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The Centralized Resource Allocation under Semi-Additive Production Technology in DEA

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Abstract. Centralized resource allocation (CRA) models are usually presented under variable returns to scales (VRS) technology. In these models, the evaluation of the efficiency of the decision-making units (DMUs) is done only on the basis of the observed DMUs. In this paper, we introduce CRA models in semi-additive production technology. In this technology, in addition to the observed DMUs, aggregation of production units is present in the process of performance evaluation using data envelopment analysis (DEA). We prove that we can solve this model only based on observational DMUs in order to reduce the number of calculations. In the following, we develop this model for a general case based on the approach provided by Fang [6]. The proposed models adjust the inputs and outputs to achieve the total input contraction by the central decision-maker (DM). We can only consider adjustments to inefficient DMUs instead of all DMUs in the CRA model. The proposed model maximizes the efficiency of individual DMUs at the same time that total input consumption is minimized or total output production is maximized. We obtain the efficient targets corresponding to all DMUs on the efficiency frontier of semi-additive production technology by solving only one model. We illustrate our approach with an empirical example.

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1. Introduction

All Traditional DEA models evaluate DMUs individually (Charnes et al. [3]; Banker et al. [2], Gerami [9], Gerami et al. [10-11]). These models depict the DMU under evaluation on the efficiency frontier by reducing inputs (input-orientated) or increasing outputs (output-orientated). However, when the evaluation is done within a centralized framework, the DM may want to evaluate all DMUs simultaneously and obtain the optimal level of inputs and outputs from all DMUs by solving a model rather than reducing the inputs of the DMUs independently. Korhonen and Syrjänen [23] proposed an interactive model by combining DEA and multiple-objective linear programming. They used it in the resource-allocation problem. Their model allocates available resources among DMUs so that the total amount of output will be maximized simultaneously. Lozano and Villa [25] proposed two CRA models under VRS technology. The first model seeks a radial reduction of the total consumption of each input by all DMUs, while the second model seeks a separate reduction

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for each input according to the preference structure. They supposed that total production output was guaranteed not to decrease. Their model projects all DMUs onto the efficient frontier. Asmild et al. [1] modified the CRA model of Lozano and Villa [25] and only considered adjustments to inefficient DMUs. They proposed a procedure that can be applied to generate alternative optimal solutions. Lozano et al. [26] proposed centralized reallocation in the DEA for emission permits. Their model supposes that firms produce desirable and undesirable outputs. Their approach has three objectives, including maximizing aggregated desirable production, minimizing undesirable total emissions, and minimizing the consumption of input resources. Du et al. [4] developed a CRA approach to minimize the total input consumed and maximize the total output produced by all DMUs simultaneously. Hosseinzadeh Lotfi et al. [18] developed a CRA model based on the enhanced Russell models. They obtain efficient targets for all DMUs non-radially. Hosseinzadeh Lotfi et al. [19] developed a CRA model for stochastic data. They allocate centralized resources where inputs and outputs are stochastic. Mar-Molinero et al. [29] proposed a simplified version of the CRA model that was proposed by Lozano and Villa [25]. They show that their model is simple to implement in many situations. Fang [6] proposed a generalized DEA model for CRA. He showed that his model extends the models of Lozano and Villa [25] and Asmild et al. [1] in the general case. They decomposed the structural efficiency into three components: the aggregate technical efficiency, the aggregate allocative efficiency and re-transferable efficiency components to discover the sources of such total input shrinkage in the generalized CRA model. Fang and Li [7] proposed CRA based on a cost and revenue analysis. He presents a centralized approach for reallocating resources to DMUs in a centralized decision-making environment based on revenue efficiency. Fang [8] proposed a CRA model based on efficiency analysis for step-by-step improvement path analysis in a centralized decision-making environment. Hakim et al. [17] developed a bi-level formulation for CRA in DEA under efficiency constraints. They investigated that DMUs are controlled by a central DM that has the authority to allocate limited resources to them; in this way, overall organization effectiveness is maximized. Sadeghi and Dehnokhalaji [31] presented a comprehensive method for the CRA in DEA. They developed two planning ideas. The first idea is to maximize the outputs produced with future planned resources and eliminate all input inefficiencies as much as possible while all units are highly efficient. The second idea is to optimize the revenue and total cost functions to achieve the best system performance. It is assumed that all DMUs can adjust their input consumption and output production in the current production possibility set (PPS). Momeni et al. [30] developed a centralized DEA-based reallocation of emission permits under cap and trade system based on countries efficiencies. Their model considers DMUs together and improves the efficiency score of them by reducing total emission permit as undesirable outputs. Kamyab et al. [20] proposed a CRA model based on the ratio-based DEA model for a two-stage incentive system. They evaluated commercial banks when data were ratios. Lozano and Contreras [28] proposed a CRA model by using lexicographic goal programming. They applied it to the Spanish public university system. Their approach proposed using three priority levels: (1) aggregated input consumption and output production goals. (2) the input and output goals of the individual operating units, and (3) the technical efficiency of the computed targets. Lozano and Villa [27] proposed a Multiobjective centralized DEA approach to Tokyo 2020 Olympic games, they obtain suitable targets in centralized management. Xiong et al. [32] proposed a parallel DEA-based approach for multi-period CRA among all DMUs by considering individual periods as segments operating in parallel. They introduce the concept of cross-efficiency for balancing the goals of the organization and the individual preferences of each DMU in resource allocation. Zhu et al. [34] proposed a DEA model for partial centralization of resource allocation among independent subsets of DMUs. Their approach was to improve the output of each DMU in the subset and reduce the myopia effect by rationally optimizing the allocation of resources. Their model can directly calculate the resource optimization results without modifying the model again. An improvement path to guide DMUs to achieve their ultimate goals on the efficient frontier in a workable and realistic manner. Yang [33] proposed a CRA-DEA model for target setting in a two-stage production process. He applied his model to determine the optimal centralized resource allocation and target setting plan for 27 retailers belonging to an automobile parts supplier in Taiwan. Lin and Lu [24] proposed a centralized DEA model for effective allocation of shared inputs, they applied their approach in the optimizing public sector resources.

Traditional DEA model measure the efficiency of DMUs in the constant and variable returns to scale technologies. Charnes et al. [3] developed the first technology called constant returns to scales (CRS) technology. Banker et al. [2] proposed another technology under variable returns to scales (VRS). Koopmans [19] presented a technology under the title of non-increasing efficiency technology. Deprins et al. [4] did not consider the convexity axiom and introduced free disposal hull (FDH) technology. Green and Cook [16] proposed additive technology based on observed DMUs and aggregation DMUs and proposed a new PPS. The additive axiom states that the two observed DMUs such as A and B can aggregate their activities to create a new DMU called unit aggregation as $A+B$. By considering the additive axiom, we suppose that if observed DMU A and B are able to product, then the unit aggregation as A+B is also able to product. This axiom is applied in the relevant articles as additive and semi additive axioms. If we want to distinguish between these two assumptions, we can say that by considering the as additive axiom, if an observed DMU such as A belongs to the production technology and has the possibility of activity, then new aggregated units such as $2A$ and $3A$ also belong to the set of production technology. However, the semi additive axiom states that the new aggregation DMU of units A and B as $A+B$ belong to production technology if $A \neq B$. However, according to the additive axiom, these units can be the same and have the same inputs and outputs (Ghiyasi [13]; Ghiyasi and Cook [14]).

Ghiyasi [10] proposed a DEA production technology under the title semi-additive technology and applied it for the incorporation of collaboration in efficiency analysis. He stated that each unit must compete not only with individual units but also with aggregated units and provide a more competitive efficiency frontier. Also, it is a generalized technology from which traditional DEA technologies can be derived. Ghiyasi and Cook [14] proposed the semi-additive production technology in DEA and developed a new model that decreases the computational complexity of models in the semi-additive production technology significantly. They proved that the proposed semi-additive methodology allows the number of variables to decrease and the complexity of the algorithm to also be reduced. Karami Khorramabadi et al. [21] proposed a cost-efficiency evaluation DEA model by considering undesirable outputs in the semi-additive production technology. Ghiyasi and Cook [14] developed a semi-additive integer-valued production technology for analyzing public hospitals in Mashhad. Gerami [12] proposed strategic alliances and partnerships in DEA based on the semi-additive production technology in DEA. They applied the proposed approach to strategic alliances and partnerships in banking. They showed that with the semi-additive production technology, more favorable targets can be achieved for the units in the partnership process.

It can be said that the main contribution of this paper is as follows: In this paper, we apply DEA in centralized settings that units operate in the same organization. We introduce a new CRA model in the semi-additive production technology in DEA. We assume that all DMUs operate under the supervision of a central unit to introduce input and output targets for all DMUs in the next production. The proposed model, instead of solving an independent LP model that projects each DMU in turn, projects all DMUs simultaneously. Instead of reducing the inputs of each DMU, the goal is to reduce the total input consumption of DMUs. We develop the model for the general state of CRA in the semi-additive production technology based on the idea of Fang [6]. We developed an extension to the CRA model in the semi-additive production technology for incorporating non-adjustable input variables and non-transferable outputs. The results of models in this paper show that our approach in the semi-additive production technology can better realize the optimal allocation of DMUs resources that has important application significance.

The structure of the rest of this paper is as follows: The second section presents the CRA model in semi-additive production technology based on the models of Lozano and Villa [21] and Asmild et al. [1]. The third section presents the general case of CRA in the semi-additive production technology that includes the models in Section 2. The fourth section illustrates models with a numerical example. The fifth section proposes an application of the CRA model in the semi-additive production technology for allocating resources in a set of chain stores, and at the end, we present the results of the research.

2. CRA in semi-additive production technology

As we know, the radial models in the DEA (input-orientated) obtain the projection of the DMU under evaluation onto the efficiency frontier of the production technology during two phases. In the first phase, they obtain a reduction in all input components, and in the second phase, they obtain an additional reduction of each input or expansion of each output. But there are two differences between CRA models and the standard DEA model. First, instead of solving an independent model that depicts each DMU separately, the CRA model depicts all DMUs on the efficiency frontier simultaneously. Second, instead of reducing inputs from each of the units separately, the total consumption input from all DMUs is reduced simultaneously (Lozano and Villa [25]). In this section, we propose the CRA model in the semi-additive production technology of DEA.

Let n DMUs are in the production process as $DMU_j = (X_j, Y_j)$, $j = 1, \dots, n$. Each DMU_j consume input vector $X_j = (x_{1j}, \dots, x_{mj})^T \in R_+^m$ for producing the vectors output as $Y_j = (y_{1j}, \dots, y_{sj})^T \in R_+^s$. Also, assume, $j, l = 1, \dots, n$, be indexes of DMUs. The index for inputs and outputs are $i = 1, \dots, m$ and $r = 1, \dots, s$, repectively. θ_{SA}^{CRA} show radial contraction of total input vector. Also, put $(\mu_{1l}, \mu_{2l}, \dots, \mu_{nl})$ be vector for projecting of DMU_l, $l = 1, \dots, n$.

Let $N = \{1, \dots, n\}$ is indexes set of DMUs. $N' = P(N) - \emptyset$ show the power set of N that we exclude the origin. N' is includes the index of all observed DMUs and the aggregation units corresponding to them, then this set has $2^n - 1$ member. Now, we present the PPS in order to CRA as follows.

$$T_{SA}^{CRA} = \{(X,Y) \mid \textstyle\sum_{j\in N'} \sum_{l\in N'} \mu_{jl} X_j \le X, \sum_{j\in N'} \sum_{l\in N'} \mu_{jl} Y_j \ \ \ge Y, \sum_{j\in N'} \mu_{jl} = 1, \mu_{jl} \ge 0\}. \quad (1)$$

The CRA model in semi-additive production technology proposed as follows.

$$\begin{aligned} Min \quad & \theta_{SA}^{CRA} \\ s.t. \quad & \left(\theta_{SA}^{CRA} \textstyle\sum_{j\in N'} X_j, \sum_{j\in N'} Y_j \ \right) \in T_{SA}^{CRA}. \end{aligned} \qquad (2)$$

According to the definition of the set T_{SA}^{CRA}, model (2) becomes as follows.

$$\begin{aligned} Min \quad & \theta_{SA}^{CRA} \\ s.t. \quad & \textstyle\sum_{j\in N'} \sum_{l\in N'} \mu_{jl} x_{ij} \le \theta_{SA}^{CRA} \left(\sum_{j\in N'} x_{ij}\right),\ i = 1, \dots, m, \\ & \textstyle\sum_{j\in N'} \sum_{l\in N'} \mu_{jl} y_{rj} \ge \sum_{j\in N'} y_{rj},\ r = 1, \dots, s, \\ & \textstyle\sum_{j\in N'} \mu_{lj} = 1,\ l \in N', \\ & \qquad \mu_{lj} \ge 0,\ l, j \in N'. \end{aligned} \qquad (3)$$

Model (2) has $\left((2^n-1)\times(2^n-1)\right)+1$ variables and $m+s+(2^n-1)$ constraints. As can be seen, for any arbitrary number n of DMUs, solving problem (2) is a formidable task. Also, to apply this model we need to create the aggregates of all subsets of observed DMUs. For this purpose, based on the idea of Ghiyasi and Cook [14], we present the set T_{SA}^{CRA} only based on the observed DMUs and as follows.

$$T_{MSA}^{CRA}=\left\{(X,Y)\middle|\sum_{j\in N}\sum_{l\in N}\mu_{jl}X_j\le X,\sum_{j\in N}\sum_{l\in N}\mu_{jl}Y_j\ge Y,\sum_{j\in N}\mu_{jl}\ge 1,0\le\mu_{jl}\le 1\right\}. \quad (4)$$

Theorem 2.1 The sets T_{SA}^{CRA} and T_{MSA}^{CRA} are equivalent, that is $T_{SA}^{CRA}=T_{MSA}^{CRA}$.

Proof. To show that these two sets are equal, we prove that $T_{SA}^{CRA}\subseteq T_{MSA}^{CRA}$ and $T_{MSA}^{CRA}\subseteq T_{SA}^{CRA}$.

First, we show that $T_{SA}^{CRA}\subseteq T_{MSA}^{CRA}$. Let $(X,Y)\in T_{SA}^{CRA}$, according to the definition of the set T_{SA}^{CRA}, there is a vectors $\mu_l=(\mu_{1l},\dots,\mu_{nl})$, $l\in N'$, such that $\sum_{j\in N'}\sum_{l\in N'}\mu_{jl}X_j\le X,\sum_{j\in N'}\sum_{l\in N'}\mu_{jl}Y_j\ \ \ge Y,\sum_{j\in N'}\mu_{jl}=1,\mu_{jl}\ge 0,\ l\in N',$ $j\in N'$.

In this way, we have two sets of DMUs belong to set N', observed and aggregated DMUs, by considering these two sets of DMUs, we show that $T_{SA}^{CRA}\subseteq T_{MSA}^{CRA}$. For the first state, suppose $\mu_{jl}>0$ only for $l\in N,\ j\in N$ namely for observed DMUs, in this state, we have $\sum_{j\in N}\sum_{l\in N}\mu_{jl}X_j\le X,\sum_{j\in N}\sum_{l\in N}\mu_{jl}Y_j\ge Y,\sum_{j\in N}\mu_{jl}=1,0\le\mu_{jl}$, therefore we conclude that $\sum_{j\in N}\sum_{l\in N}\mu_{jl}X_j\le X,\sum_{j\in N}\sum_{l\in N}\mu_{jl}Y_j\ge Y,\sum_{j\in N}\mu_{jl}\ge 1,0\le\mu_{jl},\ l\in N,$ $j\in N$, to prove that $T_{SA}^{CRA}\subseteq T_{MSA}^{CRA}$, so it is enough show only that $\mu_{jl}\le 1,\ l\in N,\ j\in N$, this hold since $\sum_{j\in N}\mu_{jl}=1$ and $0\le\mu_{jl},\ l\in N,\ j\in N$. Then $T_{SA}^{CRA}\subseteq T_{MSA}^{CRA}$.

In second state, suppose that there is a $\mu_{jl}>0,\ l\in N'-N$ or $j\in N'-N$, namely this index is belong to a aggregated DMUs as $(X,Y)\in T_{SA}^{CRA}$. Then

$\sum_{j\in N'}\sum_{l\in N'}\mu_{jl}X_j\le X,\sum_{j\in N'}\sum_{l\in N'}\mu_{jl}Y_j\ \ \ge Y,\sum_{j\in N'}\mu_{jl}=1,\mu_{jl}\ge 0,\ l\in N',\ j\in N'$.

Let $\widetilde{N}$ show a subset of N including index of observed DMUs that generate (X,Y). In other words, $\sum_{j\in\widetilde{N}}\sum_{l\in\widetilde{N}}\mu_{jl}X_j=X,\sum_{j\in\widetilde{N}}\sum_{l\in\widetilde{N}}\mu_{jl}Y_j=Y$, we define

$$\tilde{\mu}_{jl}=\begin{cases}1 & j\in\widetilde{N}\ \ and\ \ l\in\widetilde{N}\\ \mu_{jl} & j\in N-\widetilde{N}\ \ or\ \ l\in N-\widetilde{N}\end{cases},$$

then we have $0\le\tilde{\mu}_{jl}\le 1,\ l\in N,\ j\in N$ and $\sum_{j\in N}\tilde{\mu}_{jl}\ge 1$. These intensity variable satisfies in the $\sum_{j\in N}\sum_{l\in N}\tilde{\mu}_{jl}X_j\le X,\sum_{j\in N}\sum_{l\in N}\tilde{\mu}_{jl}Y_j\ge Y$. Therefore $(X,Y)\in T_{MSA}^{CRA}$ and we conclude that $T_{SA}^{CRA}\subseteq T_{MSA}^{CRA}$.

To prove that $T_{MSA}^{CRA}\subseteq T_{SA}^{CRA}$. Let $(X,Y)\in T_{MSA}^{CRA}$, according to the definition of the set T_{MSA}^{CRA}, there is a vectors $\mu_l=(\mu_{1l},\dots,\mu_{nl})$, $l\in N$, such that $\sum_{j\in N}\sum_{l\in N}\mu_{jl}X_j\le X$, $\sum_{j\in N}\sum_{l\in N}\mu_{jl}Y_j\ge Y,\sum_{j\in N}\mu_{jl}\ge 1,0\le\mu_{jl}\le 1$. We consider two different state, for the first state, suppose $\sum_{j\in N}\mu_{jl}=1,\ l\in N$ for the later intensity variable of T_{MSA}^{CRA}. Then same vector works for T_{SA}^{CRA}. Let

$$\bar{\mu}_{jl}=\begin{cases}0 & j\in N'-N\ \ or\ \ l\in N'-N\\ \mu_{jl} & j\in N\ \ and\ \ l\in N\end{cases}$$

Then we have $\sum_{j\in N'}\sum_{l\in N'}\bar{\mu}_{jl}X_j\le X,\sum_{j\in N'}\sum_{l\in N'}\bar{\mu}_{jl}Y_j\ \ \ge Y,\sum_{j\in N'}\bar{\mu}_{jl}=1,\bar{\mu}_{jl}\ge 0,\ \in N',\ j\in N'$. Therefore $T_{MSA}^{CRA}\subseteq T_{SA}^{CRA}$.

In the second state, we have $\sum_{j\in N}\mu_{jl}>1,\ l\in N$, consider DMUs with $0\le\mu_{jl},\ j\in N\ \ and\ \ l\in N$, we define the new intensity variable

$$\bar{\mu}_{jl} = \begin{cases} 0 & j \in N' - N \quad or \quad l \in N' - N \\ \frac{\mu_{jl}}{\sum_{j\in N} \mu_{jl}} & j \in N \quad and \quad l \in N \end{cases}$$

Therefore, it is established that $\sum_{j\in N'} \bar{\mu}_{jl} = 1,\ \bar{\mu}_{jl} \geq 0,\ l \in N',\ j \in N'$. Also, $\sum_{j\in N'} \sum_{l\in N'} \mu_{jl} X_j \leq X, \sum_{j\in N'} \sum_{l\in N'} \mu_{jl} Y_j \geq Y$. This show the $(X,Y) \in T_{SA}^{CRA}$. Therefore $T_{MSA}^{CRA} \subseteq T_{SA}^{CRA}$. So, the proof is complete.■

According to Theorem 1, the CRA model in semi-additive production technology can be presented based on the definition of set T_{MSA}^{CRA} as follows.

$$\begin{aligned} & Min \quad \theta_{MSA}^{CRA} \\ & s.t. \quad \left(\theta_{MSA}^{CRA} \textstyle\sum_{j\in N} X_j , \sum_{j\in N} Y_j \right) \in T_{MSA}^{CRA}. \end{aligned} \tag{5}$$

According to the definition of the set T_{MSA}^{CRA}, model (5) becomes as follows.

$$\begin{aligned} \theta_{MSA}^{CRA^*} = Min \quad & \theta_{MSA}^{CRA} \\ s.t. \quad & \textstyle\sum_{j\in N} \sum_{l\in N} \mu_{jl}\, x_{ij} \leq \theta_{MSA}^{CRA} \left(\sum_{j\in N} x_{ij}\right), \quad i = 1, \dots, m, \\ & \textstyle\sum_{j\in N} \sum_{l\in N} \mu_{jl}\, y_{rj} \geq \sum_{j\in N} y_{rj}, \quad r = 1, \dots, s, \\ & \textstyle\sum_{j\in N} \mu_{jl} \geq 1,\ l \in N, \\ & 0 \leq \mu_{jl} \leq 1,\ l,j \in N. \end{aligned} \tag{6}$$

In contrast to model (2), model (6) has $n^2 + 1$ variables and $m + s + n + n^2$ constraints. The advantage model (6) compared to model (2) is the fact that it avoids many calculations related to solving the model (2). Also, in model (6), we consider only observed DMUs explicitly, although all aggregate units are implicitly checked.

In order to obtain efficient targets on the efficiency frontier from the PPS, we continue to obtain the mix inefficiency by finding slacks in the input and output components. Assume α_i and β_r are slacks of components of inputs and outputs that show slack along the input and additional increase along the output respectively. Suppose $\theta_{MSA}^{CRA^*}$ is the optimal objective function of model (6). For this purpose, we solve the model (7) as follows.

$$\begin{aligned} Max \quad & \left(\textstyle\sum_{i=1}^{m} \alpha_i + \sum_{r=1}^{s} \beta_r\right) \\ s.t. \quad & \textstyle\sum_{j\in N} \sum_{l\in N} \mu_{jl}\, x_{ij} + \alpha_i = \theta_{MSA}^{CRA^*} \left(\sum_{j\in N} x_{ij}\right),\ i = 1, \dots, m, \\ & \textstyle\sum_{j\in N} \sum_{l\in N} \mu_{jl}\, y_{rj} - \beta_r = \sum_{j\in N} y_{rj},\ r = 1, \dots, s, \\ & \textstyle\sum_{j\in N} \mu_{jl} \geq 1,\ l \in N, \\ & 0 \leq \mu_{jl} \leq 1,\ l,j \in N. \end{aligned} \tag{7}$$

Suppose $\left(\mu_{jl}^*, \alpha_i^*, \beta_r^*,:\ l \in N,\ j \in N,\ i = 1, \dots, m,\ r = 1, \dots, s\ \right)$ is an optimal solution of model (6). We define the operating point or efficient target corresponding to $DMU_l,\ l \in N$ as follows.

$$\begin{aligned} & X_l = \left(\textstyle\sum_{j\in N} \mu_{jl}^* x_{1j} , \sum_{j\in N} \mu_{jl}^* x_{2j} , \dots, \sum_{j\in N} \mu_{jl}^* x_{mj}\right), \quad l \in N, \\ & Y_l = \left(\textstyle\sum_{j\in N} \mu_{jl}^* y_{1j} , \sum_{j\in N} \mu_{jl}^* y_{2j} , \dots, \sum_{j\in N} \mu_{jl}^* y_{sj}\right), \quad l \in N. \end{aligned} \tag{8}$$

Model (7) obtain additional reduction of each input and expansion of each output.

Theorem 2.2 The operating point resulting of model (7) that is defined in relation (8) is Pareto efficient.

Proof. By contradiction, suppose the operating point (X_l, Y_l) be inefficient. Then there exists a vector $(\bar{X}, \bar{Y}) \in T_{MSA}^{CRA}$ so that $\bar{X} \leq X_l$ and $\bar{Y} \geq Y_l$, then according to $(\bar{X}, \bar{Y}) \in T_{MSA}^{CRA}$, there exists a vector $(\bar{\mu}_{jl}: l \in N, j \in N)$ so that

$$\bar{x}_{il} = \sum_{j\in N} \bar{\mu}_{jl} x_{ij} \leq x_{il},\ l \in N, \quad i = 1, \dots, m, \tag{9}$$
$$\bar{y}_{rl} = \sum_{j\in N} \bar{\mu}_{jl} y_{rj} \geq y_{rl},\ l \in N, \quad r = 1, \dots, s.$$

At least one of the inequalities in (9) is strictly established. Considering the constraints in model (7), it can be seen that vector $\left(\bar{\mu}_{jl}, \bar{\alpha}_i = \alpha_i^* + x_{il} - \bar{x}_{il},\ \bar{\beta}_r = \beta_r^* + \bar{y}_{rl} - y_{rl}: l \in N, j \in N, i = 1, \dots, m,\ r = 1, \dots, s\right)$ is a feasible solution for model (7). The objective function value of model (7) for this solution is as follows.

$$\sum_{i=1}^{m} \bar{\alpha}_i + \sum_{r=1}^{s} \bar{\beta}_r = \sum_{i=1}^{m} (\alpha_i^* + x_{il} - \bar{x}_{il}) + \sum_{r=1}^{s} (\beta_r^* + \bar{y}_{rl} - y_{rl}) = \sum_{i=1}^{m} \alpha_i^* + \sum_{r=1}^{s} \beta_r^* + \sum_{i=1}^{m} (x_{il} - \bar{x}_{il}) + \sum_{r=1}^{s} (\bar{y}_{rl} - y_{rl}) > \sum_{i=1}^{m} \alpha_i^* + \sum_{r=1}^{s} \beta_r^*.$$

which is contradictory to the optimality of the solution $\left(\mu_{jl}^*, \alpha_i^*, \beta_r^*,: l \in N,\ j \in N,\ i = 1, \dots, m,\ r = 1, \dots, s\right)$ for model (8). Therefore, the contradiction is invalid and the proof is complete.■

3. A generalized CRA model in the semi-additive production technology

The CRA models proposed by Lozano and Villa [25] and Asmild et al. [1] suppose that the centralized DM can allocate inputs and outputs across all DMUs. In many cases, some DMUs are geographically dispersed, or it may be impossible to reallocate inputs or transfer outputs among DMUs due to adjustment costs, regulations, or indivisibility. Then, inputs may be reallocated or outputs may be transferred, but only in some DMUs, not all. To consider the said items, Fang [6] proposed a generalized CRA model under VRS technology. He shows that the proposed general CRA model extends the models of Lozano and Villa [25] and Asmild et al. [1] to a more general case. In this section, we used Fang's [6] idea and propose a general CRA model in the semi-additive production technology in order to consider whether inputs may be reallocated or outputs may be transferred in the CRA model. The general CRA model in semi-additive production technology is proposed as follows:

$$\begin{aligned} \theta_{MSA}^{GCRA^*} = Min \quad & \theta_{MSA}^{GCRA} \\ s.t. \quad & \sum_{j\in N} \sum_{l\in N} \mu_{jl}\, x_{ij} \leq \theta_{MSA}^{GCRA} \left(\sum_{j\in N} t_j x_{ij}\right), \quad i = 1, \dots, m, \\ & \sum_{j\in N} \sum_{l\in N} \mu_{jl}\, y_{rj} \geq \sum_{j\in N} t_j y_{rj}, \quad r = 1, \dots, s, \\ & \sum_{l\in N} \mu_{jl} \geq t_j,\ j \in N, \\ & \sum_{j\in N} \mu_{jl} \geq \sigma_l,\ l \in N, \\ & 0 \leq \mu_{jl} \leq 1,\ l, j \in N. \end{aligned} \tag{10}$$

The variable $t_j \in \{0,1\},\ j \in N$ is a binary variable that the central DM should decide which DMUs to be consider in the overall optimization or not. If we put $t_j = 1$ then DMU_j is in the overall optimization process of CRA, otherwise by putting $t_j = 0$, we do not consider it.

Also, the central DM maybe decide that DMU_j not to be used as peers, it may follow a different business strategy and it not comparable with other DMUs. Then the central DM decide that exclude DMU_l from the reference set of other DMUs and puts $\sigma_l = 0$ in the model (10).

In model (10), by putting $t_j = 1, \quad j \in N$ and $\sigma_l \geq 0, \quad l \in N$, we obtain the CRA model (6).

Suppose we consider the set of efficient DMU in the semi-additive technology as set

ESD and inefficient DMUs to ISD in this paper. Then $N = ESD \cup ISD$. We can only consider adjustments of the subset ISD of inefficient DMUs using the efficient points of subset ESD as peers. By considering $t_j = 1$, for $j \in ISD$ and $\sigma_l = 0$, for $l \notin ESD$. We proposed CRA model for inefficient DMUs in the semi-additive technology as follows.

$$
\begin{aligned}
Min \quad & \theta_{MSA-IE}^{CRA} \\
s.t. \quad & \textstyle\sum_{j\in ISD}\sum_{l\in ESD}\mu_{jl}\,x_{il} \le \theta_{MSA-IE}^{CRA}\left(\sum_{j\in ISD}x_{ij}\right), \quad i = 1,\dots,m, \\
& \textstyle\sum_{j\in ISD}\sum_{l\in ESD}\mu_{jl}\,y_{rl} \ge \sum_{j\in ISD}y_{rj}, \quad r = 1,\dots,s, \\
& \textstyle\sum_{l\in ESD}\mu_{jl} \ge 1,\ j \in ISD, \\
& 0 \le \mu_{jl} \le 1,\ l \in ESD,\ j \in ISD.
\end{aligned} \tag{11}
$$

In the CRA model (10), the central DM adjusts the levels of all inputs or transfers outputs between DMUs. But, in some cases, some input variables cannot be adjusted or some output variables cannot be transferred between DMUs. Then, we extend the CRA model (10) to consider non-adjustable input variables and non-transferable outputs. For this purpose, we define index sets as including input variables that can be non-adjustable variables as INA and index sets as including output variables that can be non-transferable as ONT. We modify the CRA model (10) as follows:

$$
\begin{aligned}
{\theta_{MSA-AT}^{GCRA}}^{*} = Min \quad & \theta_{MSA-AT}^{CRA} \\
s.t. \quad & \textstyle\sum_{j\in N}\sum_{l\in N}\mu_{jl}\,x_{ij} \le \theta_{MSA-AT}^{GCRA}\left(\sum_{j\in N}t_j x_{ij}\right), \quad i \notin INA, \\
& \textstyle\sum_{j\in N}\sum_{l\in N}\mu_{jl}\,x_{ij} \le x_{ij}, \qquad i \in INA, \\
& \textstyle\sum_{j\in N}\sum_{l\in N}\mu_{jl}\,y_{rj} \ge \sum_{j\in N}t_j y_{rj}, \qquad r \notin INT, \\
& \textstyle\sum_{j\in N}\sum_{l\in N}\mu_{jl}\,y_{rj} \ge y_{rj},\ r \in INT, \\
& \textstyle\sum_{l\in N}\mu_{jl} \ge t_j,\ j \in N, \\
& \textstyle\sum_{j\in N}\mu_{jl} \ge \sigma_l,\ l \in N, \\
& 0 \le \mu_{jl} \le 1,\ l, j \in N.
\end{aligned} \tag{12}
$$

The second constraint indicates that only input reduction is allowed for non-adjustable input variables, and the fourth constraint indicates that only output increments are allowed for non-transferable output variables.

4. Numerical example

In this section, with a simple numerical example, we describe the approach presented in this paper geometrically. Consider three DMUs according to Tables 1 and 2.

Table 1. The data set and the results of the evaluation DMUs in the numerical example.

DMUs	Input	Output	The efficiency scores under VRS technology	The efficiency scores under semi-additive technology
A	3	0.75	1	1
B	4	1.5	1	1
C	6	2	1	0.9167

Table 2. The targets of CRA model in the numerical example.

DMUs	The targets of CRA model under VRS technology		The targets of CRA model under semi-additive technology	
A	3.67	1.25	4	1.5
B	4	1.5	3.67	1.25
C	4	1.5	4	1.5

The PPS under VRS technology is shown in Figure 1. All three DMUs are efficient under VRS technology, as seen in Figure 1.

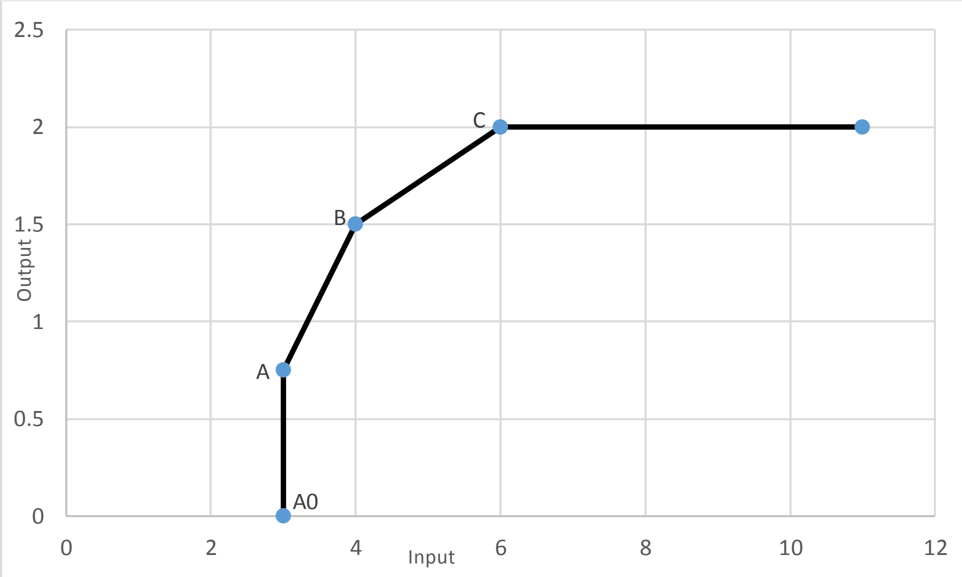

Figure 1. The PPS under VRS technology.

To illustrate PPS in semi-additive technology geometrically, we have three observed DMUs according to Table 1, as follows.

$A = (3, 0.75)$ ، $B = (4, 1.5)$ ، and $C = (6, 2)$.

Now, we create aggregated DMUs, as follows.

$D = A + B = (7, 2.25)$, $F = A + C = (9, 2.75)$, $G = B + C = (10, 3.5)$, $E = A + B + C = (13, 4.25)$.

To illustrate PPS, we consider horizontal axis as input-axis and vertical axis as output-axis. We consider DMUs in three different technologies including constant returns to scale (CRS), VRS, and semi-additive technology geometrically. The CRS technology is the biggest technology that includes all the other technologies. The PPS under CRS technology is included the region restricted by the input-axis and the right-hand side of the line starting from the origin and passing the B in the first area of the coordinate system. The PPS under VRS technology is included the bounded region by the input-axis starting from A_0 and the segment A–B–C and the horizontal extension from C. The PPS in the semi-additive technology is bounded by the input-axis starting from A_0 passing the segment of A–B–G–E and horizontal extension from E as it is shown in Figure 2. As we expect, the PPS under semi-additive assumption is bigger than the PPS of the BCC model. The DMU C is efficient DMU under VRS technology. However, it is an inefficient DMU in semi-additive technology. DMU $F = A + C = (9, 2.75)$ is an aggregated DMU, it an inefficient DMU in the semi-additive technology. To assess the efficiency score of DMU F, we can project it on efficient frontier of semi-additive technology. We depict DMU F at point F_1 on the efficiency frontier of the PPS corresponding to semi-additive technology radially in the input oriented. According to Fig. 2, the efficiency score is calculated as the ratio $\left|\frac{OF_1^X}{OF^X}\right| = 0.8611$. F^X and F_1^X, represent the image of the points F and F_1 on the input-axis, respectively.

The results related to the efficiency scores of the original DMUs in semi-additive technology are given in the last column of Table 2, as it can be seen that units A and B are efficient and unit C is inefficient. Also, we obtain the targets corresponding to the original

DMUs with the CRA approach under VRS and semi-additive technologies. The results show in the Table 2. Now, we obtain the targets corresponding to the aggregated DMUs based on the CRA approach in semi-additive technology. The results are shown in Table 3. The DMUs A, B, and C project onto the efficient frontier of PPs of semi-additive technology in point G. DMUs E, F, and G project at point B. DMU B project at a different point on the efficient frontier.

Figure 3, shows the projection of different DMUs by the CRA model. DMUs project on the Pareto efficiency frontier of semi-additive technology, although Pareto efficient DMUs are not always predicted on their own.

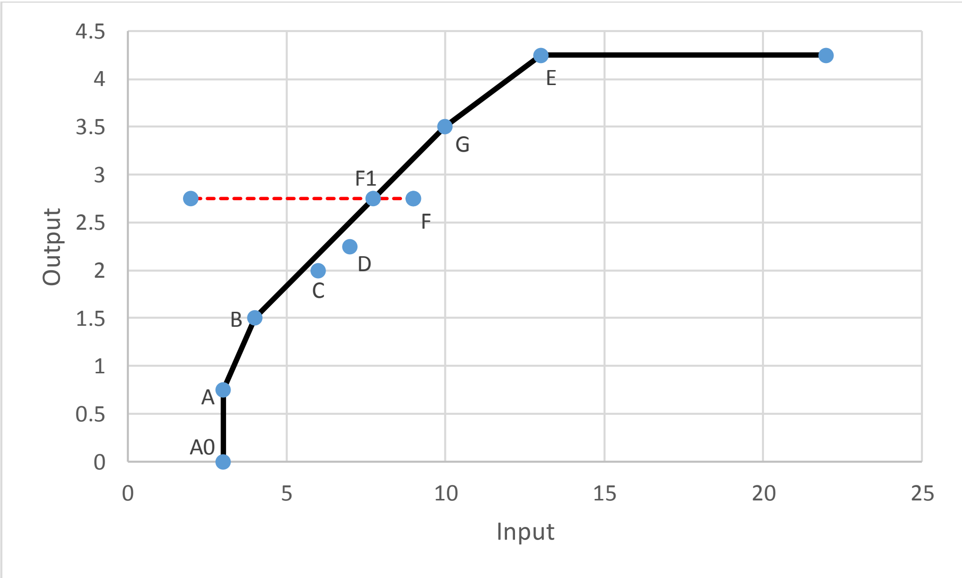

Figure 2. The PPS of semi-additive technology for the data set in the Table 1.

Table 3. The targets of CRA model under semi-additive technology.

DMUs	Input	Output
A	10	3.5
B	10	3.5
C	10	3.5
D	5.5	2
E	4	1.5
F	4	1.5
G	4	1.5

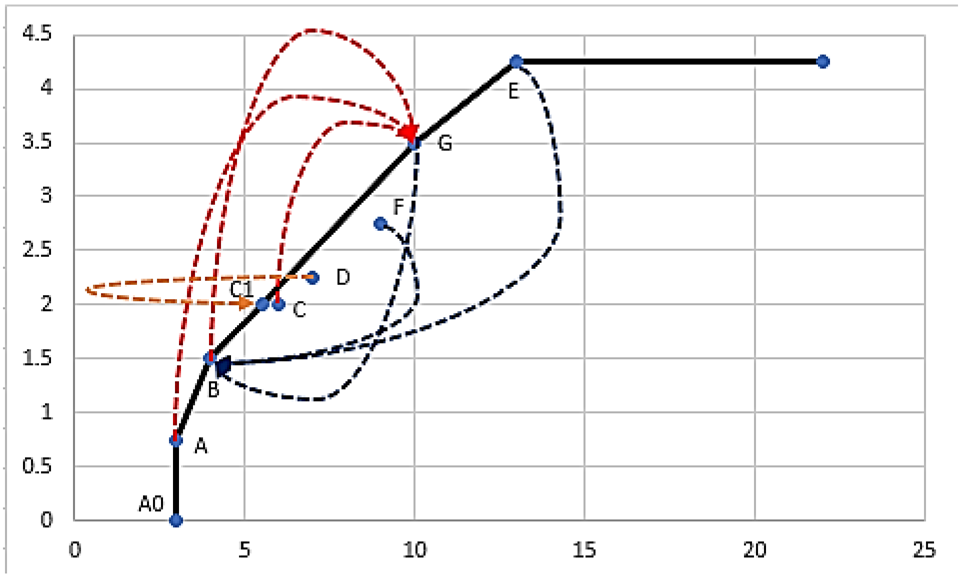

Figure 3. The projections of the aggregated DMUs onto efficient frontier of semi-additive technology.

5. Case study

In this section, to analyses the results of the CRA models presented in this paper, we analyses an empirical dataset consisting of 30 chain stores in Iran. Chain stores with the same brand are actually run by a single manager and usually offer similar products. In fact, chain stores in Iran and other countries around the world were developed in order to lower distribution costs, reduce product prices, and also reduce commuting in cities. With the development of chain stores, many items needed by people on a daily basis were offered at reasonable and lower prices. Also, in most of these stores, there are various products in addition to food, such as clothes, household appliances, etc., which reduces the costs of distribution and commuting for people in the city to buy these items. We use this data set to compare the results of the models in this paper in semi-additive technology with the results of the models presented by Lozano and Villa [25], Asmild et al. [1], and Fang [6] in VRS technology. These supermarkets belong to a chain with a central unit that has the authority and power to monitor the performance of all branches and allocate resources between them.

Table 4 shows the input and output data set for these 30 chain stores. In this evaluation, for each chain store as a DMUs, two inputs including man-hours ($10^3 h$) and size ($10^3 m^2$) and two output variables including sales (billion rials) and profit (billion rials) are considered. Man-hours refers to the labor force used in a certain period and the total size of the retail space of each chain store.

Table 4. Data set for 30 Chain stores.

Chain stores	Man-hours	Size	Sales	Profit	Efficiency (VRS)	Efficiency (semi-additive)
CHS1	129.5	10.8	211.8	10.5	**0.995**	**0.679**
CHS2	38.9	4	88.4	3.21	0.89	0.89
CHS3	134.5	11.8	221.8	11.5	**1**	**0.66**
CHS4	35.6	17.44	196.4	17.8	1	1
CHS5	96.7	5.21	78.6	3.3	0.516	0.516
CHS6	119.5	5.8	59.7	15.1	1	1
CHS7	79.3	13.21	98.4	1.23	0.398	0.398
CHS8	33.6	6.1	55.3	1.88	0.872	0.872

CHS9	59.4	8.94	87.9	2.85	0.512	0.512
CHS10	102.4	11.56	197.5	7.23	**0.849**	**0.583**
CHS11	45.7	2.78	65.5	2.45	0.894	0.894
CHS12	152.6	6.45	166.7	3.66	**0.751**	**0.749**
CHS13	95.2	5.31	133.1	1.98	**0.725**	**0.721**
CHS14	29.3	3.97	78.2	5.34	1	1
CHS15	98.9	14.87	68.3	4.21	0.296	0.296
CHS16	110.5	8.5	72.7	13.21	**0.751**	**0.681**
CHS17	54.1	11.88	154.8	8.27	0.711	0.711
CHS18	117.8	13.27	78.6	3.65	0.271	0.271
CHS19	41.2	2.22	77.8	0.98	1	1
CHS20	56.2	6.97	89	0.12	0.569	0.569
CHS21	101.5	4.26	75.8	6.87	0.778	0.778
CHS22	111.5	10.6	65.7	18.87	**1**	**0.911**
CHS23	123.4	12.6	100.8	3.54	0.296	0.296
CHS24	115.5	7.8	69.2	14.87	**0.884**	**0.791**
CHS25	42.3	3.28	63.2	8.21	1	1
CHS26	45.3	4.88	168	3.95	1	1
CHS27	85.6	9.2	108.5	1.43	0.431	0.431
CHS28	109.4	5.1	86.7	10.8	**0.979**	**0.895**
CHS29	67.8	19.7	74.2	12.94	0.55	0.55
CHS30	125.5	9.8	61.7	12.87	**0.622**	**0.578**
Total	2558.7	258.3	3154.3	212.82	0.751	0.708

As can be seen in the last two columns of Table 4, chain stores CHS4, CHS6, CHS14, CHS19, CHS25, and CHS26 are efficient in two technologies. But chain stores CHS3 and CHS22 are efficient in VRS technology, while they are inefficient in semi-additive technology. Chain stores that have different efficiency scores are shown in the last two columns of Table 4 in bold.

Table 5 shows input and output targets for every chain store by the conventional input-orientated DEA under the semi-additive production technology. According to Table 5, it can be seen that the reduction of the total first and second inputs is equal to 1014.55, 93.97, respectively, and decreases to 60.35% and 61.62%, respectively, when each chain store becomes technically efficient independently in the semi-additive production technology. The increment of the total first and second outputs is equal to 191.78 and 22.21, respectively. In other words, after solving the radial model and using the model solution in the second phase, considering the inefficiency slack values in the input and output components, the first and second outputs increased by 5.53% and 9.45%, respectively.

Table 5. Input and output targets of chain stores in the semi-additive production technology.

Chain stores	Man-hours	Size	Sales	Profit
CHS1	77.85	7.34	211.8	10.5
CHS2	34.63	3.56	88.4	3.9
CHS3	83.42	7.79	221.8	11.5
CHS4	35.6	17.44	196.4	17.8
CHS5	41.76	2.69	78.6	3.3
CHS6	119.5	5.8	59.7	15.1
CHS7	31.59	5.26	98.4	6.29
CHS8	29.3	3.97	78.2	5.34
CHS9	30.42	4.58	87.9	5.78
CHS10	59.7	6.74	197.5	7.23
CHS11	40.86	2.49	75.25	2.45
CHS12	51.13	4.83	166.7	3.66
CHS13	56.11	3.83	133.1	2.28

CHS14	29.3	3.97	78.2	5.34
CHS15	29.3	3.97	78.2	5.34
CHS16	75.24	5.79	81.41	13.21
CHS17	38.49	8.45	154.8	8.47
CHS18	31.95	3.6	78.6	4.39
CHS19	41.2	2.22	77.8	0.98
CHS20	31.99	3.97	89	4.89
CHS21	42.47	3.31	75.8	6.87
CHS22	101.62	9.66	107.41	18.87
CHS23	36.54	3.73	100.8	3.82
CHS24	91.31	6.17	88.66	14.87
CHS25	42.3	3.28	63.2	8.21
CHS26	45.3	4.88	168	3.95
CHS27	36.86	3.96	108.5	4.08
CHS28	68.64	4.56	86.7	10.8
CHS29	37.27	10.83	135.2	12.94
CHS30	72.5	5.66	80.05	12.87
Total	1544.15	164.33	3346.08	235.03

Since these chain stores are controlled by the CEO of the company and belong to the same organization, he controls all these DMUs simultaneously. Therefore, the central DM can simultaneously monitor these 30 chain stores to minimize the total input consumption by all DMUs, leading to the use of the CRA approach in models (6) and (7). Table 6 proposes the results of input and output targets for chain stores of models (6) and (7). As can be seen in Table 6, according to the CRA models (6) and (7) in the semi-additive production technology, the stores CHS1-CHS15 chooses store CHS11 as their target. CHS17-CHS30 choose the store CHS8 as their target (projection point). Store CHS16 obtains a different projection point on the efficiency frontier of semi-additive production technology, which is a virtual unit on this frontier.

Now we compare the results of models (6) and (7) in Table 6 with the results of the conventional non-centralized approach in semi-additive production technology in Table 5. As can be seen, the total man-hours and size have been further reduced than if they were reduced based on a conventional, non-centralized approach. This difference is equal to 347.421 and 33.114, respectively. According to the conventional non-centralized approach, the reduction of input components, namely man-hours and size, for each of the chain stores is done individually, while the reduction of input components is simultaneously based on the CRA modes (6) in the semi-additive production technology.

Table 6. Results of input and output targets for chain stores by models (6) and (7).

Chain stores	Man-hours	Size	Sales	Profit
CHS1	45.7	2.78	65.5	2.45
CHS2	45.7	2.78	65.5	2.45
CHS3	45.7	2.78	65.5	2.45
CHS4	45.7	2.78	65.5	2.45
CHS5	45.7	2.78	65.5	2.45
CHS6	45.7	2.78	65.5	2.45
CHS7	45.7	2.78	65.5	2.45
CHS8	45.7	2.78	65.5	2.45
CHS9	45.7	2.78	65.5	2.45
CHS10	45.7	2.78	65.5	2.45
CHS11	45.7	2.78	65.5	2.45

CHS12	45.7	2.78	65.5	2.45
CHS13	45.7	2.78	65.5	2.45
CHS14	45.7	2.78	65.5	2.45
CHS15	45.7	2.78	65.5	2.45
CHS16	40.829	4.116	61.394	2.221
CHS17	33.6	6.1	55.3	1.88
CHS18	33.6	6.1	55.3	1.88
CHS19	33.6	6.1	55.3	1.88
CHS20	33.6	6.1	55.3	1.88
CHS21	33.6	6.1	55.3	1.88
CHS22	33.6	6.1	55.3	1.88
CHS23	33.6	6.1	55.3	1.88
CHS24	33.6	6.1	55.3	1.88
CHS25	33.6	6.1	55.3	1.88
CHS26	33.6	6.1	55.3	1.88
CHS27	33.6	6.1	55.3	1.88
CHS28	33.6	6.1	55.3	1.88
CHS29	33.6	6.1	55.3	1.88
CHS30	33.6	6.1	55.3	1.88
Total	1196.729	131.216	1818.094	65.291

Table 7 shows the results of input and output targets for chain stores in VRS technology according to the approach of Lozano and Villa [25]. Now we compare the results of the CRA model in VRS (Table 7) and semi-additive production technologies (Table 6). As can be seen from the last row of Tables 6 and 7, the total man-hours are 1195.788 and 1196.729 in the VRS and semi-additive production technologies, respectively. This reduction in VRS technology is more than 0.941. In contrast, total size (second input) is 131.475 and 131.216 in the VRS and semi-additive production technologies, respectively. This reduction in semi-additive production technology is more than 0.259. The difference in total sales and profit (as total outputs) in VRS and semi-additive production technologies is 0.793 and 0.045, respectively. Then increase total sales and profit (total outputs) in the semi-additive production technology. As can be seen in Tables 6 and 7, the resulting projection points for chain stores from the approach of Lozano and Villa [25] in VRS technology and our approach in semi-additive production technology (models 6 and 7) are the same. The projection point of store CHS16 is different between the two technologies.

Table 7. Results of input and output targets for chain stores in VRS technology by approach of Lozano and Villa [25].

Chain stores	Man-hours	Size	Sales	Profit
CHS1	45.7	2.78	65.5	2.45
CHS2	45.7	2.78	65.5	2.45
CHS3	45.7	2.78	65.5	2.45
CHS4	45.7	2.78	65.5	2.45
CHS5	45.7	2.78	65.5	2.45
CHS6	45.7	2.78	65.5	2.45
CHS7	45.7	2.78	65.5	2.45
CHS8	45.7	2.78	65.5	2.45
CHS9	45.7	2.78	65.5	2.45
CHS10	45.7	2.78	65.5	2.45
CHS11	45.7	2.78	65.5	2.45
CHS12	45.7	2.78	65.5	2.45

CHS13	45.7	2.78	65.5	2.45
CHS14	45.7	2.78	65.5	2.45
CHS15	45.7	2.78	65.5	2.45
CHS16	39.888	4.375	60.601	2.176
CHS17	33.6	6.1	55.3	1.88
CHS18	33.6	6.1	55.3	1.88
CHS19	33.6	6.1	55.3	1.88
CHS20	33.6	6.1	55.3	1.88
CHS21	33.6	6.1	55.3	1.88
CHS22	33.6	6.1	55.3	1.88
CHS23	33.6	6.1	55.3	1.88
CHS24	33.6	6.1	55.3	1.88
CHS25	33.6	6.1	55.3	1.88
CHS26	33.6	6.1	55.3	1.88
CHS27	33.6	6.1	55.3	1.88
CHS28	33.6	6.1	55.3	1.88
CHS29	33.6	6.1	55.3	1.88
CHS30	33.6	6.1	55.3	1.88
Total	1195.788	131.475	1817.301	65.246

Now, suppose that the central DM considers minimizing the total inputs consumed by the inefficient chain stores. We apply the subset of efficient chain stores as peers. Then we use the CRA model (11). The results of the CRA model (11) in the semi-additive production technology are shown in Table 8. The results show the total man-hours and size of all the inefficient chain stores are 1022.491 and 100.501, respectively. The total man-hours are reduced by 60.04%. Also, the total size is reduced by 61.1%.

As can be seen in Table 8, according to the CRA model (11) in the semi-additive production technology, the stores CHS1-CHS5, CHS13, CHS23, and CHS24 choose store CHS26 as their target. Also, the stores CHS8-CHS12, CHS15-CHS20, CHS22, CHS27, and CHS28 chose store CHS25 as their target. In other words, these points are depicted on the efficiency chain store CHS24 at the efficiency frontier of semi-additive production technology. Other stores, namely CHS7, CHS29, and CHS30, project at a different efficient point from the efficiency frontier.

Table 8. Results of input and output targets for inefficient chain stores by model (11).

Chain stores	Man-hours	Size	Sales	Profit
CHS1	45.3	4.88	168	3.95
CHS2	45.3	4.88	168	3.95
CHS3	45.3	4.88	168	3.95
CHS5	45.3	4.88	168	3.95
CHS7	44.347	4.372	134.704	5.303
CHS8	42.3	3.28	63.2	8.21
CHS9	42.3	3.28	63.2	8.21
CHS10	42.3	3.28	63.2	8.21
CHS11	42.3	3.28	63.2	8.21
CHS12	42.3	3.28	63.2	8.21
CHS13	45.3	4.88	168	3.95
CHS15	42.3	3.28	63.2	8.21
CHS16	42.3	3.28	63.2	8.21
CHS17	42.3	3.28	63.2	8.21
CHS18	42.3	3.28	63.2	8.21

CHS20	42.3	3.28	63.2	8.21
CHS21	45.3	4.88	168	3.95
CHS22	42.3	3.28	63.2	8.21
CHS23	45.3	4.88	168	3.95
CHS24	45.3	4.88	168	3.95
CHS27	42.3	3.28	63.2	8.21
CHS28	42.3	3.28	63.2	8.21
CHS29	33.398	3.753	73.472	6.245
CHS30	32.446	10.696	137.224	11.562
Total	1022.491	100.501	2511	161.44

The results of input and output targets for chain stores in VRS technology by the approach of Asmild et al. [1] are given in Table 9. Now we compare the results of the CRA model in VRS technology by the inefficient chain stores (according to Table 9) and the results of the CRA model (11) in semi-additive production technology by the inefficient chain stores (according to Table 8). As can be seen from the last row of Tables 9 and 10, the total man-hours are 877.015 and 1022.491 in the VRS and semi-additive production technologies, respectively. This reduction in VRS technology is more than the amount of 145. 476. Also, the total sizes are 86.982 and 100.501 in VRS and semi-additive production technologies, respectively. This reduction in VRS technology is more than 13.519. The difference in total sales and profit (as total outputs) in VRS and semi-additive production technologies is 287.499 and 30.37, respectively. Then increase total sales and profit (total outputs) in the semi-additive production technology is more. It should be noted that this comparison was made in order to analyses the results of two technologies. Because the number of efficient and inefficient stores is different between the two technologies, Chain stores CHS4, CHS6, CHS14, CHS19, CHS25, and CHS26 are efficient in two technologies. However, stores CHS3 and CHS22 are efficient in VRS technology, while they are inefficient in semi-additive technology. Chain stores that have different efficiency scores are shown in the last two columns of Table 4. As can be seen in Tables 8 and 9, the resulting projection points from the approach of Asmild et al. [1] in VRS technology and our approach in the semi-additive production technology (model 11) by the inefficient chain stores are different. The projection point of store CHS16 is different between the two technologies. According to the approach of Asmild et al. [1] in VRS technology in Table 9, the inefficient stores CHS1, CHS2, CHS8, CHS9, and CHS10 choose store CHS14 as their target. The inefficient stores CHS7, CHS12, and CHS18 chose store CHS25 as their target. Also, the inefficient stores CHS21-CHS30 chose store CHS26 as their target. Other stores, namely CHS5, CHS11, and CHS20, project at a different efficient point from the efficiency frontier.

Table 9. Results of input and output targets for inefficient chain stores in VRS technology by approach of Asmild et al. [1].

Chain stores	Man-hours	Size	Sales	Profit
CHS1	29.3	3.97	78.2	5.34
CHS2	29.3	3.97	78.2	5.34
CHS5	32.644	3.478	78.088	4.115
CHS7	42.3	3.28	63.2	8.21
CHS8	29.3	3.97	78.2	5.34
CHS9	29.3	3.97	78.2	5.34
CHS10	29.3	3.97	78.2	5.34
CHS11	42.152	3.137	65.167	7.236
CHS12	42.3	3.28	63.2	8.21

CHS13	42.3	3.28	63.2	8.21
CHS15	42.3	3.28	63.2	8.21
CHS16	42.3	3.28	63.2	8.21
CHS17	42.3	3.28	63.2	8.21
CHS18	42.3	3.28	63.2	8.21
CHS20	42.519	3.397	70.846	7.899
CHS21	45.3	4.88	168	3.95
CHS23	45.3	4.88	168	3.95
CHS24	45.3	4.88	168	3.95
CHS27	45.3	4.88	168	3.95
CHS28	45.3	4.88	168	3.95
CHS29	45.3	4.88	168	3.95
CHS30	45.3	4.88	168	3.95
Total	877.015	86.982	2223.501	131.07

In the CRA models (6), (7) and (11) propose in this paper, all the efficient chain stores are can be considered as the benchmarks. We consider only some efficient chain stores as benchmarks. The store CHS26 is an efficient store in the semi-additive technology from Table 4. Assume store CHS26 follows a different business strategy, and then we should exclude it from the benchmark set for chain stores. Therefore, we remove this store from the set of benchmark stores and solve model (11). For this purpose, we put $\delta_{26} = 0$. Table 10 shows the results of input and output targets for inefficient chain stores by model (11) regardless of the store CHS26 as a benchmark store. Therefore, for solving CRS model (11) in the semi-additive technology, we let $\delta_{26} = 0$. The inefficient stores CHS2, CHS3 and CHS5 consider virtual store $(83.5,5.5,141,9.19)$ as their target or benchmark. The stores CHS9-CHS22 select efficient store CHS14 as their target or benchmark. Other stores choose different virtual units on the efficiency frontier of semi-additive production technology. As can be seen in the last row of Table 9, the total man-hours and size decreased to 1399.886 (54.71%) and 144.401 (55.9%) compared to the first total man-hours and size of stores in Table 4, respectively.

Table 10. Results of input and output targets for inefficient chain stores by model (11) with $\delta_{26} = 0$.

Chain stores	Man-hours	Size	Sales	Profit
CHS1	80.809	5.355	135.919	9.126
CHS2	83.5	5.5	141	9.19
CHS3	83.5	5.5	141	9.19
CHS5	83.5	5.5	141	9.19
CHS7	41.2	2.22	77.8	0.98
CHS8	31.46	3.652	78.127	4.549
CHS9	29.3	3.97	78.2	5.34
CHS10	29.3	3.97	78.2	5.34
CHS11	29.3	3.97	78.2	5.34
CHS12	29.3	3.97	78.2	5.34
CHS13	29.3	3.97	78.2	5.34
CHS15	29.3	3.97	78.2	5.34
CHS16	29.3	3.97	78.2	5.34
CHS17	29.3	3.97	78.2	5.34
CHS18	29.3	3.97	78.2	5.34
CHS20	29.3	3.97	78.2	5.34
CHS21	29.3	3.97	78.2	5.34

CHS22	29.3	3.97	78.2	5.34
CHS23	70.5	6.19	156	6.32
CHS24	71.6	7.25	141.4	13.55
CHS27	71.6	7.25	141.4	13.55
CHS28	70.5	6.19	156	6.32
CHS29	48.545	5.462	106.954	9.075
CHS30	70.5	6.19	156	6.32
Total	1158.814	113.899	2511	161.44

Similarly, we can remove a specific efficient store from the set of benchmark stores. Tables 11 and 12 show the results of solving model (11) in the semi-additive production technology by removing stores CHS25 and CHS14 from the set of target or benchmark stores, respectively.

Table 11. Results of input and output targets for inefficient chain stores by model (11) with $\delta_{25} = 0$.

Chain stores	Man-hours	Size	Sales	Profit
CHS1	119.5	5.8	59.7	15.1
CHS2	119.5	5.8	59.7	15.1
CHS3	119.5	5.8	59.7	15.1
CHS5	73.956	5.235	126.175	8.256
CHS7	45.3	4.88	168	3.95
CHS8	45.3	4.88	168	3.95
CHS9	45.3	4.88	168	3.95
CHS10	45.294	4.88	167.969	3.95
CHS11	60.189	7.298	192.756	8.033
CHS12	29.3	3.97	78.2	5.34
CHS13	29.3	3.97	78.2	5.34
CHS15	29.3	3.97	78.2	5.34
CHS16	29.3	3.97	78.2	5.34
CHS17	29.3	3.97	78.2	5.34
CHS18	29.3	3.97	78.2	5.34
CHS20	29.3	3.97	78.2	5.34
CHS21	29.3	3.97	78.2	5.34
CHS22	29.3	3.97	78.2	5.34
CHS23	29.3	3.97	78.2	5.34
CHS24	29.3	3.97	78.2	5.34
CHS27	29.3	3.97	78.2	5.34
CHS28	74.6	8.85	246.2	9.29
CHS29	29.3	3.97	78.2	5.34
CHS30	29.3	3.97	78.2	5.34
Total	1158.639	113.883	2511	161.439

Table 12. Results of input and output targets for inefficient chain stores by model (11) with $\delta_{14} = 0$.

Chain stores	Man-hours	Size	Sales	Profit
CHS1	45.3	4.88	168	3.95
CHS2	45.3	4.88	168	3.95
CHS3	45.3	4.88	168	3.95
CHS5	45.3	4.88	168	3.95

CHS7	45.3	4.88	168	3.95
CHS8	45.3	4.88	168	3.95
CHS9	45.3	4.88	168	3.95
CHS10	45.3	4.88	168	3.95
CHS11	43.534	3.938	106.297	6.458
CHS12	42.3	3.28	63.2	8.21
CHS13	42.3	3.28	63.2	8.21
CHS15	42.3	3.28	63.2	8.21
CHS16	42.3	3.28	63.2	8.21
CHS17	42.3	3.28	63.2	8.21
CHS18	42.3	3.28	63.2	8.21
CHS20	42.3	3.28	63.2	8.21
CHS21	42.3	3.28	63.2	8.21
CHS22	42.3	3.28	63.2	8.21
CHS23	42.3	3.28	63.2	8.21
CHS24	42.3	3.28	63.2	8.21
CHS27	42.3	3.28	63.2	8.21
CHS28	42.3	3.28	63.2	8.21
CHS29	41.537	2.545	73.327	3.195
CHS30	37.046	13.51	165.776	13.457
Total	1034.417	101.673	2511	161.44

As stated earlier, the size (as input) is actually a non-adjustable input in real life, which means that the space of the store rental floor cannot be changed in a practical sense. To better illustrate such real-world situations, suppose that the size (as input) is non-adjustable, then we apply the CRA model (12) with a size non-adjustable.

Table 13 propose the results of input and output targets for chain stores of model (12) when we consider the input size as non-adjustable. As can be seen, the stores CHS1-CHS22 and CHS29 select the efficient store CHS14 as their target. The stores CHS24-CHS28 and CHS30 select the efficient store CHS4 as their target or benchmark. The store CHS23 chose a different virtual unit on the efficiency frontier of semi-additive production technology as a benchmark. The total man-hours as adjustable input decreased to 1636.618 compared to its initial value in Table 4.

Table 13. Results of input and output targets for chain stores by model (12).

The results of model (12) with size non-adjustable.			
Chain stores	Man-hours	Sales	Profit
CHS1	29.3	78.2	5.34
CHS2	29.3	78.2	5.34
CHS3	29.3	78.2	5.34
CHS4	29.3	78.2	5.34
CHS5	29.3	78.2	5.34
CHS6	29.3	78.2	5.34
CHS7	29.3	78.2	5.34
CHS8	29.3	78.2	5.34
CHS9	29.3	78.2	5.34
CHS10	29.3	78.2	5.34
CHS11	29.3	78.2	5.34
CHS12	29.3	78.2	5.34
CHS13	29.3	78.2	5.34
CHS14	29.3	78.2	5.34

CHS15	29.3	78.2	5.34
CHS16	29.3	78.2	5.34
CHS17	29.3	78.2	5.34
CHS18	29.3	78.2	5.34
CHS19	29.3	78.2	5.34
CHS20	29.3	78.2	5.34
CHS21	29.3	78.2	5.34
CHS22	29.3	78.2	5.34
CHS23	34.582	177.3	15.787
CHS24	35.6	196.4	17.8
CHS25	35.6	196.4	17.8
CHS26	35.6	196.4	17.8
CHS27	35.6	196.4	17.8
CHS28	35.6	196.4	17.8
CHS29	29.3	78.2	5.34
CHS30	35.6	196.4	17.8
Total	922.082	3154.3	245.407

In the following, we suppose that the output sales are non-transferable. Tables 14 show the results of model (12) by considering sales as non-transferable output. The stores CHS1-CHS16 obtain efficient store CHS25 as their benchmark. The stores CHS18-CHS29 select the efficient store CHS14 as their benchmark. The stores CHS17 and CHS30 choose a different virtual unit on the efficiency frontier of semi-additive production technology as a benchmark. The total man-hours and size as inputs decreased to 1456.134 and 146.997 compared to their initial values in Table 4, respectively. The total profit as transferable output does not change and is 212.82. Because model (12) is input-orientated, only inputs are reduced.

Table 14. Results of input and output targets for chain stores by model (12).

The results of model (12) sales non-transferable.			
Chain stores	Man-hours	Size	Profit
CHS1	42.3	3.28	8.21
CHS2	42.3	3.28	8.21
CHS3	42.3	3.28	8.21
CHS4	42.3	3.28	8.21
CHS5	42.3	3.28	8.21
CHS6	42.3	3.28	8.21
CHS7	42.3	3.28	8.21
CHS8	42.3	3.28	8.21
CHS9	42.3	3.28	8.21
CHS10	42.3	3.28	8.21
CHS11	42.3	3.28	8.21
CHS12	42.3	3.28	8.21
CHS13	42.3	3.28	8.21
CHS14	42.3	3.28	8.21
CHS15	42.3	3.28	8.21
CHS16	42.3	3.28	8.21
CHS17	30.009	3.932	5.496
CHS18	29.3	3.97	5.34
CHS19	29.3	3.97	5.34
CHS20	29.3	3.97	5.34

CHS21	29.3	3.97	5.34
CHS22	29.3	3.97	5.34
CHS23	29.3	3.97	5.34
CHS24	29.3	3.97	5.34
CHS25	42.3	3.28	8.21
CHS26	29.3	3.97	5.34
CHS27	29.3	3.97	5.34
CHS28	29.3	3.97	5.34
CHS29	29.3	3.97	5.34
CHS30	31.157	7.941	9.014
Total	1102.566	111.303	212.82

6. Conclusion

The traditional DEA models obtain targets for each DMU separately; the CRA is a different approach that projects all the DMUs simultaneously onto the efficiency frontier. In this paper, we propose a new CRA model for the semi-additive production technology in DEA. The semi-additive production technology considers the aggregation of DMUs in the process of efficiency analysis. We showed that the CRA model in the semi-additive production technology obtains the projection of the DMUs based only on the observed DMUs, and there is no need to consider all aggregations of DMUs. In the first stage, the CRA model seeks radial reductions in the total consumption of all the inputs. In the second stage, the CRA model seeks to maximize the value of inefficiency slacks in the input and output components. In this model, we suppose that the central DM aims to minimize the total input consumption by all DMUs in the organization. We develop this model for the general case of CRA in the semi-additive production technology based on the idea of Fang [6]. We developed a CRA model for considering adjustments to inefficient units by using efficient units in the semi-additive production technology based on the idea of Asmild et al. [1]. In the following, we propose the CRA model in semi-additive production technology for incorporating non-adjustable input variables and non-transferable outputs. Finally, for the reader's better understanding of CRA models, we gave a simple numerical example. Geometrically, we showed how DMUs are depicted in semi-additive production technology. An application of the approach presented in this paper to a set of data from a set of 30 chain stores in Iran that operate under central management was also presented. The performance of each model in determining the benchmark or operational points was explained. We have shown that the total value of each input is reduced by the CRA model in the semi-additive production technology. In each case, this percentage reduction was determined. It was also found that some stores may choose other, more efficient stores as targets. Thus, by solving only one model, we can provide efficient targets for all stores on the efficiency frontier of semi-additive production technology. The CRA model significantly reduces the number of calculations. As a further work, we develop the models presented in this paper for non-radial models in DEA in the semi-additive production technology. Also, the models in this paper can be developed for the case where the inputs and outputs are in the form of imprecise data, such as fuzzy or robust data. These models can be proposed for cost-fixed allocation in DEA.

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