A Novel Framework for Efficiency Assessment in Multi-Divisional Systems: Application of Stackelberg Game Theory and Dynamic Network DEA

ABSTRACT

Intermediate products in network DEA interconnect the divisions that make up DMUs, while carry-over activities are responsible for establishing connections across multiple periods in dynamic DEA. These variables offer managers more detailed insights into inefficiencies within the organizations in different periods. A major challenge in performance evaluation is the dual role of these measures. Intermediate measures act as outputs for one division and inputs for another, creating a conflict that hinders managers from accurately assessing inefficiencies related to these measures. This paper proposes a novel approach to address this conflict in multi-divisional production systems by utilizing Stackelberg game theory. By employing this theory, we decompose the system's overall efficiency into leader's and follower's efficiencies, providing a more detailed evaluation of performance. Our model makes a significant contribution to the literature by developing a dynamic network DEA model. This model resolves conflicts arising from the dual role of connecting measures and establishes a Stackelberg-game dynamic between periods and divisions, ensuring continuity of flows. Additionally, in real-world problems, some data change proportionally (radially), while others change non-proportionally (non-radially). This paper applies a hybrid model, combining both radial and non-radial approaches, for efficiency evaluation. To verify the proposed model, we assess the performance of 14 petrochemical units over two years. The results show that the intermediate measures linking the followers to the leader need to be fully controlled by the leader.

Keywords: Dynamic Network DEA, Stackelberg Game theory, Intermediate measures, Carry-over activities, Hybrid DEA model.

1. Introduction

Data Envelopment Analysis (DEA), introduced by Charnes et al. in 1978, serves as a robust methodology for assessing the efficiency of peer decision-making units (DMUs), which utilize the same multiple inputs to generate the same multiple outputs. Traditional DEA models view DMUs as black boxes, neglecting their internal structures (Färe and Grosskopf, 2000). Consequently, these models do not provide detailed insights for decision makers to pinpoint inefficiencies within DMUs arising from interactions among various production stages (Avkiran, 2009). To remedy this, researchers have developed Network DEA models, which evaluate efficiency scores by considering the internal structures of DMUs. This approach involves defining intermediate measures that capture the interactions between different production stages. Intermediate products, generated in one stage and utilized in subsequent stages, function both as outputs and inputs, leading to a dual role that can complicate efficiency measurement. The early stage strives to maximize its output for efficiency, while the later stage aims to minimize its input, creating a potential conflict. Scholars have proposed various solutions to this issue. Kao and Hwang (2008) combined the efficiency scores of two stages using a multiplicative (geometric) method, whereas Chen et al. (2009) used a weighted additive model. Generally, differences in these models stem from the assumptions about including intermediate measures in the constraints. Kao (2018) classified links into three types: independent, relational, and cooperative. The cooperative type is further divided into scenarios where the goal is either set at the current level or optimally determined. Kao (2019) redefined the relational case from Kao (2018) as non-cooperative and also discussed the cooperative case, which aligns with the continuity condition introduced by Tone and Tsutsui (2009). This concept has been widely adopted in subsequent studies to develop network DEA models for efficiency measurement, including works by Rasinojehdehi and Valami (2023) and Lu et al. (2016). Hassanzadeh and Mostafaee (2018) explored six scenarios related to link control among different stages for evaluating intermediate measures in Network DEA. These scenarios include control by the earlier stage, the later stage, both stages in a non-cooperative manner, both stages in a cooperative manner, no stage in a cooperative manner, and no stage in a non-cooperative manner. Tone and Tsutsui (2009) referred to the scenarios where both stages control the link cooperatively as free links and where no stage controls the link cooperatively as fixed links. Additionally, Rasinojehdehi et al. (2023) proposed a Network SBM model that endogenously determines the role of intermediate measures to optimize the objective function.

Network DEA is particularly noteworthy for its ability to investigate inefficiency sources within DMUs. However, incorporating dynamic aspects into network systems poses significant challenges. While network modeling offers a theoretical framework for analyzing the internal structure of DMUs, dynamic modeling elucidates the connections between periods through carry-over activities. Dynamic Network DEA (DNDEA) models address the complexity of efficiency evaluation by integrating multiple dynamic stages linked through network structures in each period. This approach involves comparing a series of static models (Chao et al., 2018), allowing for a comprehensive analysis. It enables the observation of changes in overall efficiency, dynamic adjustments in divisional efficiency, potential improvements, and efficiency estimates derived from a holistic assessment that considers interactions between periods and divisions (Torres & Ramos, 2024). The innovative DNDEA approach provides a deeper insight into the traditionally opaque processes of DMUs, capturing interactions across different time periods.

The seminal study by Chen (2009) marks a pioneering advancement in the research area of DNDEA. This development integrates dynamic effects into the network structure to assess the efficiency of subunits and the overall system. Subsequently, scholars and researchers have conducted numerous theoretical studies, leading to a wide range of practical applications across various sectors such as healthcare (Mirmozaffari and kamal, 2023), banking (Chaoqun et al., 2024), transportation (Losa et al., 2020; Rasinojehdehi et al., 2023; Liu et al., 2024), education (Tran, 2021; Bagherzadeh Valami & Raeinojehdehi, 2016), research and development (Nojehdehi, 2011; Tatlari et al., 2023; Lim & Kim, 2022), and energy (Alizadeh et al., 2020).

One of the most renowned models in the DNDEA literature is the one proposed by Tone and Tsutsui (2014). They developed their model based on the Slack-Based Measure (SBM) approach, building upon their earlier Network SBM (NSBM) and dynamic SBM models from 2009 and 2010, respectively (Tone & Tsutsui, 2009, 2010). Their models assume the continuity of link flows between divisions and periods. Lozano (2015) extends this framework by relaxing the intermediate product constraints of the NSBM free link model, allowing for more intermediate products to be produced internally than consumed.

Hinojosa et al. (2016) classify DEA models into different categories, such as game theory, relational, weighted additive efficiency decomposition, slack-based network measure, slack-based inefficiency measure, dynamic network, and the Malmquist index approach. Additionally, DEA network models have various extensions and variations, some utilizing the envelope model and others the multiplier model. These models differ in their approaches to handling the overall efficiency index and the relationships between processes. A literature review by Alves and Meza (2023) indicates that game theory-based DEA methods have been less extensively explored in DEA research.

Game theory, originating from the revolutionary work of John von Neumann and Oskar Morgenstern (1944), was further extended by inspiring work of John Forbes Nash (1950), who mathematically formulated strategic interactions among rational decision-makers (DMs). Game theory has applications across various social science domains and is extensively utilized in decision-making problems. Myerson (1998) defined game theory as the study of mathematical models for strategic interactions among rational decision-makers (DMs). Here, we review several studies that integrate game theory with DEA.

Liang et al. (2008) proposed a two-stage DEA model using concepts from both noncooperative and cooperative games, illustrated with a manufacturer and retailer scenario. Yu and Rakshit (2023) combined DEA and game theory in the airline industry to evaluate input and output targets. Their model facilitates impartial and rational negotiation between inputs and outputs, aiming to achieve fair and optimal solutions for both.

Shi et al. (2020) proposed a novel parallel fuzzy DEA model for assessing efficiency in parallel systems with two components, leveraging Stackelberg game theory. Their model decomposes the system's efficiency into a series of sub-system efficiency scores, simplifying the process by avoiding the need for α-cuts. They implemented this model to evaluate the performance of chemistry departments at UK universities.

Tavana and Khalili-Damghani (2014) presented a novel two-stage Stackelberg DEA model incorporating uncertainty within a two-stage network production process. The aim of their study is to measure both system and stages' efficiency in parallel networks under a fuzzy environment, utilizing Stackelberg game theory. Specifically, their model explores the leader-follower dynamics between process efficiencies.

Zare et al. (2018) introduced a hybrid model combining DEA and game theory to enhance performance measurement in healthcare centers. Their approach integrates input and output variables related to healthcare efficiency, identified through a comprehensive literature review and internal organizational data. Expert input is utilized to weight and prioritize these indicators. The model's applicability and effectiveness are demonstrated through a case study, which provides valuable insights into the efficiency levels of regional healthcare centers. This hybrid model offers a refined method for assessing and improving performance and productivity in healthcare management systems.

Ma et al. (2020) presented additive centralized and Stackelberg DEA models for evaluating two-stage systems with shared resources. Their approach addresses the inefficiency in traditional DEA models by considering the internal structure of decision-making units (DMUs). The proposed models evaluate efficiency in cooperative and non-cooperative game settings, highlighting the impact of strategic decisions on resource allocation. Through a case study in the banking sector, they demonstrated the model's ability to decompose and measure efficiency more accurately.

Lim and Song (2024) introduced a "pre-mortem Stackelberg game approach" as an alternative to traditional decentralized network DEA models, addressing the issue of infeasibility in such models. By incorporating non-radial slacks, the proposed approach optimally adjusts intermediate products and improves follower efficiency. An empirical study on a Korean life insurance company demonstrated that the model provides accurate efficiency measures, even when conventional DEA models fail. This method enhances the use of non-cooperative game theory in network DEA, especially under variable returns to scale (VRS) assumptions.

Tabasi et al. (2019) enhanced DEA by introducing multistage DEA, which accounts for internal relations between DMUs. They compare efficiencies derived from Nash, Centralized, and Stackelberg games, finding that Centralized and Stackelberg games yield higher efficiencies than Nash. They also integrate fuzzy and grey theory to handle real-world data uncertainties, with their case study focusing on Iran Khodro Company.

In this paper, we introduce a novel dynamic network DEA model that integrates game theory to assess the overall, period, and divisional efficiency scores of multi divisional network systems. This study makes three significant contributions to the literature. The first is the development of an efficiency measurement framework that combines dynamic network DEA with Stackelberg game theory, a relatively underexplored area in DEA research. The second contribution is addressing the dual role of intermediate measures and carry-over activities, incorporating their inefficiencies into the efficiency measurement process. The third contribution is the development of a novel hybrid DNDEA model that incorporates data changing both radially and non-radially.

The remainder of this paper is organized as follows: Section 2 presents foundational concepts to enhance understanding of the proposed model. Section 3 introduces the hybrid DNDEA model. In Section 4, we validate the model with a numerical example from the petrochemical industry. Finally, Section 5 concludes the paper.

2. Preliminaries

In this section, we review the key preliminaries necessary for this research.

2.1 hybrid DEA model

According to the literature, the DEA models proposed by the scholars and researchers are divided into two main categories; radial and non-radial. Radial models, such as the CCR model (Charnes et al. 1978) with CRS assumption and the BCC model (Banker et al. 1984) with VRS assumption, suffer from a significant limitation: they do not consider non-radial input or output slacks. In contrast, non-radial models, including the SBM model (Tone 2001), the additive model (Charnes et al. 1985), the Russell measure model (Russel 1985), and the Enhanced Russell Measure (ERM), (Pastor et al1999), fail to account for the radial characteristics of inputs or outputs. To address these shortcomings, Tone (2004) introduced a hybrid measure of efficiency that integrates these models into a unified framework. It is important to understand the differences in input and output characterization. Radial inputs or outputs change proportionally, while non-radial inputs or outputs change disproportionately. For instance, consider $(x_1, x_2, \dots, x_m)$ as radial inputs and $(x_{m+1}, x_{m+2}, \dots, x_n)$ as non-radial inputs.
Radial inputs are those that change proportionately, represented as $(\alpha x_1, \alpha x_2, \dots, \alpha x_m)$ with $\alpha \geq 0$. In contrast, non-radial inputs are those that change non-proportionately, with each input adjusting based on its slack. As described by Tone (2004), for representing the hybrid measure, it is assumed that the input and output data matrices are $X \in R_+^{m\times n}$ and $Y \in R_+^{s\times r}$, where n, m, and s denote the number of DMUs, inputs, and outputs, respectively. The input matrix X can be divided into radial and non-radial components, $X^R \in R_+^{m_1\times n}$ and $X^{NR} \in R_+^{m_2\times n}$, with $m = m_1 + m_2$. Similarly, the output matrix Y is divided into radial and non-radial components, $Y^R \in R_+^{s_1\times n}$ Y and $Y^{NR} \in R_+^{s_2\times n}$, with $s = s_1 + s_2$, as equation (1):

$$X = (X^R, X^{NR})^T \text{ and } Y = (Y^R, Y^{NR})^T \tag{1}$$

It is important to note that the input and output data must be positive, i.e., $X > 0$ and $Y > 0$. The production possibility set (PPS) P is defined as equation (2):

$$P = \{(x, y) \mid x \geq X\lambda, y \leq Y\lambda, \lambda \geq 0, \lambda \in R^n\} \qquad (2)$$

Consider a specific DMU under evaluation, denoted as p, with the inputs and outputs represented as $(x_p^R, x_p^{NR}, y_p^R, y_p^{NR}) \in P$. Given the distinct characteristics of inputs and outputs, we have the relationships presented in equation (3):

$$\begin{aligned} \theta x_p^R &= X^R\lambda + s^{R-} \\ x_p^{NR} &= X^{NR}\lambda + s^{NR-} \\ \phi y_p^R &= Y^R\lambda - s^{R+} \\ y_p^{NR} &= Y^{NR}\lambda - s^{NR+} \end{aligned} \qquad (3)$$

Given the conditions $\theta \leq 1, \varphi \geq 1$, and $\lambda, s^{R-}, s^{NR-}, s^{R+}, s^{NR+} \geq 0.$, the input slacks are denoted by the vectors $s^{R-} \in R^{m_1}$ and $s^{NR-} \in R^{m_2}$, which correspond to the surplus in radial and non-radial inputs, respectively. Similarly, the output slacks are represented by the vectors $s^{R+} \in R^{s_1}$ and $s^{NR+} \in R^{s_2}$, corresponding to the shortages in radial and non-radial outputs, respectively. As referenced in Tone (2004), the hybrid model is presented in equations (4) – (9):

$$\rho_p = \min \frac{1 - \left(\frac{m_1}{m}\right)(1-\theta) - \frac{1}{m}\sum_{i=1}^{m_2} \frac{s_i^{NR-}}{x_{ip}^{NR}}}{1 + \left(\frac{s_1}{s}\right)(\phi - 1) - \frac{1}{s}\sum_{r=1}^{s_2} \frac{s_i^{NR+}}{y_{rp}^{NR}}} \qquad (4)$$

s.t.

$$\theta x_p^R \geq \lambda X^R \qquad (5)$$

$$x_p^{NR} = \lambda X^{NR} + s^{NR-} \qquad (6)$$

$$\phi y_p^R \leq \lambda Y^R \qquad (7)$$

$$y_p^{NR} = \lambda Y^{NR} - s^{NR+} \qquad (8)$$

$$\theta \leq 1, \phi \geq 1, \lambda, s^{NR-}, s^{NR+} \geq 0 \qquad (9)$$

The objective function (4) is denoted as ρ_p, which is unit invariant, meaning it remains unchanged regardless of the measurement units of the data. It's important to note that the objective function is not directly influenced by $s^{R-}$and s^{R+}, reflecting the free disposability of these radial slacks. Let $(\theta^*, \varphi^*, \lambda^*, s^{*NR-}, s^{*NR+})$ be an optimal solution for the model, presented in equations (4-9). According to Tone (2004), a DMU is considered hybrid-efficient if and only if $\rho_p{}^*$ = 1. This condition is met when, $\theta^* = 1$, $\varphi^* = 1$, and $s^{*NR-} = 0$, $s^{*NR+} = 0$.

This section reviewed the fundamentals of hybrid DEA models. In the next section, we introduce our novel hybrid DNDEA model.

2.2 Stackelberg Game and Its Justification in DNDEA

The Stackelberg model, first introduced by Heinrich Freiherr von Stackelberg in 1934, is an economic strategy game that depicts a competitive scenario in which a leading firm and a following firm make sequential decisions regarding market quantity. Building on this framework, Chen and Cruz (1972) and Simaan and Cruz (1973) extended the original static, two-person, non-cooperative, nonzero-sum Stackelberg game to a dynamic game format, incorporating asymmetric information. Later, Shimizu and Aiyoshi (1981) generalized the concept into a bi-level Stackelberg game, proposing the interior penalty method, also known as the barrier method, to solve nonlinear min-max optimization problems based on the Stackelberg game. Game theory serves as a widely recognized analytical tool for studying interactions among multiple participants and has applications across a variety of fields. Games are typically categorized by characteristics such as cooperation, simultaneity, information availability, symmetry, and sum type. In this paper, we concentrate on the Stackelberg game, a non-cooperative, sequential game model with perfect information. In Stackelberg games, players act in a specific order, where the player moving first is known as the leader, and the one moving subsequently is the follower. The leader's strategy is typically made with the expectation of how the follower will respond. In turn, the follower's actions are determined based on the leader's decisions.

The justification for using Stackelberg game theory within the DNDEA framework lies in its ability to model complex decision-making processes, which are common in large multi-divisional systems. Real-world organizations often feature a dominant division (e.g., production or operations) that drives decision-making, while other divisions adjust their strategies in response. This structure creates interdependencies between divisions, which can be effectively captured using Stackelberg game theory. Additionally, the dynamic nature of efficiency changes over time in multi-period systems necessitates the use of Dynamic Network DEA, ensuring that both cross-divisional and inter-temporal interactions are fully accounted for.

In this section, we reviewed the fundamental concepts necessary for this study. In the next section, we apply these concepts to present our novel model.

3. The proposed model

In this section, we propose a novel framework for evaluating the efficiency of a multi-divisional production systems in a dynamic environment. For simplicity, we assume that the division 1 ($Div\ 1$) acts as the leader and the other divisions as the followers. The leader in the 1st division optimizes its efficiency and determines the role of intermediate measures using binary here-and-now variables in the mixed integer linear programming approach described in equations (10) – (33). Once the roles of intermediates in the first stage are established, we assess the overall score by fixing the leader's score and the optimal configuration of intermediate measures and carry-overs determined from the leader's division.

Assume there are n DMUs indexed by $j = 1, \dots, n$ across $t = 1 \dots, T$ time periods. At each period, the DMUs have K divisions indexed by k, $(k = 1, 2, \ldots, K)$, utilize m_k external inputs, consisting of m_{k1} radial inputs and m_{k2} non-radial inputs, where $m_{1k} + m_{2k} = m_k$. Additionally, at each period, $Div\ k$ of the DMUs produce s_k external outputs, including s_{1k} radial outputs and s_{2k} non-radial outputs, where $s_{1k} + s_{2k} = s_k$.

Let x_{ijt}^{kR} ,$(i = 1, \dots, m_{1k})$, and x_{ijt}^{kNR} ,$(i = 1, \dots, m_{2k})$, represent radial and non-radial inputs, respectively, for the $Div\ k$ of the jth DMU at time t. Similarly, y_{rjt}^{kR} $(r = 1, \dots, s_{1k})$, and y_{rjt}^{kNR} , $(r = 1, \dots, s_{2k})$, denote radial and non-radial outputs for the $Div\ k$ of the jth DMU at time t. $z_{djt}^{NR(k,h)}$, $\left(d = 1, \dots, {d_1}^{(k,h)}\right)$ and $z_{djt}^{NR(k,h)}$, $(d = 1, \dots, {d_2}^{(k,h)})$ denote, respectively, radial and non-radial intermediate measures between $Div\ k$ and $Div\ h$ for the jth DMU at time t.

$z_{cj(t,t+1)}^{kgR}, (c = 1, \dots, n_{1gk})$ and $z_{cj(t,t+1)}^{kbR}, (c = 1, \dots, n_{1gk})$, denote respectively, good and bad radial carry-overs of $Div\ k$ between period t and period t+1, where $n_{1gk} + n_{1bk} = n_{1k}$ and $z_{cj(t,t+1)}^{kgNR}$, $(c = 1, \dots, n_{2gk})$ and $z_{cj(t,t+1)}^{kbNR}, (c = 1, \dots, n_{2bk})$, denote respectively, good and bad non-radial carry-overs of $Div\ k$ from period t to period t+1 , where and $n_{2bk} + n_{2gk} = n_{nk}$. the model for evaluating the leader's efficiency is outlined in equations (10) to (33).

$$\chi_p^{*leader} = Min\ \chi_p^{leader} = \frac{\sum_{t=1}^{T} W^t \left[1 - \frac{m_{11}}{m_1}(1-\theta_{t1}) - \frac{n_1 b_1}{nb_1}(1-\phi_{t1}) - \frac{d_1{}^{(h,1)}}{d^{(h,1)}}(1-\sigma_{1t}) - \frac{1}{m_k}\sum_{i=1}^{m_2} \frac{s_{ipt}^{-k,NR}}{x_{iot}^{NR}} - \frac{1}{nb_k}\sum_{i=1}^{n_2 b1} \frac{s_{cp(t,t+1)}^{-k,bNR}}{z_{cp(t,t+1)}^{bNR-k}} - \frac{1}{d_2{}^{(h,1)}}\sum_{i=1}^{d_2{}^{(h,1)}} \frac{s_{dpt}^{-NR}}{z_{dpt}^{NR}}\right]}{\sum_{t=1}^{T} W^t \left[1 - \frac{s_{11}}{s_1}(1-\rho_{t1}) - \frac{n_1 g_1}{ng_1}(1-\psi_{t1}) - \frac{d_1}{\mathrm{d}}(1-\tau_{1t}) - \frac{1}{m_1}\sum_{i=1}^{m_{21}} \frac{s_{ipt}^{-k,NR}}{x_{iot}^{NR}} - \frac{1}{ng_1}\sum_{i=1}^{n_2 g1} \frac{s_{cp(t,t+1)}^{-k,bNR}}{z_{cp(t,t+1)}^{bNR-k}} - \frac{1}{d_2{}^{(1,h)}}\sum_{i=1}^{d_2{}^{(1,h)}} \frac{s_{dpt}^{-NR}}{z_{dpt}^{NR}}\right]} \quad (10)$$

s.t. $\theta_{tk} x_{ipt}^{R} \geq \sum_{j=1}^{n} \lambda_j^{t,k} x_{ijt}^{R} \quad (i = 1, \dots, m_{1k}, \forall t, \forall \mathrm{k}) \quad (11)$

$$x_{ipt}^{k,NR} - s_{ipt}^{-k,NR} = \sum_{j=1}^{n} \lambda_j^{t,k} x_{ijt}^{k,NR} \quad (i = 1, \dots, m_{2k}, \forall t, \forall \mathrm{k}) \quad (12)$$

$$\rho_{tk} y_{rpt}^{k,R} \leq \sum_{j=1}^{n} \lambda_j^{t,k} y_{rjt}^{k,R} \quad (r = 1, \dots, s_{1k}, \forall t, \forall \mathrm{k}) \quad (13)$$

$$y_{rpt}^{k,NR} + s_{rpt}^{+k,NR} = \sum_{j=1}^{n} \lambda_j^{t,k} y_{nt}^{k,NR} \quad (r = 1, \dots, s_{2k}, \forall t, \forall \mathrm{k}) \quad (14)$$

$$\psi_{tk} z_{cp(t,t+1)}^{k,gR} \leq \sum_{j=1}^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,gR} \quad (c = 1, \dots, n_{1gk}, \forall t, \forall \mathrm{k}) \quad (15)$$

$$\phi_{tk} z_{cp(t,t+1)}^{kbR} \geq \sum_{j=1}^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,gR} \quad (c = 1, \dots, n_{1bk}, \forall t, \forall \mathrm{k}) \quad (16)$$

$$z_{cp(t,t+1)}^{k,gNR} + s_{cp(t,t+1)}^{+k,gNR} = \sum_{j=1}^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,gNR} \quad (c = 1, \dots, n_{2gk}, \forall t, \forall \mathrm{k}) \quad (17)$$

$$z_{cp(t,t+1)}^{k,bNR} - s_{cp(t,t+1)}^{-k,bNR} = \sum^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,bNR} \quad (c = 1, \dots, n_{2bk}, \forall t, \forall \mathrm{k}) \quad (18)$$

$$\sum_{j=1}^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,\alpha} = \sum_{j=1}^{n} \lambda_j^{t+1,k} z_{cj(t,t+1)}^{k,\alpha} \quad (c = 1, \dots, n_{2bk}, \forall t, \forall \mathrm{k}) \quad (19)$$

$$\tau_{tk} z_{dpt}^{R(k,h)} \leq \sum_{j=1}^{n} \lambda_j^{t,k} z_{djt}^{R} \quad (d = 1, \dots, d_1{}^{(k,h)}, \forall t) \quad (20)$$

$$M(1-\delta_{ht}) + \sigma_{th} z_{dpt}^{R(k,h)} \geq \sum_{j=1}^{n} \lambda_j^{t,h} z_{djt}^{R(k,h)} \quad (d = 1, \dots, d_1{}^{(k,h)}, \forall t) \quad (21)$$

$$0 \leq \sigma_{kt} \leq \delta_{kt} \quad \forall t \quad (22)$$

$$\tau_{kt} \geq 1 - M\delta_{kt} \quad \forall t \quad (23)$$

$$z_d^{NR(k,h)} - s_{dpt}^{-NR} = \sum_{j=1}^{n} \lambda_j^{t,h} z_{djt}^{NR(k,h)} \quad (d = 1, \dots, d_2{}^{(k,h)}, \forall t) \quad (24)$$

$$s_{dpt}^{-NR} \leq M\xi_{2t} \quad (d = 1, \dots, d_2{}^{(k,h)}, \forall t) \quad (25)$$

$$z_d^{NR} \leq z_{dpt}^{NR(k,h)} - M(1-\xi_{kt}) \quad (d = 1, \dots, d_2{}^{(k,h)}, \forall t) \quad (26)$$

$$z_d^{NR(k,h)} \leq z_{dpt}^{NR(k,h)} + M(1-\xi_{kt}) \quad (d = 1, \dots, d_2{}^{(k,h)}, \forall t) \quad (27)$$

$$z_d^{NR(k,h)} + s_{dpt}^{+NR(k,h)} = \sum_{j=1}^{n} \lambda_j^{t,k} z_{djt}^{NR(k,h)} \quad (d = 1, \dots, d_2{}^{(k,h)}, \forall t) \quad (28)$$

$$s_{dpt}^{+NR(k,h)} \leq M(1-\xi_{kt}) \quad (d = 1, \dots, d_2{}^{(k,h)}, \forall t) \quad (29)$$

$$zp_d^{NR(k,h)} \geq z_{dpt}^{NR(k,h)} - M\xi_{kt} \quad (d = 1, \dots, d_2{}^{(k,h)}, \forall t) \quad (30)$$

$$zp_{dt}^{NR(k,h)} \leq zp_{dpt}^{NR(k,h)} + M\xi_{kt} \quad (d = 1, \dots, d_2{}^{(k,h)}, \forall t) \quad (31)$$

$$\sum_{j=1}^{n} \lambda_j^{t,k} z_{djt}^{\beta(k,h)} = \sum_{j=1}^{n} \lambda_j^{t,h} z_{djt}^{\beta(k,h)} \qquad (d = 1, \ldots, \mathrm{d}^{(k,h)}, \forall t) \tag{32}$$

$$\xi_{kt}, \delta_{kt} \in \{0,1\}, z_d^{NR}{:}\, free, zp_d^{NR}{:}\, free, s_{ipt}^{-k,NR} \geq 0, s_{rpt}^{+k,NR} \geq 0, s_{cp(t,t+1)}^{+k,gNR} \geq 0, s_{cp(t,t+1)}^{-k,bNR} \geq 0, s_{dpt}^{-NR} \geq 0, s_{dpt}^{+NR} \geq 0, \lambda_j^{t,k} \geq 0, \forall t, \forall \mathrm{d}, \forall \mathrm{i}, \forall \mathrm{c} \tag{33}$$

Equation (10) represents the objective function, which evaluates the leader's score. Equations (11) and (12) define the constraints for radial and non-radial external inputs, respectively. Equations (13) and (14) outline the constraints for radial and non-radial external outputs. Equations (15)-(19) describe the constraints related to carry-over activities, with Equation (19) ensuring the continuity of carry-overs from one period to the next. The symbol α stands for radial good, radial bad, non-radial good, and non-radial bad carry-overs.

Equations (20) to (32) deal with constraints related to intermediate measures. Specifically, constraint (20) considers the radial intermediate measure $z_{djt}^{R(k,h)}$ as an output from the $Divk$, while constraint (21) considers it as an input to the follower.

The here-and-now binary variable ξ_{kt} determines whether $z_{djt}^{R(k,h)}$ is treated as an input to the $Divh$ or an output from the $Divk$. When $\delta_{kt} = 0$, $z_{djt}^{R(k,h)}$ is considered a radial output from the $Divk$, making constraint (21) redundant. Conversely, when $\delta_{kt} = 1$, $z_{djt}^{R(k,h)}$is considered a radial input to the $Divh$, making constraint (20) redundant. Similarly, constraints (24) through (31) address the non-radial intermediate measure $z_{djt}^{NR(k,h)}$. The binary variable ξ_{kt} determines the role of the non-radial intermediate measure $z_{djt}^{NR(k,h)}$. Constraints (32) keeps the continuity of linking flow from the leader's division to the follower's division. The symbol β stands for radial and non-radial intermediate measures.

After obtaining the leader scores and the optimal values of the here-and-now variables, the next step is to use the leader's efficiency as constraints and fix the role of intermediate measures using the optimal values of the here-and-now variables when evaluating the overall system's efficiency ($\chi_p^{overall}$). To do this, we solve the model outlined in equations (34) – (52).

$$\chi_p^{*overall} = Min\, \chi_p^{overall} = \frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{m_{1k}}{m_k}(1-\theta_{tk}) - \frac{n_1 b_k}{n b_k}(1-\phi_{tk}) - \frac{d_1}{\mathrm{d}}(1-\sigma_t) - \frac{1}{m_k}\sum_{i=1}^{m_2} \frac{s_{ipt}^{-k,NR}}{x_{iot}^{NR}} - \frac{1}{n b_k}\sum_{i=1}^{n_2 b} \frac{s_{cp(t,t+1)}^{-k,bNR}}{z_{cp(t,t+1)}^{bNR-k}} - \frac{1}{d}\sum_{i=1}^{n_2 b} \frac{s_{dpt}^{-NR}}{z_{dpt}^{NR}}\right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{s_{1k}}{s_k}(1-\rho_{tk}) - \frac{n_1 g_k}{n g_k}(1-\psi_{tk}) - \frac{d_1}{\mathrm{d}}(1-\tau_t) - \frac{1}{m_k}\sum_{i=1}^{m_{2k}} \frac{s_{ipt}^{-k,NR}}{x_{iot}^{NR}} - \frac{1}{n g_k}\sum_{i=1}^{n_2 gk} \frac{s_{cp(t,t+1)}^{-k,bNR}}{z_{cp(t,t+1)}^{bNR-k}} - \frac{1}{d}\sum_{i=1}^{d_2} \frac{s_{dpt}^{-NR}}{z_{dpt}^{NR}}\right]} \tag{34}$$

$$\text{s.t. } \theta_{tk} x_{ipt}^{R} \geq \sum_{j=1}^{n} \lambda_j^{tk} x_{ijt}^{R} \qquad (i = 1, \ldots, m_{1k}, \forall t, \forall \mathrm{k}) \tag{35}$$

$$x_{ipt}^{k,NR} - s_{ipt}^{-k,NR} = \sum_{j=1}^{n} \lambda_j^{t,k} x_{ijt}^{k,NR} \qquad (i = 1, \ldots, m_{2k}, \forall t, \forall \mathrm{k}) \tag{36}$$

$$\rho_{tk} y_{rpt}^{k,R} \leq \sum_{j=1}^{n} \lambda_j^{t,k} y_{rjt}^{k,R} \qquad (r = 1, \ldots, s_{1k}, \forall t, \forall \mathrm{k}) \tag{37}$$

$$y_{rpt}^{k,NR} + s_{rpt}^{+k,NR} = \sum_{j=1}^{n} \lambda_j^{t,k} y_{nt}^{k,NR} \qquad (r = 1, \ldots, s_{2k}, \forall t, \forall \mathrm{k}) \tag{38}$$

$$\psi_{tk} z_{cp(t,t+1)}^{k,gR} \leq \sum_{j=1}^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,gR} \qquad (c = 1, \ldots, n_{1gk}, \forall t, \forall \mathrm{k}) \tag{39}$$

$$\phi_{tk} z_{cp(t,t+1)}^{kbR} \geq \sum_{j=1}^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,gR} \qquad (c = 1, \ldots, n_{1bk}, \forall t, \forall \mathrm{k}) \tag{40}$$

$$z_{cp(t,t+1)}^{k,gNR} + s_{cp(t,t+1)}^{+k,gNR} = \sum_{j=1}^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,gNR} \qquad (c = 1, \ldots, n_{2gk}, \forall t, \forall \mathrm{k}) \tag{41}$$

$$z_{cp(t,t+1)}^{k,bNR} - s_{cp(t,t+1)}^{-k,bNR} = \sum^{n} \lambda_j^{tk} z_{cj(t,t+1)}^{k,bNR} \qquad (c = 1, \ldots, n_{2bk}, \forall t, \forall \mathrm{k}) \tag{42}$$

$$\sum_{j=1}^{n} \lambda_j^{t,k} z_{cj(t,t+1)}^{k,\alpha} = \sum_{j=1}^{n} \lambda_j^{t+1,k} z_{cj(t,t+1)}^{k,\alpha} \qquad (c = 1, \ldots, n_{2bk}, \forall t, \forall \mathrm{k}) \tag{43}$$

$$\tau_t z_{dpt}^{R} \leq \sum_{j=1}^{n} \lambda_j^{t,1} z_{djt}^{R} \qquad (d \in \{1, \ldots, d_1 | \xi_{1dt}^{*} = 0\}, \forall t) \tag{44}$$

$$\sigma_t z_{dpt}^{R} \geq \sum_{j=1}^{n} \lambda_j^{t,2} z_{djt}^{R} \qquad (d \in \{1, \ldots, d_1 | \xi_{1dt}^{*} = 1\}, \forall t) \tag{45}$$

$$0 \le \sigma_t \le 1 \qquad \forall t \tag{46}$$

$$\tau_t \ge 1 \qquad \forall t \tag{47}$$

$$z_{dp}^{NR} - s_{dpt}^{-NR} = \sum_{j=1}^{n} \lambda_j^{t,1} z_{djt}^{R} \qquad (d \in \{1, \dots, d_2 | \xi_{2dt}^{*} = 1\}, \forall t) \tag{48}$$

$$z_{dp}^{NR} + s_{dpt}^{+NR} = \sum_{j=1}^{n} \lambda_j^{t,2} z_{djt}^{R} \qquad (d \in \{1, \dots, d_2 | \xi_{2dt}^{*} = 0\}, \forall t) \tag{49}$$

$$\sum_{j=1}^{n} \lambda_j^{t,1} z_{djt}^{\beta} = \sum_{j=1}^{n} \lambda_j^{t,2} z_{djt}^{\beta} \qquad (d = 1, \dots, \mathrm{d}, \forall t) \tag{50}$$

$$\chi_p^{leader} = \chi_p^{*leader} \tag{51}$$

$$z_d^{NR}: free, zp_d^{NR}: free, s_{ipt}^{-k,NR} \ge 0, s_{rpt}^{+k,NR} \ge 0, s_{cp(t,t+1)}^{+k,gNR} \ge 0, s_{cp(t,t+1)}^{-k,bNR} \ge 0, s_{dpt}^{-NR} \ge 0, s_{dpt}^{+NR} \ge 0, \lambda_j^{t,k} \ge 0, \forall t, \forall \mathrm{d}, \forall \mathrm{i}, \forall \mathrm{c} \tag{52}$$

Note that the model presented in equations (34) – (52) has no binary variables, and in equation (51), the leader's score is fixed at the optimal value obtained from the previous stage. The followers score can be obtained from replacing the optimal solution of the model (34) – (52) into equation (53). The symbol * denotes the optimal value of the variable.

$$\chi_p^{*follower} = \frac{\sum_{t=1}^{T} W^t \left[1 - \frac{m_{12}}{m_2}(1-\theta_{t2}{}^{*}) - \frac{n_1 b_1}{nb_1}(1-\phi^{*}{}_{t2}) - \frac{d_1}{\mathrm{d}}(1-\sigma^{*}{}_t) - \frac{1}{m_2}\sum_{i=1}^{m_2} \frac{s_{ipt}^{-2,NR*}}{x_{iot}^{NR}} - \frac{1}{nb_{22}}\sum_{i=1}^{n_2 b1} \frac{s_{cp(t,t+1)}^{-2,bNR*}}{z_{cp(t,t+1)}^{bNR-2}} - \frac{1}{d}\sum_{i=1}^{n_2 b} \frac{s_{dpt}^{-NR*}}{z_{dpt}^{NR}} \right]}{\sum_{t=1}^{T} W^t \left[1 - \frac{s_{12}}{s_2}(1-\rho_{t2}) - \frac{n_1 g_2}{ng_2}(1-\psi_{t2}) - \frac{d_1}{\mathrm{d}}(1-\tau_t) - \frac{1}{m_2}\sum_{i=1}^{m_2 k} \frac{s_{ipt}^{-k,NR*}}{x_{iot}^{NR}} - \frac{1}{ng_2}\sum_{i=1}^{n_2 gk} \frac{s_{cp(t,t+1)}^{-2,bNR*}}{z_{cp(t,t+1)}^{2,bNR}} - \frac{1}{d}\sum_{i=1}^{d_2} \frac{s_{dpt}^{-NR*}}{z_{dpt}^{NR}} \right]} \tag{53}$$

In this section, we introduced a hybrid model within dynamic network DEA to assess the efficiency of a multi divisional production system in a dynamic environment. By incorporating the leader-follower relationship from the Stackelberg game, we evaluated the overall efficiency, as well as the efficiency of the leader and the follower. In the next section, we will verify the proposed model with a numerical example.

4. Numerical Example:

The petrochemical industry, as the primary sector of the oil industry, is a leading force in Iran's industrial landscape. The products derived from this industry are among the most significant components of non-oil exports, playing a crucial role in the nation's economic prosperity. Given the significance of the petrochemical industry and the scarcity and non-renewable nature of the raw materials used, evaluating the performance of companies operating in this sector is crucial. Therefore, in this section, we will assess the performance of petrochemical companies listed on the stock exchange, using the proposed model. The efficiency of 14 petrochemical units will be evaluated based on the proposed hybrid DNDEA model.

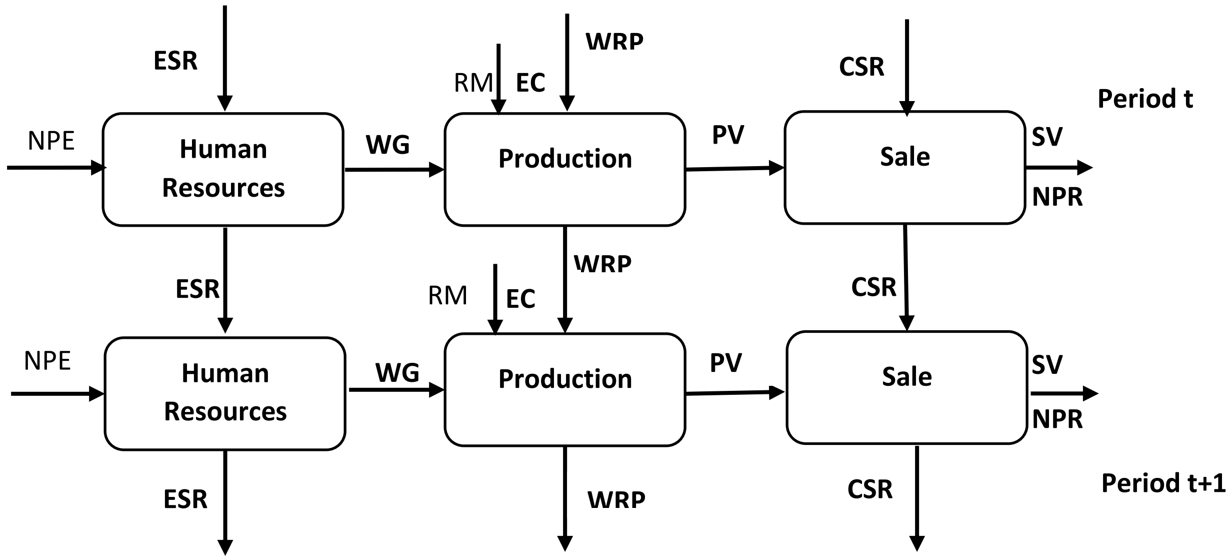

Figure 1. The network structure of petrochemical units

Figure 1 illustrates the network structure used in the efficiency analysis of the petrochemical units. The network is divided into three main divisions: Human Resources, Production, and Sales, with each handling specific inputs, intermediates, outputs, and carry-over activities. Human Resources Division manages external inputs such as the number of personnel (NPE), which impacts overall operational efficiency. Additionally, wages (WG) are treated as an intermediate measure, connecting the Human Resources and Production divisions. The employee satisfaction rate (ESR) is a carry-over activity, reflecting how employee engagement and satisfaction influence long-term performance. Production Division is responsible for converting inputs into outputs and intermediates. The raw material cost (RM) is an external input that plays a crucial role in production efficiency. Production volume (PV) acts as an intermediate measure, influencing both the internal performance of the division and the final outputs of the Sales division. Furthermore, waste reduction percentage (WRP) is treated as a carry-over activity, emphasizing the division's focus on minimizing waste and improving efficiency across periods. Sales Division handles the final outputs of the system, including sales volume (SV) and net profit (NPR), which are external outputs reflecting the efficiency of both production and sales efforts. The customer satisfaction rate (CSR), a carry-over activity, highlights the ongoing impact of production and sales on customer satisfaction over time.

Carry-over Activities ESR, CSR, and WRP represent long-term activities that carry over across periods, influencing the future performance of the divisions. These activities provide continuity in the analysis, ensuring that efficiency improvements are not limited to a single period. The flow of inputs, intermediates, and outputs across divisions is visualized by the arrows in the figure. Effective management of these flows, along with the carry-over activities, is critical to improving the overall efficiency of the petrochemical units.

In Table 1, we present the Key Performance Indicators (KPIs) used for the efficiency analysis of our operational processes. These criteria are based on insights from experts in petrochemical units. Each criterion is categorized according to its role within the production framework, whether it is an input, intermediate product, carry-over, or output. Additionally, the criteria are distinguished by their nature, indicating whether they are radial (R) or non-radial (NR).

Table 1. Key criteria used for the efficiency analysis

Criteria	Abbreviation	Role	Radial (R) or non-radial (NR)
Number of personnel	NPE	Input	NR
Employee satisfaction rate	ESR	Input	R
Wage	WG	Intermediate product	NR
Raw material cost	RM	Input	NR
Energy Cost	EC	Input	NR
Waste Reduction Percentage	WRP	Carry-over	R
Production Volume	PV	Intermediate product	NR
Customer Satisfaction Rate	CSR	Carry-over	R
Sale volume	SV	Output	NR
Net Profit	NPR	Output	NR

the criteria used for efficiency assessment are divided into radial and non-radial measures. This distinction is essential for understanding the model's hybrid approach to data changes, as some criteria vary proportionally (radial) while others change non-proportionally (non-radial).

- Radial Criteria: These are measures that change proportionally. In this study, the waste reduction percentage (WRP), employee satisfaction rate (ESR), and customer satisfaction rate (CSR) are treated as radial carry-over activities, meaning their influence on the system is considered to adjust proportionally over time. For instance, improvements or declines in these metrics reflect proportional changes in the efficiency of the divisions.
- Non-Radial Criteria: These represent measures that change non-proportionally. Inputs like the number of personnel (NPE) and raw material cost (RM) are non-radial external inputs, while wage (WG) and production volume (PV) are non-radial intermediate measures. Similarly, sales volume (SV) and net profit (NPR) are non-radial external outputs. These measures reflect changes that do not scale proportionally, and the model accounts for the different rates at which these inputs and outputs adjust.

By combining radial and non-radial criteria, the model ensures a more comprehensive efficiency evaluation, capturing both proportional and non-proportional changes in the performance of the petrochemical units.

Table 2 and Table 3 provide a detailed statistical summary of KPIs, capturing the minimum, maximum, mean, and standard deviation values for each criterion in the years 1400 and 1401, respectively.

Table 2. Descriptive statistics of the criteria in the year 1400

Criterion	Min	Max	Mean	Standard deviation
Energy Cost (Million Rials)	163095	27452354	9923337.64	10166762.732
Raw Material Costs (Million Rials)	4536276	611499056	130348296.50	173125932.866

Wages and Salaries Cost (Million Rials)	368629	10490210	2816691.36	2738148.421
Net Profit (Million Rials)	679604	263632351	78073922.79	72814944.847
Sales Volume (Operating Income)	5643364	752376245	208968170.29	207560791.737
Production Volume (Tons)	73236	5381591	2164709.50	1668683.724
Number of Personnel	232	3229	883.71	733.821
Customer Satisfaction Rate	35.5%	76.2%	68.9%	20.9%
Employee satisfaction rate	75.5%	86.2%	80.4%	13.9%
Waste Reduction Percentage	2.5%	5.6%	3.9%	1.7%

Table 3. Descriptive statistics of the criteria in the year 1401

Criterion	Min	Max	Mean	Standard deviation
Energy Cost (Million Rials)	195714	32119254.2	10915671	10675100.87
Raw Material Costs (Million Rials)	5534256.7	746028848	1.59E+08	211213638.1
Wages and Salaries Cost (Million Rials)	376001.58	11749035.2	2591356	3888170.758
Net Profit (Million Rials)	965037.68	268904998	87442794	95387577.75
Sales Volume (Operating Income)	6884904.1	759900007	2.28E+08	259450989.7
Production Volume (Tons)	89347.92	6565541.02	2640946	2035794.143
Number of Personnel	259.84	3687.518	1042.778	770.51205
Customer Satisfaction Rate	25.5%	81.2%	72.9%	26.9%
Employee satisfaction rate	55.5%	74.2%	60.8%	10.1%
Waste Reduction Percentage	2.5%	4.9%	3.5%	1.9%

According to experts in petrochemical company structures, the production division is the leader, while Human Resource and Sale division follow. This is because production drives the business, setting the pace and output capacity. Other divisions, like Human Resource and Sale, must align their strategies to support and optimize production. As discussed earlier, the first stage is to evaluate the leader's score and determine the values of the here-and-now variables to reveal the role of intermediate measures. To achieve this, the model (10)- (33) must be solved. The results obtained from of the first stage are presented in Table 4.

Table 4. The leader's score and the optimal role of intermediate measures.

1st Stage	Leader's score	Intermediate measure's Role

DMU	Production division efficiency	Wage (WG)	Production Volume (PV)
DMU1	0.73	Input to production	Output from Production
DMU2	0.88	Output from Human Resource	Output from Production
DMU3	0.68	Output from Human Resource	Input to Sale
DMU4	0.67	Input to production	Input to Sale
DMU5	0.47	Input to production	Output from Production
DMU6	1.00	Input to production	Output from Production
DMU7	0.46	Input to production	Output from Production
DMU8	1.00	Input to production	Output from Production
DMU9	1.00	Input to production	Input to Sale
DMU10	0.38	Input to production	Input to Sale
DMU11	1.00	Output from Human Resource	Input to Sale
DMU12	0.91	Output from Human Resource	Output from Production
DMU13	0.14	Output from Human Resource	Output from Production
DMU14	0.71	Output from Human Resource	Output from Production
Average	0.72	Output from Human Resource	Output from Production

Table 4. represents the optimal role of intermediate measures for evaluating the leader's score. The roles of intermediate measures, Wage and Production Volume, underscore the critical importance of production efficiency. It is evident that most roles are categorized as "Input to production" and "Output from Production," highlighting their direct connection to the production division. Consequently, leaders must focus on controlling these intermediate measures to enhance their scores by reducing inputs and increasing outputs. The results indicate that DMU6, DMU8, and DMU11 achieve perfect scores in the production division. In contrast, DMU13 has the lowest score, indicating significant room for improvement.

By applying the results from the first stage and determining the optimal values of the here-and-now variables, we proceed to the second stage to derive the overall and divisional scores, as presented in Table 5.

Table 5. Overall score of the Petrochemical units.

2nd stage / DMU	Human Resource division efficiency (follower's score)	Sale division efficiency (follower's score)	Overall efficiency
DMU1	0.30	1.00	0.68
DMU2	0.12	0.31	0.48
DMU3	0.42	0.49	0.53
DMU4	0.27	0.55	0.50
DMU5	1.00	1.00	0.82
DMU6	0.56	0.28	0.61
DMU7	0.58	0.61	0.55
DMU8	1.00	0.06	0.69
DMU9	0.43	0.33	0.59
DMU10	0.21	1.00	0.56
DMU11	0.33	0.23	0.52
DMU12	0.29	1.00	0.76
DMU13	0.17	1.00	0.43
DMU14	0.32	0.33	0.49

Average	0.43	0.58	0.59

Table 5. presents the efficiency scores of the human resource and sale divisions, along with the overall efficiency for each DMU. DMU5 stands out with perfect scores of 1.00 in both the human resource and sale divisions, resulting in the highest overall efficiency of 0.82. In contrast, DMU2 and DMU13 exhibit the lowest overall efficiencies of 0.48 and 0.43, respectively. The average scores across all DMUs are 0.72 for production efficiency, 0.43 for human resource efficiency, 0.58 for sale efficiency, and 0.59 for overall efficiency. This suggests that, on average, DMUs perform better in the production division than in the other divisions. Figure 2 illustrates the results.

Figure 2. overall and divisional scores of the petrochemical units

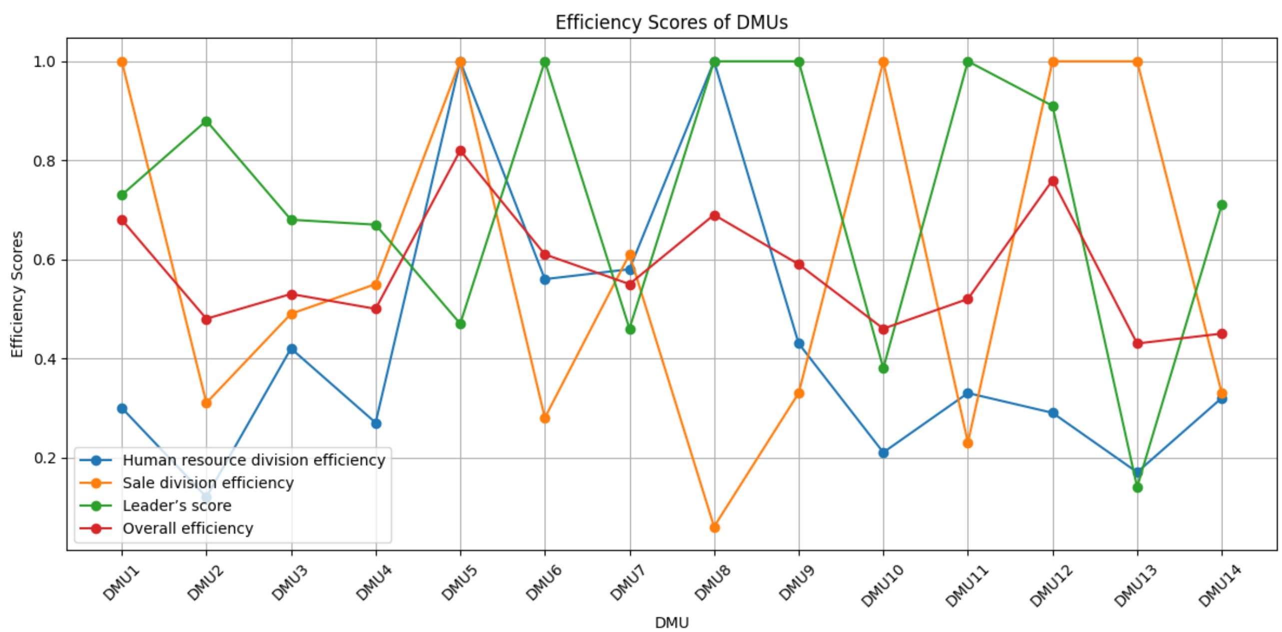

Table 6. overall and divisional period efficiencies of the petrochemical units

Division	Production		Sale		Human Resource		Overall efficiency	
Year	1400	1401	1400	1401	1400	1401	1400	1401
DMU1	0.63	0.80	1.00	1.00	0.31	0.29	0.61	0.73
DMU2	0.98	0.81	0.25	0.35	0.1	0.13	0.59	0.41
DMU3	0.58	0.75	0.39	0.56	0.49	0.37	0.51	0.54
DMU4	0.6	0.72	0.65	0.48	0.35	0.22	0.58	0.45
DMU5	0.31	0.58	1.00	1.00	1	1.00	0.71	0.89
DMU6	1	1.00	0.35	0.23	0.55	0.57	0.69	0.56
DMU7	0.4	0.50	0.69	0.56	0.65	0.53	0.51	0.58
DMU8	1	1.00	0.04	0.07	1	1.00	0.74	0.66
DMU9	1	1.00	0.29	0.36	0.37	0.47	0.69	0.52
DMU10	0.3	0.43	1.00	1.00	0.19	0.22	0.55	0.40
DMU11	1	1.00	0.35	0.15	0.39	0.29	0.63	0.45
DMU12	0.88	0.93	1.00	1.00	0.21	0.34	0.81	0.73
DMU13	0.11	0.16	1.00	1.00	0.24	0.12	0.49	0.39

DMU14	0.65	0.75	0.38	0.30	0.25	0.37	0.52	0.40
Average	0.67	0.75	0.60	0.58	0.44	0.42	0.61	0.54

The analysis of overall efficiency across the years 1400 and 1401 presented in Table 6 reveals a decline from an average score of 0.61 to 0.54. While some DMUs like DMU1 and DMU5 improved their efficiency, moving from 0.61 to 0.73 and 0.71 to 0.89 respectively, others such as DMU2 and DMU13 show significant drops. This indicates a need for better resource management and operational practices. Despite improvements in production efficiency, declines in sales and human resource management suggest areas requiring attention.

Figure 3 compares the average scores across divisional and overall scores for the years 1400 and 1401. In the Production division, there is a noticeable improvement from an average score of 0.67 in 1400 to 0.75 in 1401, indicating enhanced production efficiency. The Sale division shows a slight decrease in the average score, dropping from 0.60 in 1400 to 0.58 in 1401, suggesting a minor decline in sales performance. The Human Resource division experiences a small reduction in the average score from 0.44 in 1400 to 0.42 in 1401, pointing to a marginal decrease in HR efficiency.

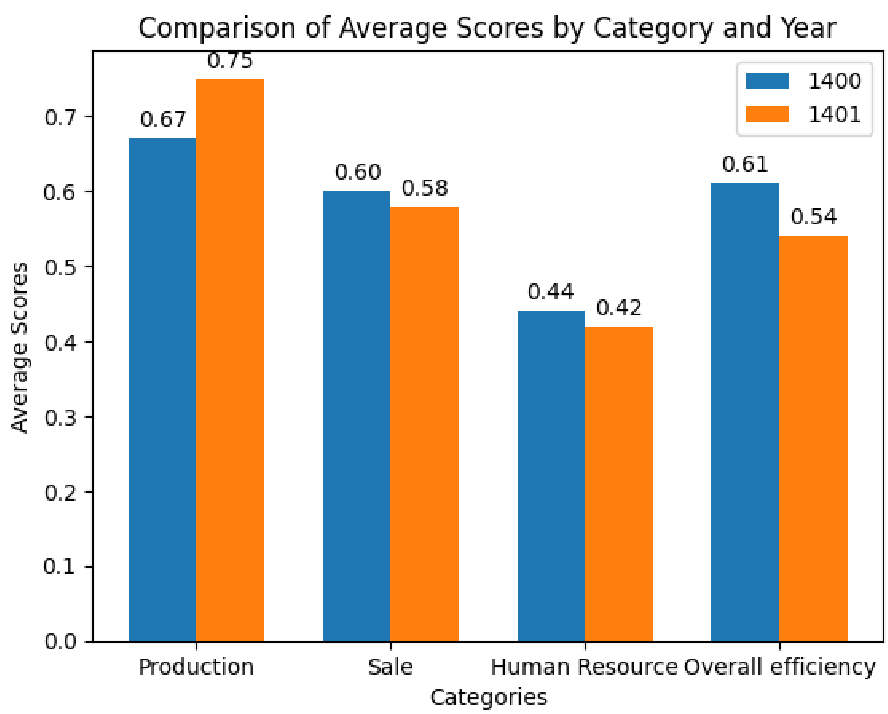

Figure 3 comparison of the average scores for the years 1400 and 1401

5. Conclusion

This study introduces a novel framework for evaluating the performance of DMUs by integrating Stackelberg game theory into a dynamic network DEA model. This approach effectively addresses the challenges posed by intermediate measures in multi-divisional systems. By adopting a leader-follower dynamic, where a specific division is designated as the leader, the framework allows for a more nuanced efficiency assessment. The process begins with optimizing the leader's efficiency, followed by configuring intermediate measures, which then influences the evaluation of the entire system's efficiency.

To verify the proposed model, it was applied to Iran's petrochemical industry, focusing on 14 units across three key divisions: Human Resources, Production, and Sales. The Production division, identified as the leader, demonstrated a significant impact on overall efficiency through its management of intermediate measures such as wages and production volume. This application underscored the pivotal role of the Production division in enhancing system efficiency and provided strategic insights for optimizing performance by aligning these measures with the organization's broader goals. The study also uncovered significant variations in efficiency across the DMUs, with some achieving high scores while others exhibited considerable inefficiencies. Despite improvements in production efficiency, a decline in overall efficiency between the years 1400 and 1401 was observed, particularly due to decreases in sales and human resource efficiency, highlighting the need for better resource management and operational practices.

This paper makes three significant contributions. First, it introduces a comprehensive efficiency measurement framework that combines dynamic network DEA with game theory, specifically the Stackelberg model, addressing a critical gap in the DEA literature. Second, it provides a practical tool for managers and decision-makers to identify and mitigate inefficiencies within complex multi-divisional and multi-period systems, thereby improving overall system performance. Third, it proposes a novel hybrid DNDEA model that considers both radial and non-radial data changes in efficiency evaluation.

Future research could extend this framework to other industries and sectors, explore more complex scenarios involving multiple leaders or multi-stage game dynamics, and validate the model's effectiveness in real-world settings to refine approaches for performance evaluation in dynamic, networked environments.

While the proposed dynamic network DEA model incorporating Stackelberg game theory provides a novel approach for assessing efficiency in multi-divisional systems, it is important to acknowledge certain limitations. First, the application of this model to only 14 petrochemical units may limit the generalizability of the findings. Expanding the dataset to include more diverse industries or sectors would provide a more comprehensive evaluation of its robustness. Second, the model's reliance on historical data assumes that external factors remain relatively constant, which might not hold true in rapidly changing environments. Future studies could explore adaptive models that incorporate real-time data and dynamic market conditions to improve the accuracy of efficiency assessments. Lastly, the assumption of a fixed leader-follower structure may oversimplify interactions within organizations. Future research could investigate more complex multi-leader or cooperative scenarios to capture a wider range of organizational dynamics.

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