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Controlling Desirable Outputs and Pollutants Using a Multi-Objective Function in an Inverse-DEA Model

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Abstract
Inverse Data Envelopment Analysis (DEA) is a mathematical technique for assessing relative efficiencies of homogeneous decision-making units (DMUs) based on multiple inputs to multiple outputs. Inverse DEA is an emerging theoretical and methodological technique continuously evolving and substantially impacting operations research, economics, and efficiency analyses. It has emerged as a valuable post-DEA sensitivity analysis approach for resource allocation and efficiency optimization. In this article, the Slacks-Based Measure (SBM) DEA model has been developed to address limitations in traditional DEA models, particularly in evaluating environmental efficiency and undesirable outputs in various applications, including environmental policy analysis and performance assessment of organizations. In the first objective, the issue of minimizing the increase in inputs is addressed while also taking into account the minimum increase in undesirable output. Hence, the models previously presented in this article attempt to control the increase of inputs and possibly reduce them by considering a multi-objective function.

Keywords: Data Envelopment analysis, Undesirable outputs, Invers DEA

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1. Introduction

Data Envelopment Analysis (DEA), with its roots tracing to the mid-1970s, is a mathematical technique for assessing relative efficiencies of homogeneous decision-making units (DMUs) based on multiple inputs to multiple outputs. The term came into play via the seminal works of Charnes et al. (1978). The first application of DEA upon higher education institutions (HEIs) accounted for by Allen et al. (1993) could inadvertently cause misunderstandings, although it proves a milestone [1]. For instance, it included a single input and output variable normally viewed as the academic model. Although not customized for the conventional DEA approach to inefficiency nature in academia, simple formulae accounted for the model while offering indistinct insights into education opportunities (a further complication in appraising efficiency in higher state institutions). Inverse DEA is an emerging theoretical and methodological technique continuously evolving and substantially impacting operations research, economics, and efficiency analyses. Inverse DEA proved applicable in different contexts and circumstances to account for decision-makers' deterministic preferences in analyzing efficient patterns by examining the tension between the practical and theoretically the best possible. These factors substantiated the usefulness of inverse DEA to facilitate conventional DEA problems based on real activities and offer opportunities for further development in the inverse DEA framework. Castro and Reson's (2015) proposition outlined the possibility of establishing economic attributes because inverse DEA focuses on studying causes as opposed to the traditional DEA modes that establish effects referenced above the firm performance [2].

Inverse Data Envelopment Analysis (DEA) has emerged as a valuable post-DEA sensitivity analysis approach for resource allocation and efficiency optimization [3]. While traditional inverse DEA models focus on radial efficiency measures, recent developments have introduced non-radial approaches, such as the inverse Slack-Based Measure (SBM) model, which considers slacks and provides more comprehensive information for decision-making [4]. Hosseininia & Saen (2020) proposed a novel inverse SBM model that maintains relative efficiency of decision-making units (DMUs) with new inputs and outputs, offering a linear programming solution to the multi-objective non-linear problem [5]. Furthermore, Lim (2016) introduced an inverse DEA method that incorporates expected frontier changes, enhancing its applicability for new product target setting [6]. These advancements in inverse DEA have expanded its potential for solving various optimization problems across sectors such as banking, energy, education, and supply chain management [3]. Hosseinzadeh Lotfi et al. (2023) proposed a basic DEA-R model without explicit inputs is formulated and the relation between output-oriented DEA models without explicit inputs and output-oriented DEA-R models is analyzed. They evaluated 41 Chinese commercial banks in DEA and DEA-R models in the input and output oriented [7]. Younesi and Hosseinzadeh Lotfi (2023) deal with an inverse data envelopment analysis (DEA) based on the non-radial slacks-based model in the presence of uncertainty employing both integer and continuous interval data. To this matter, suitable technology and formulation for the DEA are proposed using arithmetic and partial orders for interval numbers. The Slacks-Based Measure (SBM) Data Envelopment Analysis (DEA) model has been developed to address limitations in traditional DEA models, particularly in evaluating environmental efficiency and undesirable outputs [8]. Song et al. (2013)

and Wu (2010) proposed an improved SBM-DEA model (ISBM-DEA) that better accounts for undesirable outputs in efficiency assessments [9,10]. Bolós et al. (2022) addressed issues with SBM super-efficiency models, introducing a composite SBM score to resolve discontinuity problems and overestimation of efficiency scores [11]. To handle uncertain data, Mahla and Agarwal (2021) developed a fuzzy SBM DEA model using a credibility measure approach, which provides more realistic results compared to conventional DEA models [12]. These advancements in SBM-DEA models offer improved methodologies for evaluating environmental efficiency, ranking efficient decision-making units, and dealing with qualitative or uncertain data in various applications, including environmental policy analysis and performance assessment of organizations. The typical DEA mainly focuses on post-operative evaluation of an organizational performance. Sometimes economic conditions such as economic prohibitions on exports or imports are imposed on a system. These prohibitions prevent decision-making units from the best performance (efficiency one). In this case, if the system has the best performance (with a less than one efficiency score) then it will be considered as an efficient system. So, the efficiency frontier changes problem must be studied [13]. The advantage of the model presented in this article compared to previous models is that it considers the model as a bi-objective one. In the first objective, the issue of minimizing the increase in inputs is addressed while also taking into account the minimum increase in undesirable outputs. This is because increasing desirable outputs does not necessarily require an increase in inputs or undesirable outputs. In the second objective, the efficiency of the changed units should not worsen compared to before. The remaining parts of the paper are organized as follows: Section 2 contains the literature review of the inverse DEA models. Section 3 provides an improved inverse DEA mode with multi-objective. In section 4, the conclusion is given.

2. Background

Suppose that n DMUs with m inputs, s_1 desirable outputs and s_2 undesirable outputs are given and $DMU_o, o=1,\dots,n$ is the DMU under evaluation. Besides, the vectors $X_o \in \mathbb{R}^m \geq\neq 0, o=1,\dots,n$ and $Y_o{}^b \in \mathbb{R}^{s^2} \geq\neq 0, o=1,\dots,n$ are input, good $\mathbb{R}^{s^1} \in Y_o{}^g \geq\neq 0, o=1,\dots,n$ output and bad output vectors corresponding to DMU_o, respectively.

Wegener and Amin (2019) used the directional distance DEA model (1) as a baseline model for evaluating the performance of units.

$$
\begin{aligned}
\max \quad & \psi \qquad\qquad\qquad\qquad (1)\\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_j \leq x_o \\
& \sum_{j=1}^{n} \lambda_j y_j^g \geq (1+\psi) y_o^g \\
& \sum_{j=1}^{n} \lambda_j y_j^b = (1-\psi) y_o^b \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 \qquad j=1,\dots,n
\end{aligned}
$$

the optimal value of model (1), or ψ_o^*, provides a measure of the inefficiency of DMU_o.

Many oil and gas firms have the capacity to reduce their GHG emissions for their level of production, the directional distance DEA model allows us to use this capacity while simultaneously increasing production.

Let's assume we represent a set of units intended for increased production with K, and we aim to ensure that the efficiency of these units does not deteriorate during the production process. In fact, the inputs and outputs of the units in K change. We want to know how much undesirable output will be produced for a certain level of production using a set of production units located in K. To this end, we will examine two scenarios: the first scenario is when the production frontier changes after adjustments, and the second scenario is when the frontier does not change. In the first scenario, where the frontier changes, each DMU in K is represented by a convex combination of the set of efficient DMUs on F (F being the efficiency frontier before the change) and a set of DMUs in K that are located on the new frontier. However, in the case where the frontier does not change, the representation of the units in K is obtained on F.

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Assuming that there is no boundary movement after the production of additional outputs by DMUs at k, Wegener and Amin (2019) proposed the following model to minimize the increase of undesirable outputs.

$$\begin{aligned}
\min \quad & \left(\gamma_{11},\dots,\gamma_{s_2 1},\dots,\gamma_{1t},\dots,\gamma_{s_2 t}\right) && (2)\\
s.t. \quad & \\
& \sum_{j\in F}\lambda_j^k x_{ij} \le \left(x_{ik}+\alpha_{ik}\right) && \forall k\in K, i=1,\dots,m\\
& \sum_{j\in F}^{n}\lambda_j^k y_{rj}^g \ge \left(1+\hat{\psi}_k\right)\left(y_{rk}^g+\beta_{rk}\right) && \forall k\in K, r=1,\dots,s_1\\
& \sum_{j\in F}^{n}\lambda_j^k y_{rj}^b = \left(1-\hat{\psi}_k\right)\left(y_{rk}^b+\gamma_{rk}\right) && \forall k\in K, r=1,\dots,s_2\\
& \sum_{k\in k}\beta_{rk} = \bar{y}_r^g && r=1,\dots,S_1\\
& \sum_{j\in F}^{n}\lambda_j^k = 1 && \forall k\in K\\
& \lambda_j^k \ge 0,\quad \alpha_{ik}\ge 0,\quad \beta_{rk}\ge 0,\quad \gamma_{rk}\ge 0 && \forall k,\forall j
\end{aligned}$$

α_{ik} is the variable related to the amount of input added to the i-th input of DMU_K, β_{rk} represents the variable related to the increase in r-th desired output, and γ_{rk} is the variable related to the increase in r-th undesired output for producing r-th an additional desired output as much as $\bar{y}_r^g$. t is the number of units in K.

The model with the changed border is as follows:

$$\begin{aligned}
\min \quad & \left(\gamma_{11},\dots,\gamma_{s_2 1},\dots,\gamma_{1t},\dots,\gamma_{s_2 t}\right) && (3)\\
s.t. \quad & \\
& \begin{cases}\sum_{j\in F}\lambda_j^k x_{ij} + \sum_{j\in k}\hat{\lambda}_j^k (x_{ij}+\alpha_{ij})\\ \le \left(x_{ik}+\alpha_{ik}\right)\end{cases} && ,\forall k\in K, i=1,\dots,m\\
& \begin{cases}\sum_{j\in F}\lambda_j^k y_{rj}^g + \sum_{j\in k}\hat{\lambda}_j^k (y_{rj}^g+\beta_{rj})\\ \ge \left(1+\hat{\psi}_k\right)\left(y_{rk}^g+\beta_{rk}\right)\end{cases} && ,\ \forall k\in K, r=1,\dots,s_1
\end{aligned}$$

$$\begin{cases} \sum_{j\in F}\lambda_j^k y_{rj}^b + \sum_{j\in k}\hat{\lambda}_j^k y_{rj}^b \\ = \left(1-\hat{\psi}_k\right)\left(y_{rk}^b + \gamma_{rk}\right) \end{cases} \quad ,\forall k\in K, r=1,\ldots,s_2$$

$$\sum_{k\in k}\beta_{rk} = \bar{y}_r^g \qquad r=1,\ldots,S_1$$

$$\sum_{j\in F}\lambda_j^k + \sum_{j\in k}\hat{\lambda}_j^k = 1 \qquad \forall k\in K$$

$$\lambda_j^k \ge 0, \hat{\lambda}_j^k \ge 0, \alpha_{ik} \ge 0, \beta_{rk} \ge 0, \gamma_{rk} \ge 0, \forall k \forall j$$

3. Model building

3.1. Improved SBM-inverse DEA mode

However, the issue that may have been overlooked in models (2) and (3) is that an increase in the desired output in inefficient units may not require an increase in inputs or undesirable outputs. In other words, this may occur in inefficient units. Therefore, in model (3), we considered the minimization of the increase in inputs and undesirable outputs for DMU_k in the objective function. The second objective function indicates that the efficiency of DMU_k, whose output has changed, should take the maximum possible value. Hence, we presented models (4) and (5) in two scenarios.

Assuming that there is no boundary movement after the production of additional outputs by the DMUs located in k, the model will be as follows:

$$\min \quad \sum_{k\in K}\sum_{r=1}^{s_2}\gamma_{rk} + \sum_{k\in K}\sum_{i=1}^{m}\alpha_{ik} \tag{4}$$

$$\max \quad \sum_{k\in K}\hat{\psi}_k$$

$$s.t.$$

$$\begin{cases} \sum_{j\in F}\lambda_j^k x_{ij} \\ \le \left(x_{ik}+\alpha_{ik}\right) \end{cases} , \qquad \forall k\in K, i=1,\ldots,m$$

$$\begin{cases} \sum_{j\in F}\lambda_j^k y_{rj}^g \\ \ge \left(1+\hat{\psi}_k\right)\left(y_{rk}^g+\beta_{rk}\right) \end{cases} ,\forall k\in K, r=1,\ldots,s_1$$

$$\begin{cases} \sum_{j\in F}\lambda_j^k y_{rj}^b \\ = \left(1-\hat{\psi}_k\right)\left(y_{rk}^b+\gamma_{rk}\right) \end{cases} ,\forall k\in K, r=1,\ldots,s_2$$

$$\sum_{k\in K}\beta_{rk} = \bar{y}_r^g, \qquad r=1,\ldots,S_1$$

$$\hat{\psi}_k \le \psi_k^* \qquad \forall k\in K$$

$$\sum_{j\in F}\lambda_j^k = 1 \qquad \forall k\in K$$

$$\lambda_j^k \ge 0, \hat{\lambda}_j^k \ge 0, \quad \beta_{rk} \ge 0, \quad \forall k, \forall j$$

The objective of model (4) aims to maintain the efficiency of the units that participated in the additional production process close to their previous efficiency, while also considering the minimal increase in undesirable outputs and inputs.

Now, assuming that boundary movement occurs after the production of additional outputs by the DMUs at k, the model is presented as follows:

$$\min \quad \sum_{k\in K}\sum_{r=1}^{s_2}\gamma_{rk} + \sum_{k\in K}\sum_{i=1}^{m}\alpha_{ik} \tag{5}$$

$$\max \quad \sum_{k\in K}\hat{\psi}_k$$

$$s.t.$$

$$\begin{cases} \sum_{j\in F}\lambda_j^k x_{ij} + \sum_{j\in K}\hat{\lambda}_j^k (x_{ij}+\alpha_{ij}) \\ \le \left(x_{ik}+\alpha_{ik}\right) \end{cases} ,\forall k\in K, i=1,\ldots,m$$

$$\begin{cases} \sum_{j\in F}\lambda_j^k y_{rj}^g + \sum_{j\in K}\hat{\lambda}_j^k (y_{rj}^g+\beta_{rj}) \\ \ge \left(1+\hat{\psi}_k\right)\left(y_{rk}^g+\beta_{rk}\right) \end{cases} ,\forall k\in K, r=1,\ldots,s_1$$

$$\begin{cases} \sum_{j\in F}\lambda_j^k y_{rj}^b + \sum_{j\in K}\hat{\lambda}_j^k y_{rj}^b \\ = \left(1-\hat{\psi}_k\right)\left(y_{rk}^b+\gamma_{rk}\right) \end{cases} , \qquad \forall k\in K, r=1,\ldots,s_2$$

$$\sum_{k\in K}\beta_{rk}=\bar{y}_r^g, \qquad r=1,...,S_1$$
$$\hat{\psi}_k\le\psi_k^*, \qquad \forall k\in K$$
$$\sum_{j\in F}\lambda_j^k+\sum_{j\in K}\hat{\lambda}_j^k=1, \qquad \forall k\in K$$
$$\lambda_j^k\ge 0,\hat{\lambda}_j^k\ge 0, \quad \beta_{rk}\ge 0, \quad \forall k,\forall j$$

$\bar{y}_r^g$ is the total amount of increase in the r-th desired output among the DMUs that are allowed to change. $\beta_{rk}\ge 0$. However, unlike in models (2) and (3), α_{ik} and γ_{rk} allow for both increases and decreases. This is because increases in desired output and decreases in undesirable output and inputs can occur in inefficient units.

4. Conclusion

In inverse-DEA, the existence of a specific demand for more desirable outputs will undoubtedly affect the desirable inputs and outputs. Therefore, controlling undesirable outputs and pollutants is a priority for management. On the other hand, merely limiting undesirable outputs without considering the increase number of inputs, does not seem like a wise approach. Hence, the models previously presented in this article attempt to control the increase of inputs and possibly reduce them by considering a multi-objective function. As we know, in inefficient units, an increase in desirable outputs may lead to a decrease in inputs or undesirable outputs, so this issue must be accounted for in the model. Additionally, we do not want the efficiency to worsen after a certain increase in outputs by some units; therefore, we have considered a second objective function in this regard.

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