R(0)$. Therefore

$$\iint_{\bar{D}_R(0)} f(x,y) \frac{p(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} dxdy = \int_0^{2\pi} \int_0^R f(r,\theta) p(r) dr d\theta \geqslant 0,$$

which is required. ∎

Example 2.8 By choosing $a := 0$ and $b := 1$ in Example 1.3, $[(2x-1)^2 + \lambda]dx$ is a dual Steffensen-Popoviciu measure on interval $[0,1]$ for every $\lambda \geqslant -\frac{1}{6}$ and by applying Proposition 2.7 we have

$$\frac{(2\sqrt{x^2+y^2}-1)^2 + \lambda}{\sqrt{x^2+y^2}} dxdy,$$

is a dual Steffensen-Popoviciu measure on unit compact disc $\bar{D}_1(0)$.

References

[1] S.S. Dragomir, *On the Hadamard's inequlality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese journal of mathematics, 2001 Dec 1:775-88.

[2] A.M. Fink, *A best possible Hadamard inequality*, Math. Inequal. Appl, 1998;1(2):223-30.

[3] S. Nazari Pasari, A. Barani, N. Abbasi, *Generalized integral Jensen inequality*, Journal of Inequalities and Applications, 2024 Feb 20;2024(1):25.

[4] C.P. Niculescu, *On a result of G. Bennett*, Bulletin mathmatique de la Socit des Sciences Mathmatiques de Roumanie, 2011 Jan 1:261-7.

[5] C.P. Niculescu, L.E. Persson, *Convex functions and their applications*, New York: Springer, 2006.

[6] C.P. Niculescu, M.M. Stanescu, *The SteffensenPopoviciu measures in the context of quasiconvex functions*, J. Math. Inequal, 2017 Jun 1;11:469-83.

[7] T. Popoviciu, *Notes sur les fonctions convexes d'ordre suprieur (IX)*, Bulletin Mathmatique de la socit Roumaine des Sciences, 1941 Jan 1;43(1/2):85-141.

[8] H.M. Srivastava, S. Mehrez, S.M. Sitnik, *Hermite-Hadamard-type integral inequalities for convex functions and their applications*, Mathematics, 2022 Aug 31;10(17):3127.

[9] T. Weir, V Jeyakumar, *A class of nonconvex functions and mathematical programming*, Bulletin of the Australian Mathematical Society, 1988 Oct;38(2):177-89.

[10] T. Weir, B. Mond, *Pre-invex functions in multiple objective optimization*, Journal of Mathematical Analysis and applications, 1988 Nov 15;136(1):29-38.