Journal of Linear and Topological Algebra
Vol. 13*, No.* 03*,* 2024*,* 197- 216
DOR:
DOI: 10.71483/JLTA.2024.1192431

Lie symmetries, homotopy, non-standard finite difference method for solution of the linear space-fractional telegraph equation

A. Mohammadi[a], S. R. Hejazi[b,*]

[a]*Faculty of Mathematical Sciences, Shahrood University of Technology, P.O. Box 36199-95161, Shahrood, Iran.*

Received 3 December 2024; Revised 17 February 2025; Accepted 21 February 2025.

Communicated by Mohammad Sadegh Asgari

Abstract. The main purpose of this paper is to use the homotopy analytical method in order to solve the linear space-fractional telegraph differential equation. Also, non-standard finite difference method has been used to solve this equation numerically. Next, the concepts of Lie symmetry is established and the symmetries of the equation are calculated. In this article, Matlab software was used for simulation. Numerical results are presented to evaluate the efficiency and usefulness of the proposed method.

Keywords: Lie symmetry, numerical solution, homotopy, non-standard finite difference.

2010 AMS Subject Classification: 76M60, 34K34, 54C40.

1. Introduction and preliminaries

One of the most important problems when dealing with a system of differential equations is finding a way to solve it. Both PDEs and ODEs are classified into two categories: linear and non-linear equations and ordinary and partial equations. Each of these categories follows a specific method for solving.

There are several different methods for solving a given system of differential equations, such as numerical methods, homotopy methods, repetition of changes etc. [8, 9, 12]. But symmetry method based on group analysis of differential equations is a powerful method which gives general solutions of a given system. Having a symmetry group of a system has many advantages. One of them is the classification of general solutions.

*Corresponding author.
E-mail address: arashmoh2019@gmail.com (Sh. Mohammadi); r_hejazi@shahroodut.ac.ir, ra.hejazi@gmail.com (S. R. Hejazi).

Print ISSN: 2252-0201
Online ISSN: 2345-5934
© 2024 IAUCTB.
http://jlta.ctb.iau.ir

When one is confronted with a complicated system of PDEs arising from some physically important problem, the discovery of any explicit solutions whatsoever is of great interest. Explicit solutions can be used as a models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant behavior of more general types of solutions. There are several methods for finding solutions of a system of differential equations in both exact and approximate cases. The method of Lie transformations (Lie symmetries or group theory of differential equations) is a strange unlimited tool in order to find exact solutions for all kinds of systems containing both ODEs or PDEs in linear or nonlinear form. Some differential operators called symmetries are found for this purpose. These operators are the infinitesimal generators corresponding to the largest group transformations acting on the independent and dependent variables of the system with the property that they transform solutions of the system to other solutions. So, if we have the group of symmetries together with an initial solution, the wide range of solutions will be given by using this method. Symmetry method for solving differential equations has powerful tools for this goal, specially when the purpose is based on finding exact solutions of a given system. One of the most important application of symmetry method is to reduce the systems of differential equations, i.e., finding equivalent systems of differential equations of simpler form, that is called reduction process. This method provides a systematic computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system. There is a lot of papers in the literature for this process and one can find the classical reduction method in [1, 5, 6, 17].

The analytical homotopy method was first proposed by Liao. This method has been successfully used to solve homogeneous and heterogeneous equations, scientific and engineering problems. The analytical homotopy method has an auxiliary parameter that shows a useful and simple method for adjusting and controlling the convergence region and the convergence rate of the series solution. Analytical solutions to non-linear problems are also possible through analytical homotopy method [2–4, 16].

Fractional partial differential equations (FPDEs) are widely used to describe various physical effects and many complex phenomena and the other various field such as electrochemistry, quantitative biology, engineering, mechanics and etc. Also the use of fractional differentiation for the mathematical modeling of real world has been widespread at the recent years. For example, the optical soliton perturbation with fractional temporal evolution is one of the viable means to address a growing problem in telecommunication industry, namely the Internet bottleneck. This problem leads to slow Internet traffic and eventually blockage of the traffic. Several mechanisms have been proposed to address this concern. One of them is to choose time-dependent coefficients of dispersion and nonlinearity. But, a better way is to consider fractional temporal evolution. Rezazadeh et. al. have been solved the corresponding fractional equation in [19]. Also The magnetic resonance imaging (MRI) is a routine example for the application of fractional derivative in biology. In this case the common feature observed in diffusion-weighted MRI of the brain is anomalous diffusion. Fractional order models of diffusion capture this tissue complexity by incorporating fractional order time and space derivatives in the governing Bloch-Torrey equation. The other example is to consider cells, bacteria, chemicals, animals and so on as particles each of which usually moves around in a random way.

A popular and important method for finding numerical solutions of a system of differential equations is based on the use of standard finite difference methods to construct discrete models of PDEs [7]. Due to the fact that standard finite difference method generally do not transfer the qualitative characteristics of the exact solution to the numerical solution, so to eliminate this shortcoming, the use of non-standard finite difference schemes can be useful. The basis of non-standard finite difference schemes is derived from exact differential schemes for a normal differential equation or part of an integer order. Non-standard finite difference schemes, in addition to the important properties of numerical methods, i.e. consistency and stability, maintain positivity and finiteness of numerical solutions. In fact, these methods have been developed as numerical methods for numerical solution of a wide range of mathematical problem models, including algebraic, differential and biological models [7, 11, 20]. A summary of the results of these projects up to 1994 is given in Mickens [12]. After that, non-standard finite difference methods is applied for solving ODEs and PDEs of fractional order numerically were increasingly implemented in areas such as mechanical system, signal processing and control [13].

Over the past few decades, fractional calculations have gained considerable importance and necessity in a wide range of applications in fields including engineering, chemistry, finance, physics, and so on. Many scientists believe that incorrect order derivatives are suitable for describing many phenomena in nature, and also provide useful tools for describing the shelf life and hereditary properties of various processes and materials [10, 14]. In fact, it has been proven that fractional order models are more suitable for describing the actual behavior of the system than the correct order models that have already been implemented. Given that obtaining the analytical solution of PDEs is a fractional order in most cases will not be easy, so using appropriate numerical methods can be used to study the qualitative behavior of fractional order systems.

In this paper, Lie symmetries, homotopy method and non-standard finite difference method are considered for space-fractional telegraph equation:

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \qquad 0 < \alpha \leq 2, \tag{1}$$

in order to find exact and numerical solutions. For this purpose some basic definitions for fractional derivatives are given in Sec. 2. The methodology of non-standard finite difference method including homotopy method with some numerical examples are coming in the third section. Finally, in the Sec. 4 symmetry operators are stablished.

2. Preparations and requirements

Some main concepts of fractional derivatives based on Grünwald-Letnikov derivative are coming below.

2.1 *Fractional Order Derivative*

As mentioned earlier, fractional calculus is a generalized form of non-integer integral and derivative of the operator ${}_aD_t^\alpha$, where a and t are operators' bounds and α is a real number. The operator of continuous derivative-integral is defined by

$${}_\alpha D_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \alpha > 0, \\ 1 & \alpha = 0, \\ \displaystyle\int_a^t (d\tau)^\alpha & \alpha < 0. \end{cases} \tag{2}$$

Next, we introduce three concepts that are highly used in fractional order derivatives.

2.1.1 Grünwald-Letnikov Fractional Derivative

The theory of fractional order derivative by Grünwald-Letnikov is analyzed as follows [18].

Assume that $f$is a continuous function. The first order derivative of function f is defined by

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h} \ . \tag{3}$$

The same formula can be used to achieve second order derivative of function f, which is

$$\begin{aligned} f''(t) &= \frac{d^2 f}{dt^2} \\ &= \lim_{h\to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \left\{ \frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h} \right\} \\ &= \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2} \ . \end{aligned} \tag{4}$$

Using equations (3) and (4), the third order derivative of function f is achieved by

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h\to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3} \ .$$

And finally, the n-th order ($n\in\mathbb{N}$) derivative of function f through induction is achieved by

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t-rh) \ , \tag{5}$$

where binomial coefficients are given as follows for constant values of n:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!},$$

and binomial coefficients for negative values of n are defined as

$$\binom{-n}{r} = \frac{-n(-n-1)\ldots(-n-r+1)}{r!}.$$

Taking equations (3) to (5) into consideration, the derivative of real fractional order α with respect to t can be written by

$$f^{(\alpha)}(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t-rh) \ .$$

Definition 2.1 Grünwald-Letnikov fractional derivative of fractional order α of function f is given by

$$ {}_{\alpha}^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t - rh) \ , \tag{6} $$

where

$$ n = \left[\frac{t-a}{h}\right], \qquad \binom{\alpha}{r} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(\alpha-r+1)}, \qquad \binom{\alpha}{0} = 1, $$

and $\left[\frac{t-a}{h}\right]$ depicts integer part of $\frac{t-a}{h}$. Here, t and a are the bounds of ${}_{\alpha}^{GL}D_t^{\alpha} f(t)$. By replacing $\omega_r^{(\alpha)}{=}(-1)^r \binom{\alpha}{r}$, Grünwald-Letnikov fractional derivative can be rewritten as

$$ {}_{a}^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{r=0}^{\left[\frac{t-a}{h}\right]} \omega_r^{(\alpha)} f(t - rh) \ . $$

For $r \in \mathbb{N} \cup \{0\}$, $\omega_r^{(\alpha)}$ are known as Grünwald-Letnikov coefficients. Using the following equation, these coefficients can be acquired recessively:

$$ \omega_r^{(\alpha)} = \left(1 - \frac{1+\alpha}{r}\right) \omega_{r-1}^{(\alpha)}, \qquad r = 1, 2, \ldots $$

where $\omega_0^{(\alpha)} = h^{-\alpha}$.

3. Methodology

The method of fininte difference scheme and homotopy will be provided in the sequel.

3.1 *Non-standard finite difference scheme*

Due to characteristics such as the simplicity of constructing and developing non-standard finite difference schemes, these schemes are employed to numerically solve various differential equations generated in relation to engineering and natural science. Historically, for each ODE with the initial condition that it has a singular solution, Mickens proved that there is an exact difference equation with zero local truncation error [7]. Nevertheless, the exact general solution to the differential equation is needed to construct the exact difference equation. Assuming that structure of standard finite difference schemes to solve differential equations may cause non-suitable behavior for the solutions (e.g., numerical instability and disheveled behavior). Mickens presented the following rules for constructing non-standard finite difference schemes:

- Non-linear terms in the differential equation should be replaced by discrete localized approximates.
- To make the first order derivative in differential equation discrete, the current denominator of the fraction, namely h, can be replaced by more complex negative functions.

- The order of the discrete derivatives must be exactly equal to the order of the corresponding derivatives of the differential equations.
- Special solutions of the differential equations should also be special solutions of the non-standard finite difference scheme.
- The scheme should preserve the qualitative behavior of exact solutions to the problem.

To describe non-standard finite difference scheme, we assume the following differential equation

$$\frac{dy}{dt}=f\left(t,y,\lambda\right),\qquad y\left(0\right)=y_0,\qquad t\in\left[0,t_f\right],\tag{7}$$

where λ is a parameter and Euler's discretization method is one of the most straightforward techniques. In this method, the derivative term of $\frac{dy}{dt}$ is replaced by $\dfrac{y\left(t+h\right)-y\left(t\right)}{h}$. However, in the Mickens scheme, this term is replaced by $\frac{y(t+h)-y(t)}{\phi(h,\lambda)}$. Next, the following discretization will be assumed $t_k = kh$, $k = 0, 1, \ldots, N$.

The non-standard finite difference scheme is constructed by two steps. The derivative in the left side of equation (1) is replaced by discretized form

$$\frac{dy}{dt}\approx\frac{y_{k+1}-y_k}{\phi\left(h,\lambda\right)},$$

where y_k is an approximate of $y\left(t_k\right)$.

The non-linear term in equation (7) is replaced by the representation of discrete localized $F\left(t, y_{k+1}, y_k, \ldots ,\lambda\right)$ which is based on some of the previous representations.

Hence, the following scheme is achieved

$$\frac{y_{k+1}-y_k}{\phi\left(h,y\right)}=F\left(t,y_{k+1},y_k,\ldots,\lambda\right).\tag{8}$$

Discretization of derivative in the left side of equation (8) is a generalized form of discretization of classic first-order derivative where the continuous function of $\phi\left(h, y\right)$ is assumed with the constant of λ and step length of h. This function should satisfy the condition $\phi\left(h, y\right) = h + O\left(h^2\right)$, $h \to 0$. This discretization of first-order derivative can be generalized to

$$\frac{dy}{dt}=\lim_{h\to0}\frac{y\left(t+\psi_1\left(h\right)\right)-y\left(t\right)}{\psi_2\left(h\right)},\tag{9}$$

where functions of ψ_1 and ψ_2 can satisfy the following condition as long as $h \to 0$

$$\psi_i\left(h\right)=h+O\left(h^2\right),\qquad i=1,2.\tag{10}$$

The following example functions are also satisfying the above conditions:

$$h,\qquad \sin\left(h\right),\qquad e^{h-1},\qquad \frac{e^{\lambda h}-1}{\lambda},\qquad \frac{1-e^{\lambda h}}{\lambda}.\tag{11}$$

By selecting each $\psi_i\left(h\right)$, optimal results for first-order derivatives can be achieved. If the function of h is placed in equation (8) instead of $\psi_i\left(h\right)$, equation (8) is altered to the definition of the ordinary derivative. In other words, $\frac{dy}{dt} = \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}$. This method of constructing discretization of derivatives can be generalized to discretization for differential equations with partial derivatives. For instance, the following discretization can be considered for the first-order partial derivative

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{u_i^{j+1} - u_i^j}{\psi(\tau)},\\ \frac{\partial u(x,t)}{\partial x} &= \frac{u_{i+1}^j - u_i^j}{\phi(h)},\end{aligned} \tag{12}$$

where u_m^n is the numerical solution in (x_m, t_n) assumed from the meshed grid of the solution domain. Similarly, the following discretization can be considered for the second-order partial derivative

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2} &= \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\psi(\tau))^2},\\ \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\phi(h))^2},\end{aligned} \tag{13}$$

where

$$\begin{aligned}\tau &= \frac{T}{N}, \qquad t_j = j\tau, \qquad j = 0,1,\ldots,N\\ h &= \frac{X}{M}, \qquad x_i = ih, \qquad i = 0,1,\ldots,M,\end{aligned}$$

and function of ψ and ϕ should satisfy the following conditions:

$$\psi(\tau) = \tau + O(\tau^2), \qquad \phi(h) = h + O(h^2), \qquad \tau \to 0, \qquad h \to 0. \tag{14}$$

In this section, linear PDEs with local and spatial fractional derivatives are solved with the aid of a non-standard finite difference scheme. Assume that Δx and Δt are the representations of local and spatial steps. Hence,

$$\Delta x = \Delta t = h, \qquad x_i = i\Delta x, \qquad t_j = j\Delta t, \qquad i = 0,\ldots,M, \qquad j = 0,\ldots,N,$$

and the functions of $\phi_3(h), \phi_2(h)$ and $\phi_1(h)$ in the denominator are chosen in the following form

$$\phi_1(h) = 2\left(e^h - 1\right), \qquad \phi_2(h) = 4\sin^2\left(\frac{h}{2}\right), \qquad \phi_3(h) = 2\sin(h), \tag{15}$$

where they satisfy the following conditions:

$$\phi_1(h) = 2h + O(h^2), \qquad \phi_2(h) = h^2 + O(h^4), \qquad \phi_3(h) = 2h + o(h^2), \qquad h \to 0.$$

3.1.1 Numerical solution of space-fractional telegraphs equation

Consider the equation (1) with respect to the following initial and boundary conditions:

$$u(x,0)=e^{x},\quad \frac{\partial u(x,0)}{\partial t}=-e^{x},\quad 0<x<1,\quad u(0,t)=e^{-t},\quad u(1,t)=e^{1-t}\ ,\ t\geq 0. \tag{16}$$

The exact solution of (16) is given as follows [15]:

$$u(x,t)=e^{x-t}\quad for\quad \alpha=1.$$

After discretization, (16) is rewritten by

$$\sum_{k=0}^{j}\omega_k^{(\alpha)}u_i^{j-k}=\frac{u_i^{j+1}-2u_i^{j}+u_i^{j-1}}{\phi_2(h)}+\frac{u_i^{j+1}-u_i^{j-1}}{\phi_1(h)}+2(\frac{u_{i+1}^{j}-u_{i-1}^{j}}{2})-u_i^{j+1}.$$

Hence,

$$u_{i+1}^{j}=\frac{\left(-\beta_1-\left(2\phi_1\omega_0^{(\alpha)}+\phi_1\phi_2)\omega_0^{(\alpha)}\right)u_i^{j}+\phi_1\phi_2(u_{i+1}^{j}+u_{i+1}^{j}\right)-(\phi_2-\phi_1)\,u_i^{j-1})}{\phi_1\phi_2-(\phi_1+\phi_2)},$$

where

$$\beta_2=\phi_1\phi_2\left(\sum_{k=1}^{j}\omega_k^{(\alpha)}u_{i-k}^{j}\right),\qquad \omega_0^{(\alpha)}=\left(\frac{\phi_1(h)}{2}\right)^{-\alpha}. \tag{17}$$

Now, the numerical solutions of the non-standard finite difference scheme are presented, and we can compare the exact solutions.

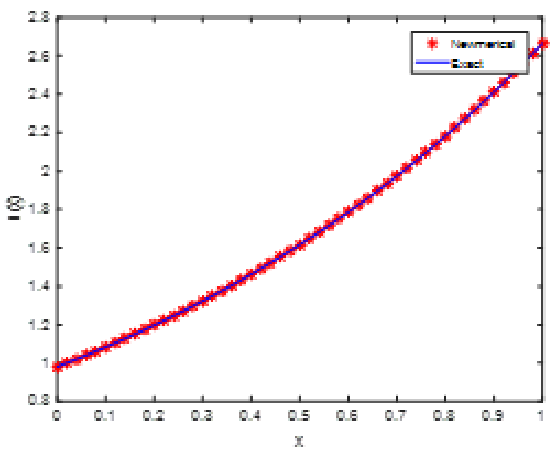

Figure 1. The comparison between numerical and exact solutions, with respect to t=0.02 and α=1.

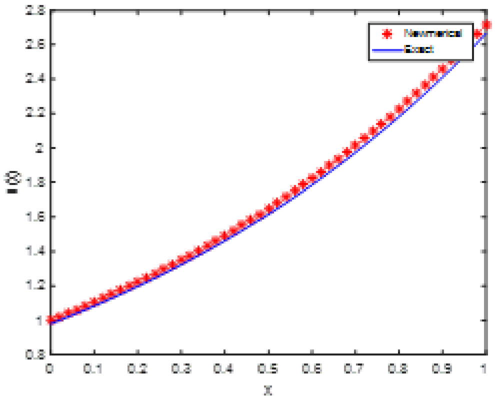

Figure 2. The comparison between numerical and exact solutions, with respect to t=0.5 and α=1.

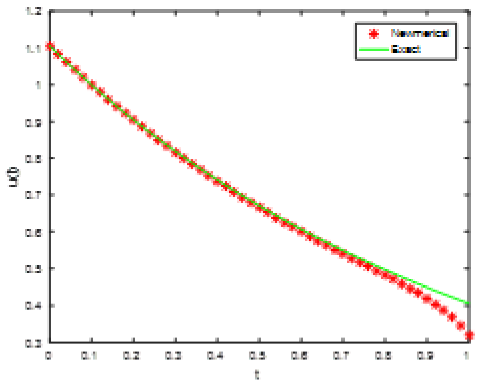

Figure 3. The comparison between numerical and exact solutions, with respect to x=0.1 and α=1.

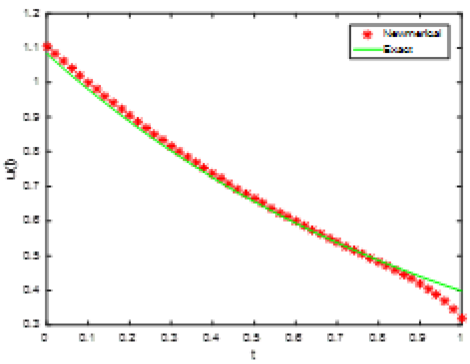

Figure 4. The comparison between numerical and exact solutions, with respect to t=0.5 and α=1.

3.2 Homotopy method

Suppose f and g are continuous function of x into y and a function like $F : X \times \{1\} \rightarrow Y$ is existed so that

$$\begin{aligned} F(x, 0) &= f(x) \\ F(x, 1) &= g(x). \end{aligned} \tag{18}$$

Then f is a homotopy to g. For example we can consider the following homotopy

$$F(x, q) = qg(x) + (1 - q) f(x), \tag{19}$$

when $q = 0$ then $F(x, 0) = f(x)$ and when $q = 1$ then $F(x, 1) = g(x)$. Hence change in parameter q from 0 to 1 makes a change in $F(x, q)$ from $f(x)$ to $g(x)$.

3.2.1 Explanation of homotopy analysis method

In 1992 professor S.J. Liao from China for the first time introduced homotopy analysis method (HAM) as an analytical method for solving linear and non-linear problems [3]. This method has effective application in linear and non-linear problems. Now we explain the method.

3.2.2 Metamorphosis zero order equation

In most cases a non-linear problem is described by a set of equations, early and boundary conditions. Here, for brevity, we consider a non-linear equation in the form of [16],

$$N[u(r, t)] = 0. \tag{20}$$

So that N is a non-linear function, $u(r, t)$ is an unknown function, t, r are accordingly dependent variable to place and time, $u(r, t)$ is the primary guess of precise solution, $h \neq 0$, $u(r, t)$ is auxiliary parameter, $H(r, t) \neq 0$ auxiliary function and L is auxiliary linear operator written by the characteristics $L[f(r, t)] = 0$. Then

$$f(r, t) = 0. \tag{21}$$

By using embedded parameter $q \in [0, 1]$ we created a homotopy as follow:

$$H[\phi(r, t; q), U_0(r, t), h, q] = (1 - q) L\ [\phi(r, t; q) - U_0(r, t)] - qhH(r, t) N[\phi(r, t; q)]. \tag{22}$$

It should be noted that having h as auxiliary parameter and auxiliary function $H(r, t)$ are very important in above homotopy. The auxiliary parameter h and the auxiliary function $H(r, t)$ play a very important role inside of HAM format.

We have to emphasize that we have a big degree of freedom in deciding about initial approximation of $u(r, t), L, h$ and $H(r, t)$. By having the above equation equal to zero, we will have metamorphosis zero order equation

$$(1-q) L\ [\phi(r, t;q) - u.(r, t)] - qhH(r, t) N[(r, t;q)] = 0. \tag{23}$$

So that $\phi(r, t; q)$ is the solution not only depends on $u_0(r, t), L, h$ and $H(r, t)$ but also depends on $q \in [0,1]$. When $q = 0$, (23) turns to

$$L \ [\phi(r,t;0) - u_0(r,t)] = 0. \tag{24}$$

By using the characteristics of operator L we conclude

$$\phi(r,t;0) = u.(r,t). \tag{25}$$

In fact is same primary function (20), provided that

$$N[\phi(r,t;1)] = u(r,t), \ \ \phi(r,t;1) = u(r,t). \tag{26}$$

In this case according to (26) when q increases from 0 to 1, $\phi(r,t;q)$ from $u(r,t)$ to precise solution $u(r,t)$ is changing. Now, we define derivative of metamorphosis of order m as

$$u^{[m]}(r,t) = \frac{\partial^m \phi(r,t;q)}{\partial q^m}|q = 0. \tag{27}$$

By Taylor proposition, we can expand $\phi(r,t;q)$ in a power of q as the following identity

$$\phi(r,t;q) = \phi(r,t;0) + \sum_{m-1}^{\infty} \frac{u^{[m]}(r,t)}{m!} q^m. \tag{28}$$

By considering

$$u^{[m]}(r,t) = \frac{u^{[m]}(r,t)}{m!} = \frac{1}{m!} \ \frac{\partial^m \phi(r,t;q)}{\partial q^m}|q = 0, \tag{29}$$

the power series (28) converts to

$$\phi(r,t;q) = u.(r,t) + \sum_{m-1}^{\infty} u_m(r,t) q^m. \tag{30}$$

Considering the vast degrees of freedom that we have in choosing $u_0(r,t), L, h$ and $H(r,t)$, we assume that all of them have been chosen rightfully, which

(1) The solution of $\phi(r,t;q)$ from metamorphosis zero order equation (23) for each $q \in [0,1]$ is existed.
(2) Metamorphosis derivative $u^{[m]}(r,t)$ for each $m = 0, 1, \ldots, \ \infty$ is existed.
(3) Power series (28) in $q = 1$ is convergent.

Then from (26) and (28) and by having these assumptions we have the solutions to $u(r,t)$ per $q = 1$ as follows

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t). \tag{31}$$

3.2.3 High order metamorphosis equation

Consider the vector

$$\overrightarrow{u_n} = \{u(r,t), u_1(r,t), \ldots,\ u_n(r,t)\}. \tag{32}$$

By considering equations (23) and (22), we derive m times relative to q and divide the solution to $m!$, then we will have m order metamorphosis equation such as

$$L[u_m\,(r,t) - \chi_m u_{m-1}(r,t)] = hH(r,t)R_m\overrightarrow{u_{m-1}}, \tag{33}$$

so that

$$R_m\overrightarrow{u_{m-1}} = \frac{1}{(m-1!)}\ \frac{\partial^{m-1}N\phi(r,t;q)}{\partial q^m}|q{=}\ 0, \tag{34}$$

where

$$\chi_m = \begin{cases} 0 & m\leq 1, \\ 1 & m>1. \end{cases} \tag{35}$$

By inserting (28) to (32) we conclude

$$R_m\overrightarrow{u_{m-1}} = \frac{1}{(m-1)!}\ \left\{\frac{\partial^{m-1}}{\partial q^m}N\left[\sum_{m-1}^{\infty} u_n(r,t)q^m\right]\right\}|q{=}\ 0. \tag{36}$$

It should be noted that high order metamorphosis equation of (33) is under the effect of operator L and $R_m\overrightarrow{u_{m-1}}$ can easily be just dependent on $\overrightarrow{u_{m-1}}$. Therefore we solve accordingly $u_1\,(r,t)\ \ u_2\,(r,t)\,,\ \ldots$ by the solution of (31). We can get m order approximation from $u(r,t)$ through the following equation

$$u\,(r,t) = \sum_{m-1}^{\infty} u_k(r,t). \tag{37}$$

Now we can create a more general zero order metamorphosis equation from (23).

Let us assume $A(q)$ and $B(q)$ are analytical complex functions in $|q|\leq 1$ area, called the embedded functions which satisfy

$$A\,(0) = B\,(0) = 0\ \ ,\ A\,(1) = B\,(1) = 1. \tag{38}$$

Maclaurin series for $A(q)$ and $B(q)$ are

$$A\,(q) = \sum_{k=1}^{\infty} a_k q^k \qquad B\,(q) = \sum_{k=1}^{\infty} b_k q^k. \tag{39}$$

Because $A(q)$ and $B(q)$ are analytical in $|q|\leq 1$ area, we conclude from (38)

$$\sum_{k=1}^{\infty} a_k = 1 \quad ,\ \sum_{k=1}^{\infty} b_k = 1. \tag{40}$$

Now we create a more general zero order metamorphosis equation as follows

$$[(1-B(q)]\{L\ [\phi(r,t;q)-u.(r,t)]\}=A(q)hH(r,t)N[\phi(r,t;q)], \tag{41}$$

and the general form of high order metamorphosis equation is

$$L[U_m(r,t)-\sum_{k-1}^{m-1} b_k U_{m-k}(r,t)]= hH(r,t)\,k_m\overrightarrow{u_{m-1}}\ (r,t), \tag{42}$$

so that

$$k_m\overrightarrow{u_{m-1}}\ (r,t)=\sum_{k-1}^{m} a_k\delta_{m-k}(r,t), \tag{43}$$

where

$$\delta_n(r,t)=\frac{1}{n!}\frac{\partial^n N[\phi(r,t,q)]}{\partial q^n}|q=0. \tag{44}$$

We can clearly see that zero order metamorphosis equation (32) and high order metamorphosis equation (33) accordingly are special kind of (41) and (42) when $B(q)=A(q)=q$.

3.2.4 Convergence of the method

In general condition we can prove that the series solution of (31) which is derived through analytical homotopy method is convergent only when it's a solution for a nonlinear problem.

Theorem 3.1 Convergence TheoremSeries $u(r,t)+\sum_{m=1}^{\infty} u_m(r,t)$ is convergence iff it is a solution of (20).

Proof. Ssee (21) for proof. ∎

3.3 *Solution of space-fractional telegraphs equation by homotopy method*

Consider the equation (10) with the following initial and boundary conditions:

$$u(x,0)=1+x,\quad 0<x<1,\qquad \frac{\partial u(0,t)}{\partial x}=-e^{-t},\qquad u(0,t)=e^{-t},\qquad t>0. \tag{45}$$

The $u_0(x,t)$ is selected as follows [15]:

$$u_0(x,t)=e^{-t}(1+x). \tag{46}$$

Consider the following linear operators

$$L\ \left[\phi(x,t;p)=D_{*x}^{0}[\phi(x,\ y;p)\right], \tag{47}$$

and

$$N\ \left[\phi\left(x,y,t;p\right)=D_{*x}^{\alpha}[\phi\left(x,\ t;p\right)]-\frac{\partial^{2}}{\partial t^{2}}\left[\phi\left(x,\ t;p\right)\right]-\frac{\partial}{\partial t}\left[\phi\left(x,\ t;p\right)\right]-\phi\left(x,\ t;p\right).\quad(48)$$

Let construct a zero-order metamorphic equation

$$\left(1-p\right)L\ \left[\phi\left(x,t;p\right)-u.\left(x,t\right)\right]=hN[\phi\left(x,t;p\right).\quad(49)$$

When $p=1,p=0$ we have

$$\phi\left(x,t;0\right)=u_{0}\left(x,t\right),\quad\phi\left(x,\ t;1\right)=u\left(x,t\right).\quad(50)$$

The m-order metamorphic equation is as follows

$$L[L[u_{m}\left(x,t\right)-\chi_{m}u_{m-1}(x,t)]=hR_{m}(u_{m-1},x,t),\quad(51)$$

where

$$\chi_{m}=\begin{cases}0 & m\leq 1,\\ 1 & m>1.\end{cases}$$

and

$$R_{m}(u_{m-1},x,t)=D_{*x}^{\alpha}u_{m-1}\left(x,t\right)-\frac{\partial^{2}u_{m-1}\left(x,t\right)}{\partial t^{2}}-\frac{\partial u_{m-1}\left(x,t\right)}{\partial t}-u_{m-1}\left(x,t\right).\quad(52)$$

By using I^{α} on both sides (51) and replacing (47) and (52) we have

$$\begin{aligned}u_{m-1}\left(x,t\right)=&(\chi_{m}+h)u_{m-1}(x,t)-(\chi_{m}+h)u_{m-1}(0,t)\\&-\left(\chi_{m}+h\right)\frac{\partial u_{m-1}\left(0,t\right)}{\partial x}x-hI^{\alpha}\left[\frac{\partial^{2}u_{m-1}}{\partial t^{2}}-\frac{\partial u_{m-1}}{\partial t}+u_{m-1}\right].\end{aligned}\quad(53)$$

Finally

$$u\left(x,t\right)=u_{0}\left(x,t\right)+\sum_{m=1}^{\infty}u_{m}(x,t),\quad(54)$$

and

$$u_0\,(x,t) = e^{-t}\,(1+x)\,,$$
$$u_1\,(x,t) = -he^{-t}[\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}],$$
$$u_2\,(x,t) = (1+h)u_1\,(x,t) - he^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],$$
$$\vdots$$
$$u\,(x,t) = \Bigg[e^{-t}\Bigg(1+x-\left(3x+3x^2+x^3\right)\left(\frac{x^{\alpha}}{\Gamma\,(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma\,(\alpha+2)}\right.$$
$$\left.-\left(2x+x^2\right)\left(\frac{x^{2\alpha}}{\Gamma\,(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma\,(2\alpha+2)}\right)\right)-x\left(\frac{x^{3\alpha}}{\Gamma\,(3\alpha+2)}+\frac{x^{3\alpha+1}}{\Gamma\,(3\alpha+2)}\right)\Bigg]+\cdots. \tag{55}$$

By insering $h = 1$ in (55) we have

$$u\,(x,t) = e^{-t}\left(1+x+\frac{x^{\alpha}}{\mathit{\Gamma}\,(\alpha+1)}+\frac{x^{\alpha+1}}{\mathit{\Gamma}\,(\alpha+2)}+\frac{x^{2\alpha}}{\mathit{\Gamma}\,(2\alpha+1)}+\frac{x^{2\alpha+1}}{\mathit{\Gamma}\,(2\alpha+2)}+\dots\right). \tag{56}$$

Similarly by inserting $\alpha = 2$ in (56) we have

$$u\,(x,t) = e^{-t}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right) = e^{x-t}. \tag{57}$$

The solution obtained for $\alpha = 2$ in (57) is an exact solution for the space-fractional telegraph equation. Figures 5 to 8 demonstrate the solutions of equation (1) can be seen for different states.

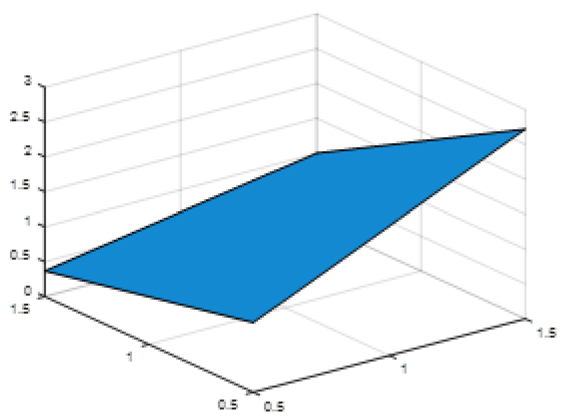

Figure 5. The exact solution for α=2 , t=1, x=0.5.

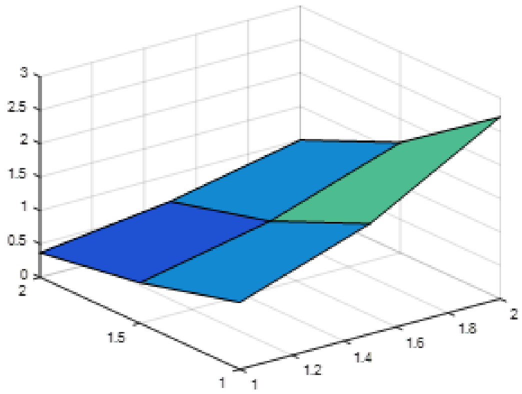

Figure 6. The exact solution for α=2 , t=0.5, x=1.

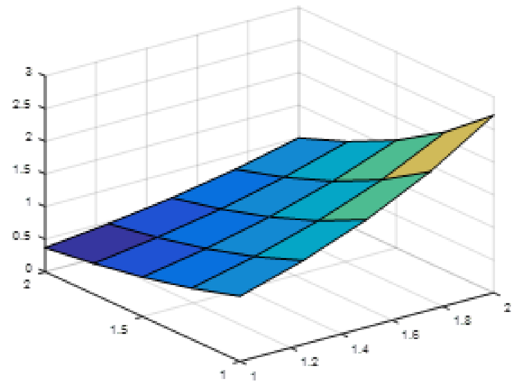

Figure 7. The exact solution for α=2 , t=0.25, x=1.

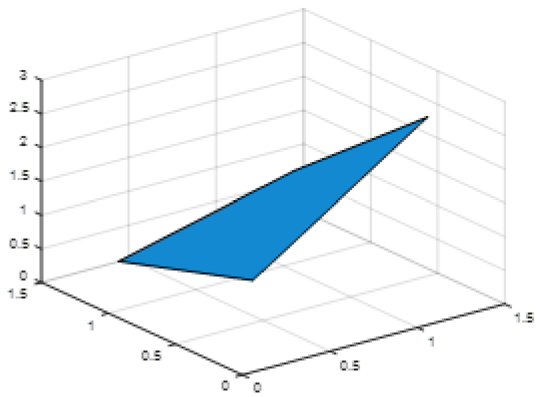

Figure 8. The exact solution for α=2 , t=1, x=0.25.

4. Lie symmetry operators

Consider the following FDE

$$\partial_t^{\alpha} u = F\left(t, x, y, u, u_x, u_{xx}, u_{xxx}, \dots\right), \qquad \Delta = \partial_t^{\alpha} u - F, \tag{58}$$

where x and t are independent variables and u is a dependent variable, $u_x, u_{xx}, u_{xxx}, \dots$ are partial derivatives with respect to x and ∂_t^{α} is the symbol of fractional derivative with respect to time.

Suppose G is a local group of transformations applied on $\mathcal{O}$ and v is an extremely small generator of it, where $\mathcal{O}$ is an open subset of the total space of (58) and v is a vector field on $\mathcal{O}$ with a one-parameter group of $\exp(\varepsilon v)$. The extent of (α, n)-order is shown as $v^{(\alpha,n)}$, which is a vector field on (α, n)-order jet space, and it is called an extremely small generator of the one-parameter group of $[exp\,(\varepsilon v)\ ]^{(\alpha,n)}$. Meaning that

$$v^{(\alpha,n)}\Big|_{(x,u^{(n)})} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [\exp(\varepsilon v)\ ]^{(\alpha,n)} \left(x, u^{(n)}\right). \tag{59}$$

Considering the above-mentioned concepts, Lie's one-parameter group of G is parameter ε of extremely small transformations given by

$$\overline{t} = \varphi(t, x, u, \varepsilon), \qquad \overline{x} = \psi(t, x, u, \varepsilon), \qquad \overline{u} = \pi(t, x, u, \varepsilon). \tag{60}$$

According to the theory of Lie Group, constructing a asymmetry group of G yields,

$$\begin{aligned}
&\overline{t} = t + \varepsilon\tau(t, x, u) + O\left(\varepsilon^2\right),\\
&\overline{x} = x + \varepsilon\xi(t, x, u) + O\left(\varepsilon^2\right),\\
&\overline{u} = u + \varepsilon\eta(t, x, u) + O\left(\varepsilon^2\right),\\
&\overline{u}_{\overline{t}}^{\alpha} = u_t^{\alpha} + \varepsilon\eta^{\alpha,t}(t, x, u) + O\left(\varepsilon^2\right),\\
&\overline{u}_{\overline{x}} = u_x + \varepsilon\eta^{x}(t, x, u) + O\left(\varepsilon^2\right),\\
&\overline{u}_{\overline{xx}} = u_{xx} + \varepsilon\eta^{xx}(t, x, u) + O\left(\varepsilon^2\right),\\
&\overline{u}_{\overline{xxx}} = u_{xxx} + \varepsilon\eta^{xxx}(t, x, u) + O\left(\varepsilon^2\right).
\end{aligned} \tag{61}$$

Lie algebra, which depends on the group (61), is generated by the vector field

$$v = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{62}$$

and its components are calculated by

$$\frac{d\overline{t}}{d\varepsilon}\Big|_{\varepsilon=0} = \tau(t, x, u), \qquad \frac{d\overline{x}}{d\varepsilon}\Big|_{\varepsilon=0} = \xi(t, x, u), \qquad \frac{d\overline{u}}{d\varepsilon}\Big|_{\varepsilon=0} = \zeta(t, x, u). \tag{63}$$

According to the invariance criterion, the transformation group of (62) will be approved by equation (61) as the asymmetry group; if and only if the effect of the extended generator (which is the order of highest order extension of derivative in the equation) becomes zero on the manifold of its solutions. In other words,

$$v^{(\alpha,n)}\left(\Delta\right)\Big|_{\Delta=0}=0,\qquad \alpha\in\left(0,1\right). \tag{64}$$

The extent of (α, n)-order of field vector v equals to

$$v^{(\alpha,n)}=v+\eta^{\alpha,t}\partial_{\partial_t^\alpha u}+\eta^{x}\partial_{u_x}+\eta^{xx}\partial_{u_{xx}}+\eta^{xxx}\partial_{u_{xxx}}+\eta^{xxxx}\partial_{u_{xxxx}}+\ldots, \tag{65}$$

and the extent coefficients of η^x, η^{xx}, η^{xxx}, and... are calculated by

$$\begin{aligned}
\eta^{x}&=D_x\left(\eta\right)-u_tD_x\left(\tau\right)-u_tD_x\left(\xi\right),\\
\eta^{xx}&=D_x\left(\eta^{x}\right)-u_{tx}D_x\left(\tau\right)-u_{xx}D_x\left(\xi\right),\\
\eta^{xxx}&=D_x\left(\eta^{xx}\right)-u_{txx}D_x\left(\tau\right)-u_{xxx}D_x\left(\xi\right),\\
\eta^{xxxx}&=D_x\left(\eta^{xxx}\right)-u_{txxx}D_x\left(\tau\right)-u_{xxxx}D_x\left(\xi\right).
\end{aligned} \tag{66}$$

Symbols of D_t and D_x are the representation of total derivatives with respect to t, x and they are presented as

$$\begin{aligned}
D_t&=\partial_t+u_t\partial_u+u_{tt}\partial_{ut}+u_{xt}\partial_{uz}+\cdots,\\
D_x&=\partial_x+u_x\partial_u+u_{tx}\partial_{ut}+u_{xx}\partial_{ux}+\cdots.
\end{aligned} \tag{67}$$

The coefficient of $\eta^{\alpha,t}$ in operator (43) is calculated using the following system

$$D^{\alpha,t}=D_t^\alpha\left(\eta\right)+\xi D_t^\alpha\left(u_x\right)-D_t^\alpha\left(\xi u_x\right)+D_t^\alpha\left(D_t\left(\tau\right)u\right)-D_t^{\alpha+1}\left(\tau u\right)+\tau D_t^{\alpha+1}\left(u\right).$$

The generalized Leibniz formula is given by

$$D_t^\alpha\left(f\left(t\right)g\left(t\right)\right)=\sum_{n=0}^{\infty}\binom{\alpha}{n}D_t^nf\left(t\right)D_t^{\alpha-n}g\left(t\right), \tag{68}$$

where

$$\binom{\alpha}{n}=\frac{\mathit{\Gamma}\left(\alpha+1\right)}{\mathit{\Gamma}\left(n+1\right)\mathit{\Gamma}\left(\alpha+1-n\right)},\quad D_t^0f\left(t\right)=f\left(t\right),\quad D_t^{n+1}f\left(t\right)=D_t\left(D_t^nf\left(t\right)\right).$$

Thus we conclude that

$$\xi D_t^\alpha\left(u_x\right)-D_t^\alpha\left(\xi u_x\right)=-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n\left(\xi\right)D_t^{\alpha-n}\left(u_x\right),$$

and

$$D_t^\alpha\left(uD_t\left(\tau\right)\right)-D_t^{\alpha+1}\left(\tau u\right)+\tau D_t^{\alpha+1}\left(u\right)=-\alpha D_t\left(\tau\right)D_t^\alpha u-\sum_{n=1}\binom{\alpha}{n+1}D_t^{n+1}\left(\tau\right)D_t^{\alpha-n}\left(u\right).$$

Hence,

$$\eta^{\alpha,t} = D_t^{\alpha}(\eta) - \alpha D_t(\tau) D_t^{\alpha} u - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u). \tag{69}$$

The generalized chain rule for the composite function is

$$\frac{d^{\alpha} f(g(t))}{dt^{\alpha}} = \sum_{n=0}^{\infty} \frac{U_n}{n!} \frac{d^n f(z)}{dz^n}\bigg|_{z=g(t)}, \tag{70}$$

where U_n is defined by

$$U_n = \sum_{k=0}^{m} (-1)^k \binom{n}{k} g^k(t)\, \partial_t^{\alpha}\left(g^{n-k}(t)\right).$$

After executing the calculations, we acquire the following explicit term for fractional differential operator

$$D_t^{\alpha} = \partial_t^{\alpha}\eta + \eta_u \partial_t^{\alpha} u - u\partial_t^{\alpha}\eta_u + \sum_{n=1}^{\infty} \binom{\alpha}{n} \partial_t^n \eta_u \partial_t^{\alpha-n} u + \mu, \tag{71}$$

and the following equation is utilized to calculate μ in the above formula:

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1} \binom{\alpha}{n}\binom{n}{m}\binom{k}{l} \frac{t^{n-\alpha}}{k!\Gamma(n+1-\alpha)} (-u)^l \frac{\partial^m}{\partial t^m}\left(u^{k-l}\right) \frac{\partial^{n-m+k}}{\partial t^{n-m}\partial u^k}. \tag{72}$$

It is worth noting that if η is linear with respect to the dependent variable of u, the value of μ will be zero and this is due to the existence of derivative $\frac{\partial^k}{\partial u^k}$, $k \geq 2$ in μ equation. Ultimately, the following explicit formula for $\eta^{\alpha,t}$ is acquired by epitomizing the mentioned reasonings:

$$\eta^{\alpha,t} = \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right]\partial_t^{\alpha-n} u - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi)\,\partial_t^{\alpha-n}(u_x) + \partial_t^{\alpha}\eta + (\eta_u - \alpha D_t(\tau))\,\partial_t^{\alpha} u - u\partial_t^{\alpha}\eta_u + \mu. \tag{73}$$

To determine which symmetries are approved by equations (61), the differential operator of $v^{(\alpha,n)}$ is applied to equation (61) and then categorized the equations according to their dependent variables and their derivatives; next, we place their coefficients equal to zero. As a result, we achieved a group of characteristic equations, namely PDEs and FDEs. We will acquire equation symmetries under discussion by solving these characteristic equations.

Consider (1). In this section, we intend to find symmetries in this fractional equation with the presented method. Using equation (65), by inserting terms of $\eta^{(\alpha,t)}$, η^{xx}, and η^{x} achieved in equations (67) and (68), we conclude that the above equation depends on

$D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$, *and* u_x, u_{xx}, u_{xt}, $u_t, \ldots (n = 1, 2, \ldots)$. We will achieve an over determined system of fractional and space-fractional equations by placing various powers of u derivatives. By solving this system, the functions of ξ_x, ξ_t, and η are achieved as follows:

$$\xi_x = F_1(x), \quad \xi_t = F_2(x), \qquad \eta = F_3(x,t) + u\ F_4(x). \tag{74}$$

5. Conclusion

In this paper, the basic ideas of a new kind of analytical technique, namely the homotopy analysis method is implemented. The non-standard finite difference method was then used to numerically solve several practical examples for PDEs with spatial fractional derivatives, and the numerical results of these approximate methods were compared with the exact solution of the problem. The analytical homotopy method shows a useful and simple method for adjusting and controlling the convergence region and the convergence velocity of the series solution. Homotopy analysis is also an efficient and useful method for solving FDEs. Finally, an attempt has been made to solve the equation considered in this paper with an algebraic view of the concept of Lie symmetry and the methods derived from it.

References

[1] G. W. Bluman, A. F. Cheviakov, S. C. Anco, Application of Symmetry Methods to Partial Differential Equations, Springer-Verlage, New York, 2000.
[2] E. Botton, J. B. Greenberg, A. Arad, D. Katoshevski, V. Vaikuntanathan, M. Ibach, B. Weigand, An investigation of grouping of two falling dissimilar droplets using the homotopy analysis method, Appl. Math. Model. 104 (2) (2022), 486-498.
[3] S. E. Fadugba, Homotopy analysis method and its applications in the valuation of European call options with time-fractional Black-Scholes equation, Chaos Solitons Fractals. 141 (2020), 2:110351.
[4] M. Ganjiani, Solution of nonlinear fractional differential equations using homotopy analysis method, Appl. Math. Model. 34 (6) (2010), 1634-1641.
[5] N. H. Ibragimov, Transformation Group Applied to Mathematical Physics, Riedel, Dordrecht 1985.
[6] N. H. Ibragimov, A. V. Aksenov, V. A. Baikov, Vladimir A. Chugunov, R. K. Gazizov, A. G. Meshkov, CRC Handbook of Lie Group Analysis of Differential Equations, Applications in Engineering and Physical Sciences, CRC Press, 1995.
[7] C. Kailash, P. Kapil, K. Sharma, ε-Uniformly convergent non-standard finite difference methods for singularly perturbed differential difference equations with small delay, Appl. Math. Comput. 175 (1) (2006), 864-890.
[8] M. Kashif, K. D. Dwivedi, T. Som, Numerical solution of coupled type fractional order Burgers' equation using finite difference and Fibonacci collocation method, Chinese J. Phys., 77 (2022), 2314-2323.
[9] S. Liao, Homotopy analysis method: A new analytical technique for nonlinear problems, Commun. Nonlinear Sci. Numer. Simul. 2 (2) (1997), 95-100.
[10] R. E. Mickens, Applications of Nonstandard Finite Difference Schemes, World Scientific, 2000.
[11] R. E. Mickens, Exact solutions to a finite-difference model of a nonlinear reaction-advection equation: Implications for numerical analysis, Numer. Methods Partial Differential Equations, 5 (4) (1989), 313-325.
[12] R. E. Mickens, Nonstandard Finite Difference Models of Differential Equations, World scientific, 1994.
[13] K. S. Miller, An Introduction to Fractional Calculus and Fractional Differential Equations, John. Wiley and Sons, New York, 1993.
[14] K. Moaddy, S. Momani, I. Hashim, The non-standard finite difference scheme for linear fractional PDEs in fluid mechanics, Comput. Math. Appl. 61 (4) (2011), 1209-1216.
[15] S. Momani, Analytic and approximate solutions of the space and time-fractional telegraph equations, Appl. Math. Comput. 170 (2) (2005), 1126-1134.
[16] Z. Odibat, On the optimal selection of the linear operator and the initial approximation in the application of the homotopy analysis method to nonlinear fractional differential equations, Appl. Numer. Math. 137 (4) (2019), 203-212.
[17] P. J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlage, New York, 1993.
[18] I. Podlubny, Fractional Differential Equations, Academic Press, USA, 1999.
[19] H. Rezazadeh, M. S. Osman, M. Eslami, M. Ekici, A. Sonmezoglu, M. Asma, W. A. M. Othman, B. R. Wong, M. Mirzazadeh, Q. Zhou, A. Biswas, M. Belick, Mitigating internet bottleneck with fractional temporal evolution of optical solitons having quadratic-cubic nonlinearity, Optik. 164 (2018), 84-92.
[20] A. M. Vargas, Finite difference method for solving fractional differential equations at irregular meshes, Math. Comput. Simul. 193 (2022), 204-216.