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Quasi-nonexpansive mappings with respect to orbits in Banach spaces and weak fixed point property

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Abstract. In the present work, we introduce quasi-nonexpansive mappings with respect to orbits on the Banach space. Then we show that a Banach space $\mathcal{A}$ has weak normal structure if and only if $\mathcal{A}$ has the weak fixed point property for quasi-nonexpansive mappings with respect to orbits.

Keywords: Fixed point property, normal structure, nonexpansive mappings.

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1. Introduction and preliminaries

Suppose that $\mathcal{B}$ is a nonempty subset of a Banach space $\mathcal{A}$, and $\mathcal{F} : \mathcal{B} \to \mathcal{A}$ is a mapping. If $\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b} \| \leqslant \| \mathfrak{a} - \mathfrak{b} \|$ for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$, then we say that $\mathcal{F}$ is a nonexpansive map.

In this work, we denote $O_{\mathcal{F}}(\mathfrak{b})$ by the set of all points $\mathcal{F}^n\mathfrak{a}$ for all $n = 0, 1, \dots$ where $\mathfrak{a} \in \mathcal{A}$, i.e., $O_{\mathcal{F}}(\mathfrak{b}) = \{\mathcal{F}^n\mathfrak{a} : \ n = 0, 1, \dots\}$. Also, for any subset $\mathcal{C}$ of Banach space $\mathcal{A}$ and $\mathfrak{a} \in \mathcal{A}$, put $r_{\mathfrak{a}}(\mathcal{C}) = \sup_{\mathfrak{b} \in \mathcal{C}} \| \mathfrak{a} - \mathfrak{b} \|$.

In the setting of Banach spaces, the fixed point theory of nonexpansive mappings has been extensively studied by many authors (for example, see [2, 3, 7, 11] and the references therein). In 2016, Amini et al. [1] defined nonexpansive mappings in Banach spaces as follows:

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Definition 1.1 Let $\mathcal{B}$ be a nonempty subset of a Banach space $\mathcal{A}$. A mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is said to be nonexpansive with respect to (wrt) orbits if $\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{a} \| \leqslant r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big)$ for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$.

Also, they defined that a Banach space $\mathcal{A}$ has the weak fixed point property for nonexpansive mapping wrt orbits if every nonexpansive mapping wrt orbits $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point, where $\mathcal{B}$ is an arbitrary weakly compact, convex subset of $\mathcal{A}$. Further, they proved that Banach space weak normal structure and the weak fixed point property for nonexpansive mappings wrt orbits are equivalent in a Banach space $\mathcal{A}$.

Theorem 1.2 [1] Let $\mathcal{A}$ be a Banach space. Then, $\mathcal{A}$ has weak normal structure if and only if every nonexpansive mapping wrt orbits $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point, where $\mathcal{B}$ is an arbitrary weakly compact, convex subset of $\mathcal{A}$.

A mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is said to be a quasi-contraction wrt orbits if

$$\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b} \| \leqslant \beta \max\big\{ r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big), r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big), r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big) \big\},$$

where $0 \leqslant \beta < 1$ and $O_{\mathcal{F}}(\mathfrak{b}) = \{\mathcal{F}^n\mathfrak{b}, n \geqslant 0\}$.

A point $\mathfrak{a} \in \mathcal{C}$ is said to be nondiametral if $r_{\mathfrak{a}}(\mathcal{C}) < diam\ \mathcal{C}$. A convex subset $\mathcal{B}$ of $\mathcal{A}$ is said to have normal structure if each convex, bounded subset $\mathcal{C}$ of $\mathcal{B}$ with $diam\ \mathcal{C} > 0$ contains a nondiametral point. A Banach space $\mathcal{A}$ has weak normal structure if every weakly compact, convex subset $\mathcal{B}$ of $\mathcal{A}$ with $diam\ \mathcal{B} > 0$ contains a nondiametral point. We say that a Banach space $\mathcal{A}$ has a weak fixed point property for nonexpansive mappings (w-fpp) if every nonexpansive mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point, where C is a nonempty weakly compact, convex subset of $\mathcal{B}$. In 1965, Kirk [6] proved that every Banach space X with weak normal structure has the w-fpp. However, two Banach spaces c_0 and $(l_2, \| a \|_{\sqrt{2}} = \max\{\| a \|_{l_2}, \sqrt{2} \| a \|_\infty\})$ have the w-fpp but fails to have weak normal structure [5].

The main purpose of this paper to introduce quasi-nonexpansive mappings wrt orbits in a Banach space. Also, we show that weak normal structure and weak fixed point property for nonexpansive mappings wrt orbits are equivalent geometric properties in a Banach space $\mathcal{A}$.

Following Amini et al. [1, Definition 1.1], we define the concept of quasi-nonexpansive wrt orbits for self-map $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ on a Banach space.

Definition 1.3 Let $\mathcal{B}$ be a nonempty subset of a Banach space $\mathcal{A}$. A mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is said to be a quasi-nonexpansive wrt orbits if

$$\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b} \| \leqslant \max\big\{ r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big), r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big), r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big) \big\}, \tag{1}$$

where $0 \leqslant \beta < 1$.

We clearly see that every nonexpansive wrt orbits in a Banach space is a quasi-nonexpansive wrt orbits and every nonexpansive mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is a quasi-nonexpansive wrt orbits. However, there are mappings which are quasi-nonexpansive wrt orbits but fail to be nonexpansive. Similar to [9, Example 5.2], we will express the following example.

***Example* 1.4** Suppose that $\mathcal{F} : [0,1] \to [0,1]$ is defined as

$$\mathcal{F}(\mathfrak{a}) = \begin{cases} \frac{\mathfrak{a}}{2}, \ \mathfrak{a} \geqslant \frac{1}{2} \\ \frac{\mathfrak{a}}{4}, \ \mathfrak{a} < \frac{1}{2}. \end{cases}$$

Then $\mathcal{F}$ is not continuous and so it cannot be nonexpansive. We show that $\mathcal{F}$ is a quasi-nonexpansive wrt orbits. Let $\mathfrak{a}, \mathfrak{b} \geqslant \frac{1}{2}$ and $\mathfrak{b} \leqslant \mathfrak{a}$. Then $\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{a} \| = \| \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{2} \|$. On the other hand,

$$r_{\mathfrak{a}}(O_{\mathcal{F}}(b)) = \mathfrak{a} \geqslant \| \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{2} \| \quad \text{and} \quad r_{\mathfrak{b}}(O_{\mathcal{F}}(a)) = \max\left\{\mathfrak{b}, \mathfrak{a} - \mathfrak{b}\right\} \geqslant \| \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{2} \| .$$

So

$$\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{a} \| = \| \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{2} \| \leqslant r_{\mathfrak{a}}(O_{\mathcal{F}}(b)) \leqslant \max\left\{r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big), r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big), r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big)\right\}.$$

Otherwise, if $\mathfrak{a} \geqslant \frac{1}{2}$ and $\mathfrak{b} < \frac{1}{2}$, we have $\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{a} \| = \| \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{4} \|$ and similarly

$$r_{\mathfrak{a}}(O_{\mathcal{F}}(b)) = \mathfrak{a} \geqslant \| \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{4} \| \quad \text{and} \quad r_{\mathfrak{b}}(O_{\mathcal{F}}(a)) = \max\left\{\mathfrak{b}, \mathfrak{a} - \mathfrak{b}\right\} \geqslant \| \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{4} \| .$$

Then (1) holds, and as a result, $\mathcal{F}$ is a quasi-nonexpansive wrt orbits.

2. Main results

Following Amini et al. [1], we define the concept of weak fixed point property for quasi-nonexpansive mappings wrt orbits in a Banach space.

Definition 2.1 A Banach space $\mathcal{A}$ is said to has the weak fixed point property for a quasi-nonexpansive mapping wrt orbits whenever each quasi-nonexpansive mapping wrt orbits $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point in which $\mathcal{B}$ is a given weakly compact, convex subset of $\mathcal{A}$.

Now, we are ready to state our main result in this section.

Theorem 2.2 Let $\mathcal{A}$ be a Banach space. Then, $\mathcal{A}$ has weak normal structure if and only if $\mathcal{A}$ has the weak fixed point property for quasi-nonexpansive mappings wrt orbits.

Proof. First, assume that $\mathcal{A}$ has weak normal structure and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is a nonexpansive mapping wrt orbits, where $\mathcal{B}$ is a nonempty, weakly compact, convex subset of $\mathcal{A}$. By Zorn's lemma, there exists a nonempty, weakly compact convex subset $\mathcal{C} \subseteq \mathcal{B}$ which is minimal $\mathcal{F}$-invariant. We will show that $\mathcal{C}$ is singleton and so $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point. On the contrary, assume that $diam\ \mathcal{C} > 0$. For each $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}$, we have

$$\begin{aligned} \| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b} \| &\leqslant \beta \max\left\{r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big), r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big), r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big)\right\} \\ &\leqslant \beta r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big) \\ &\leqslant r_{\mathfrak{a}}(\mathcal{C}). \end{aligned} \tag{2}$$

From the minimality of $\mathcal{C}$, we have $\overline{co}\ \mathcal{F}\mathcal{C} = \mathcal{C}$ and we obtain from (2) that

$$r_{\mathcal{F}\mathfrak{a}}(\mathcal{C}) = r_{\mathcal{F}\mathfrak{a}}(\overline{co}\ \mathcal{F}\mathcal{C}) = co\ \mathcal{F}\mathcal{C} \leqslant r_{\mathfrak{a}}(\mathcal{C}) \tag{3}$$

for all $\mathfrak{a} \in \mathcal{C}$. Since $\mathcal{C}$ has normal structure, then there exist $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{C}$ such that

$$r_{\mathfrak{a}_1} = r_{\mathfrak{a}_1}(\mathcal{C}) = r_{\mathfrak{a}_2}(\mathcal{C}) = r_{\mathfrak{a}_2}.$$

Now, we set $\mathcal{C}_0 = \left\{ \mathfrak{b} \in \mathcal{C} : \ r_{\mathfrak{b}}(\mathcal{C}) \leqslant \frac{r_{\mathfrak{a}_1} + r_{\mathfrak{a}_2}}{2} \right\}$. Then $\mathcal{C}_0 \subseteq \mathcal{C}$ is a nonempty, closed bounded and convex set which, by (3), is $\mathcal{F}$-invariant, and this contradicts the minimality of $\mathcal{C}$ (note that $\mathcal{C}_0 \neq \mathcal{C}$). Thus, $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point.

Now, assume that $\mathcal{A}$ has the weak fixed point property for nonexpansive mappings wrt orbits. On the contrary, assume that $\mathcal{A}$ has no weak normal structure and let $\mathcal{B} \subseteq \mathcal{A}$ is a weakly compact, convex set with $diam\ \mathcal{B} > 0$ which fails to have normal structure. Then, by the BrodskĭiMiĺman result [3], $\mathcal{B}$ contains a diametral sequence $\{z_n\}$ of distinct elements. Then, we have

$$\lim_{n\to\infty} d\big(z_{n+1}, co\big(\{z_i\}_{i=1}^{n}\big)\big) = diam\big(\{z_n\}\big) > 0. \tag{4}$$

Note that since $\mathcal{B}$ is weakly compact, then by the EberleinŠmulian theorem, $\{z_n\}$ has a weakly convergent subsequence. Moreover, every subsequence of a diametral sequence is again diametral, and so we may assume that the diametral sequence $\{z_n\}$ is weakly convergent. Now, we set $\mathcal{C} = \overline{co}\{z_n\}$, then by the Krein-Šmulian theorem, $\mathcal{C}$ is a nonempty weakly compact convex subset of $\mathcal{A}$. It is clear that $diam\ \mathcal{C} = diam(\{z_n\})$ and we obtain from (4) that

$$\lim_{n\to\infty} \| z_n - \mathfrak{a} \| = diam\ \mathcal{C} \quad \text{for each} \quad \mathfrak{a} \in \mathcal{C}. \tag{5}$$

We define $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{F}\mathfrak{a} = \begin{cases} z_1, & \mathfrak{a} \in \{z_n : \ n \notin \mathbb{N}\} \\ z_{n+1}, & \mathfrak{a} = z_n \ \text{ for some } \ n \in \mathbb{N}. \end{cases}$$

It is easy to conclude from (5) that

$$r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big), r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big) \leqslant r_{\mathfrak{a}}(O_{\mathcal{F}}(\mathfrak{b})) = diam\ \mathcal{C} \tag{6}$$

for each $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}$. Thus, we have from (1) and (5) that

$$\begin{aligned} \| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b} \| &\leqslant \beta \max\big\{ r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big), r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big), r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big) \big\} \\ &\leqslant \beta r_{\mathfrak{a}}(O_{\mathcal{F}}(\mathfrak{b})) \\ &\leqslant r_{\mathfrak{a}}(O_{\mathcal{F}}(\mathfrak{b})) = diam\ \mathcal{C} \end{aligned}$$

for each $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}$. Then $\mathcal{F}$ is a fixed point free quasi-nonexpansive mapping wrt orbits, a contradiction. ∎

Since nonexpansive mappings wrt orbits are not necessarily continuous (see Example 1.4), Schauder's theorem cannot be used to state the existence of a fixed point whenever $\mathcal{B} \subseteq \mathcal{A}$ is a nonempty, compact convex set and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is quasi-nonexpansive wrt orbits. However, any nonempty, compact set has normal structure. Then, by Theorem 2.2, we get the following corollary.

Corollary 2.3 Let $\mathcal{B}$ be a nonempty, compact convex subset of a Banach space $\mathcal{A}$ and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ be a quasi-nonexpansive mapping wrt orbits. Then, $\mathcal{F}$ has a fixed point.

For many years, the problem of the failure of the fixed point property for nonexpansive mappings defined on unbounded sets has attracted the interest of many researchers (see, for instance, [3, 4, 8, 10]). One could also consider this problem for mappings which are quasi-nonexpansive wrt orbits. For instance, we have the following result.

Theorem 2.4 Assume that $\mathcal{A}$ is a reflexive Banach space and $\mathcal{B}$ is a closed convex subset of $\mathcal{A}$. If every quasi-nonexpansive mapping wrt orbits $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point, then $\mathcal{B}$ is bounded.

Proof. Assume that $\mathcal{B}$ is unbounded. Choose a sequence $\{\mathfrak{a}_n\}$ formed by distinct points such that $\| \mathfrak{a}_n \| \to \infty$. Define $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ by $\mathcal{F}\mathfrak{a} = \mathfrak{a}_1$ if $\mathfrak{a} \notin \{\mathfrak{a}_n : n \in \mathbb{N}\}$ and $\mathcal{F}\mathfrak{a}_n = \mathfrak{a}_{n+1}$. It is clear that $\mathcal{F}$ is a fixed-point free nonexpansive mapping wrt orbits because $\max\left\{r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big), r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big), r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big)\right\}$ for every $\mathfrak{b} \in \mathcal{B}$. ■

3. Conclusion

Taking some applied concepts such as quasi-nonexpansive mappings, we gave some new definitions and several fixed point property theorems. Indeed, the main purpose of this paper was to define quasi-nonexpansive mappings to prove some fixed point property theorems. Several consequences that could show the importance of these results were as well introduced. Studying weak fixed point property for various contractions wrt to orbits in Banach spaces $\mathcal{A}$ and its relationship with where Banach space $\mathcal{A}$ has weak normal structure can be an attractive subject to follow so that other researchers may use these results in analysis and applications.

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