Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.3, No.3, (2024), pp-pp
DOI: http://doi.org/ 10.xxxxx/tfss.2022.yyyyy.zzzz

Article Type: Original Research Article
(Journal Pre-proof)

Some Orders in Groupoids and its Applications to Fuzzy Groupoids

Akbar Rezaei , **Choonkil Park** , **Hee Sik Kim***

(This paper is dedicated to Professor ”John N. Mordeson” on the occasion of his 91st birthday.)

Abstract. In this paper, we continue the investigation started in [1]. We obtain new results derived from novel concepts developed in analogy with others already established, e.g., the fact that leftoids $(X, *)$ for φ are super-transitive if and only if $\varphi(\varphi(x)) = \varphi(x)$ for all $x \in X$. In addition we apply fuzzy subsets in this context and we derive a number of results as consequences.

AMS Subject Classification 2020: 20N02; 03E72
Keywords and Phrases: Below, Above, Transitive, Fuzzy, Contractive, Contained, $(\alpha-, \beta-, \gamma_\alpha-, \gamma_\beta-)$order-preserving (reversing).

1 Introduction

In developing a general theory of groupoids one seeks to define concepts and obtain information that applies to as general a class of groupoids as possible. Thus, e.g., the observation that $(Bin(X), \Box)$ is a semigroup with identity is one such of this type. Another one is the description of the notions of order for all groupoids $(X, *)$. Here one does not expect there to be a “precise” answer of this type one often expects. Here we use the relations β (below: $x*y = y$) and α (above: $x*y = x$) which are then combined with $\leq$ ($x \leq y : x\beta y, y\alpha x$) which compares with other definitions of $\leq$ made for certain classes of groupoids (e.g., BCK-algebras ([7, 15]), pogroupoids ([9, 10, 17]). The work done in [1] was convincing enough to suggest that a follow up paper might be in order, and that in this paper it might also be proper to open the door to introduce ideas that are both related to the material in [1] and to the general subject of “fuzzification” of crisp algebraic theories. Hopefully this effort has been successful.

Zadeh [20] introduced the notion of a fuzzy subset as a function from a set into unit interval, and Rosenfeld [18] applied this concept to the theory of groupoids and groups. Mordeson and Malik [16] published a book, *Fuzzy commutative algebra*, which are fuzzifications of several classical algebras, and Ahsan et al. [2] publisehed a book, *Fuzzy semirings with applications to automata theory.* Kim and Neggers [10] applied it to pogroupoids which are algebraic representations of partially ordered sets, and obtained an equivalent condition for some relation to be transitive for any fuzzy subset. Han et al. [6] discussed on linear fuzzifications of groupoids with special emphasis on BCK-algebras. Liu et al. [14] studied the notion of hyperfuzzy groupoids as a natural extension of the basic concepts of fuzzy groupoids.

*__Corresponding Author:__ Hee Sik Kim, Email: heekim@hanyang.ac.kr, ORCID: 0000-0001-5321-5919
Received: 13 May 2024; **Revised**: 23 July 2024; **Accepted**: 25 July 2024; **Available Online**: 7 August 2024; **Published Online**: dd mmmm yyyy.

How to cite: Rezaei A, Park C, Kim HS. Some orders in groupoids and its applications to fuzzy groupoids. *Transactions on Fuzzy Sets and Systems.* 2024; 3(3): pp-pp. DOI: http://doi.org/10.xxxxx/xxxx.yyyyy.zzzz

We recall that the notion of the semigroup $(Bin(X), \square)$ was introduced by Kim and Neggers [11]. Shin et al. [19] introduced the notion of abelian fuzzy subsets on a groupoid, and discussed diagonal symmetric relations, convex sets, and fuzzy center on $Bin(X)$. Ahn et al. [1] studied fuzzy upper bounds in $Bin(X)$.

Allen et al. [4] studied several types of groupoids related to semigroups, i.e., twisted semigroups. Allen et al. [3] developed a theory of companion d-algebras, and they showed that if $(X, *, 0)$ is a d-algebra, then $(Bin(X), \oplus, \diamond_0)$ is also a d-algebra. Kim et al. [12] introduced the notions of generalized commutative laws in algebras, and investigated their relations by using Smarandache disjointness. Moreover, they showed that every pre-commutative BCK-algebra is bounded. Hwang et al. [8] generalized the notion of implicativity which was discussed in BCK-algebras, and applied it to several groupoids, BCK/BCI-algebras and their generalizations.

2 Preliminaries

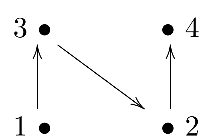
Let $(X, *)$ be a groupoid, i.e., a non-empty set X and a binary operation "$*$" on X, and let $x, y, z \in X$. x is said to be *below* y, denoted by $x\beta y$, if $x * y = y$; x is said to be *above* y, denoted by $x\alpha y$, if $x * y = x$. An element $z \in X$ is said to be *β-between x and y*, denoted by $z \in < x, y >_\beta$, if $x\beta z, z\beta y$; an element z is said to be *α-between x and y*, denoted by $z \in < x, y >_\alpha$, if $x\alpha z$ and $z\alpha y$.

We refer to [5] for basic concepts of the graph theory.

Example 2.1. [13] Let $D = (V, E)$ be a digraph and let $(V, *)$ be its associated groupoid, i.e., $*$ is a binary operation on V defined by

$$x * y := \begin{cases} x & \text{if } x \to y \notin E, \\ y & \text{otherwise.} \end{cases}$$

Let $D = (V, E)$ be a digraph with the following graph:



Then its associated groupoid $(V, *)$ has the following table:

$*$	1	2	3	4
1	1	1	3	1
2	2	2	2	4
3	3	2	3	3
4	4	4	4	4

It is easy to see that there are no elements $x, y \in V$ such that both $x\alpha y$ and $x\beta y$ hold simultaneously. Note that the relations α and β need not be transitive. In fact, $1 \to 3, 3 \to 2$ in E, but not $1 \to 2$ in E imply that $1\beta 3, 3\beta 2$, but not $1\beta 2$. Similarly, $1\alpha 4, 4\alpha 3$, but not $1\alpha 3$.

Remark 2.2. *In Example 2.1, $z \in < x, y >_\beta$ means that $x\beta z, z\beta y$, i.e., $x \to z \to y$ in E. Similarly, $z \in < x, y >_\alpha$ means that $x\alpha z, z\alpha y$, i.e., there is no arrow from x to z, and no arrow from z to y in E.*

Example 2.3. [13] Let $\mathbf{R}$ be the set of all real numbers and let $x, y \in \mathbf{R}$. If we define a binary operation "$*$" on $\mathbf{R}$ by $x * y := y^2$, then $(\mathbf{R}, *)$ is not a semigroup. In fact, $(x * y) * z = z^2$, while $x * (y * z) = z^4$. If $x\beta y$ and $y\beta z$, then $z = y * z = z^2$ and hence $z = 0$ or $z = 1$, which implies that $x * z = z$, i.e., $x\beta z$, proving that β is transitive.

Let $(X,*)$ be a groupoid and let $x,y \in X$. Define a binary relation "$\le$" on X by $x \le y \iff x\beta y, y\alpha x$. Then it is easy to see that $\le$ is anti-symmetric.

Note that the order "$x \le y$" defined by $x * y = 0$ in BCK-algebras is a partial order.

Let $(X,*)$ be a groupoid and let $x,y \in X$. We define an interval as follows:

$$[x,y] := \{q \in X \,|\, x \le q \le y\}.$$

Proposition 2.4. *[13] Let $(X,*)$ be a groupoid and let $x,y \in X$. Then $z \in [x,y]$ if and only if $z \in < x,y >_\beta$ and $z \in < y,x >_\alpha$.*

Given a set X and a function $\varphi : X \to X$, we consider a groupoid $(X,*,\varphi)$ where the multiplication is given by the formula

$$x * y = \varphi(x).$$

We call such a groupoid $(X,*,\varphi)$ a *leftoid* for φ. In particular, if $\varphi(x) = id_X(x) = x$, then $(X,*,id_X)$ has a multiplication

$$x * y = x$$

and the groupoid $(X,*)$ is referred to as a *left zero semigroup*. Similarly, we define a rightoid and right zero semigroup, i.e., $x * y := \varphi(y)$ for all $x,y \in X$.

Given a non-empty set X, we let $Bin(X)$ denote the collection of all groupoids $(X,*)$. Given groupoids $(X,*)$ and $(X,\bullet)$, we define a binary operation "$\Box$" on $Bin(X)$ by

$$(X,\Box) := (X,*)\,\Box\,(X,\bullet)$$

where

$$x\,\Box\,y = (x * y) \bullet (y * x)$$

for any $x,y \in X$. Using that notion, Kim and Neggers proved the following theorem.

Theorem 2.5. *[11] $(Bin(X), \Box)$ is a semigroup, i.e., the operation "$\Box$" is associative. Furthermore, the left-zero semigroup is the identity for this operation.*

3 Below and above in groupoids

Proposition 3.1. *Let $(X,*)$ be a leftoid for φ and let $x,y_1,y_2 \in X$. If $x\beta y_1$ and $x\beta y_2$, then $y_1 = y_2$.*

Proof. If $x\beta y_i$ $(i = 1,2)$, then $\varphi(x) = x * y_1 = y_1$ and $\varphi(x) = x * y_2 = y_2$. Since φ is a mapping, we obtain $y_1 = y_2$. □

In case of the rightoid $(X,*)$ for φ, if $x\beta y$, then $\varphi(y) = y$, i.e., there is no element $x \in X$ such that $x\beta y$ and $\varphi(y) \neq y$.

Proposition 3.2. *If $(X,*)$ is a leftoid for φ and $x\alpha y$, then x is a fixed point of φ.*

Proof. If $x\alpha y$, then $x = x * y = \varphi(x)$, i.e., x is a fixed point of φ. □

Proposition 3.3. *If $(X,*)$ is a leftoid (rightoid) for φ and $x \le y$, then $x \in \varphi^{-1}(y)$ and y is a fixed point of φ.*

Proof. The proof is straightforward. □

Proposition 3.4. *Let $(X,*)$ be a leftoid for φ, and let β be transitive. If $x\beta z, z\beta y$, then $y = z$.*

Proof. Since β is transitive, if $x\beta z, z\beta y$, then $x\beta y$ and $x * z = z$. Since $(X, *)$ is a leftoid for φ, we obtain $z = \varphi(x)$ and hence $y = x * y = \varphi(x) = z$. □

The following property can be easily proved.

Proposition 3.5. *Let $(X, *)$ be a leftoid for φ. If β is transitive, then either $< x, y >_\beta = \emptyset$ or $< x, y >_\beta = \{y\}$ for all $x, y \in X$.*

Given a mapping $\varphi : X \to X$, we define a set by

$$\mathrm{Fix}(\varphi) := \{x \in X \mid \varphi(x) = x\ \}.$$

Theorem 3.6. *Let $(X, *)$ be a rightoid for φ and let $y \in \mathrm{Fix}(\varphi)$. If $x \in X$, then $< x, y >_\beta = \mathrm{Fix}(\varphi)$.*

Proof. If $z \in < x, y >_\beta$, then $x\beta z, z\beta y$. Since $(X, *)$ is a rightoid, we have $z = x * z = \varphi(z)$, and hence $z \in \mathrm{Fix}(\varphi)$. If $z \in \mathrm{Fix}(\varphi)$, then $z = \varphi(z)$. For any $x \in X$, since $(X, *)$ is a rightoid, we have $x * z = \varphi(z) = z$, i.e., $x\beta z$. Moreover, $z * y = \varphi(y) = y$, since $y \in \mathrm{Fix}(\varphi)$, i.e., $z\beta y$. This shows that $z \in < x, y >_\beta$ for all $x \in X$. □

Theorem 3.6 shows that if $(X, *)$ is a rightoid for φ, then $< x, y >_\beta = < x', y' >_\beta = \mathrm{Fix}(\varphi)$ for all $x, x' \in X$ and $y, y' \in \mathrm{Fix}(\varphi)$.

Note that if $x\alpha z, z\alpha y$, where $(X, *)$ is a rightoid for φ, then $x = x * z = \varphi(z)$ and $z = z * y = \varphi(y)$. This shows that x is uniquely determined by y under φ,

Theorem 3.7. *Let $(X, *)$ be a leftoid for φ and let $x, y \in X$. Then*

(i) $|[x, y]| \leq 1$,

(ii) *if $z \in [x, y]$, then $y, z \in \mathrm{Fix}(\varphi)$,*

(iii) *if $x \in [x, y]$, then $x = y \in \mathrm{Fix}(\varphi)$.*

Proof. (i) Assume that there exist $z_1, z_2 \in [x, y]$. Then $x\beta z_1$ and $x\beta z_2$. This shows that $z_i = x * z_i = \varphi(x)$ where $i = 1, 2$. Since φ is a mapping, we obtain $z_1 = z_2$.
(ii) If $z \in [x, y]$, then $y\alpha z, z\alpha x$ and hence $y = y * z = \varphi(y)$ and $z = z * x = \varphi(z)$, proving that $y, z \in \mathrm{Fix}(\varphi)$.
(iii) If $x \in [x, y]$, then $x \in \mathrm{Fix}(\varphi)$ by (ii). We claim that $x = y$. Assume $x \neq y$. Since $x \leq y$, we have $x\beta y, y\alpha x$. It follows that $y = x * y = \varphi(x)$. Since $\varphi(x) = x$, we have $x = y$, a contradiction. □

Proposition 3.8. *If $(X, *)$ is a rightoid for φ and $x, y \in X$, then $[x, y] \subseteq \mathrm{Fix}(\varphi)$, and $[x, y] = \{y\}$ when $[x, y] \neq \emptyset$.*

Proof. If $z \in [x, y]$, then $z\alpha x$ and hence $z = x * z = \varphi(z)$, proving that $z \in \mathrm{Fix}(\varphi)$. Now, $y\alpha z$ implies $y = y * z = \varphi(z) = z$, since $z \in \mathrm{Fix}(\varphi)$. □

4 Transitivity in groupoids

Given a groupoid $(X, *)$, the relation β (below) is given by $x\beta y$ iff $x * y = y$ ([13]). Now, if β is transitive, then $(x\beta y) * (y\beta z) = x\beta z$, i.e., $(x * y) * (y * z) = x * z$ when $x * y = y, y * z = z, x * z = z$. Thus, if $(x * y) * (y * z) = x * z$, then this identity reflects a transitivity-like property which in any case is more general than a transitivity in the β-relation. Of course, we can argue the same way for the α-relation (above) given by $x\alpha y$ iff $x * y = x$. Thus the condition $(x * y) * (y * z) = x * z$ also generalizes the α-relation in the same manner. Since α and β are definitely not the same, we shall consider the following.

A groupoid $(X, *)$ is said to be *super-transitive* if for all $x, y, z \in X$,

$$(x * y) * (y * z) = x * z.$$

Every left-zero semigroup as well as every right-zero semigroup is therefore super-transitive as well. Moreover, every Boolean group $(X, *)$ (i.e., $x^2 = e_X$ for all $x \in X$) is super-transitive, since $(x * y) * (y * z) = x * (y * y) * z = x * z$ for all $x, y, z \in X$.

Theorem 4.1. *Let $(X, *)$ be a leftoid for φ. Then $(X, *)$ is super-transitive if and only $\varphi(\varphi(x)) = \varphi(x)$ for all $x \in X$.*

Proof. If $(X, *)$ is super-transitive, then $(x * y) * (y * z) = x * z$ for all $x, y, z \in X$. Since $(X, *)$ is a leftoid, we have $(x * y) * (y * z) = \varphi(x) * \varphi(y) = \varphi(\varphi(x))$ and $x * z = \varphi(x)$. Assume $\varphi(\varphi(x)) = x$ for all $x \in X$. Then $(x * y) * (y * z) = \varphi(x) * \varphi(y) = \varphi(\varphi(x)) = \varphi(x) = x * z$, proving that $(X, *)$ is super-transitive. □

Corollary 4.2. *Let $(X, *)$ be a rightoid for φ. Then $(X, *)$ is super-transitive if and only $\varphi(\varphi(x)) = x$ for all $x \in X$.*

Proof. The proof is similar to the proof of Theorem 4.1. □

Note that super-transitive groupoids with homomorphisms form a category, since super-transitivity is expressed in identity form.

A groupoid $(X, *)$ is said to be *α-transitive* if $x\alpha y, y\alpha z$ implies $x\alpha z$, and a groupoid $(X, *)$ is said to be *β-transitive* if $x\beta y, y\beta z$ implies $x\beta z$. A groupoid $(X, *)$ is *transitive* if it is both α-transitive and β-transitive.

Example 4.3. Let $X := \{x, y, z\}$ be a set with the following table.

*	x	y	z
x	y	z	x
y	z	x	y
z	z	x	y

Then $(X, *)$ is trivially β-transitive, since $\beta = \{(u, v) | u * v = v\} = \emptyset$. But $\alpha = \{(x, z), (y, z), (z, x)\}$. This shows that $x\alpha z, z\alpha x$, but not $x\alpha x$, proving that $(X, *)$ is not α-transitive.

Proposition 4.4. *Every super-transitive groupoid is transitive.*

Proof. Let $(X, *)$ be a super-transitive groupoid. Assume that $x\alpha y, y\alpha z$. Then $x * y = x, y * z = y$, and hence $x * z = (x * y) * (y * z) = x * y = x$, i.e., $x\alpha z$, proving that $(X, *)$ is α-transitive.

Assume that $x\beta y, y\beta z$. Then $x * y = y, y * z = z$. Since $(X, *)$ is super-transitive, we obtain $x * z = (x * y) * (y * z) = y * z = z$, i.e., $x\beta z$, proving that $(X, *)$ is β-transitive. □

Corollary 4.5. *Let $(X, *)$ be a transitive groupoid. If $x \le y, y \le z$, then $x \le z$.*

Proof. The proof is straightforward. □

The converse of Proposition 4.4 need not be true in general.

Example 4.6. Let $X := \{0, 1, 2, 3\}$ be a set with the following table.

*	0	1	2	3
0	0	1	2	1
1	1	1	2	1
2	2	2	2	1
3	1	2	1	3

Then it is easy to see that $(X, *)$ is transitive, but it is not super-transitive, since $(2*1)*(1*3) = 2 \neq 1 = 2*3$.

5 Applications to fuzzy subgroupoids

In this section, we apply the concept of fuzzy subsets to groupoid theory mentioned in the above sections.

Let $(X,*) \in Bin(X)$. A mapping $\mu : X \to [0,1]$ is said to be a *fuzzy subgroupoid* of X if, for all $x, y \in X$,

$$\mu(x*y) \geq \min\{\mu(x), \mu(y)\}.$$

A mapping $\mu : X \to [0,1]$ is said to be a *contractive fuzzy subgroupoid* of X if, for all $x, y \in X$,

$$\mu(x*y) \leq \min\{\mu(x), \mu(y)\}.$$

Proposition 5.1. *Let $(X,*)$ be a left-zero semigroup.*

(i) *every fuzzy subset $\mu : X \to [0,1]$ is a fuzzy subgroupoid of X,*

(ii) *if $\mu : X \to [0,1]$ is contractive, then it is a constant mapping.*

Proof. (i) Given $x, y \in X$, since $(X,*)$ is a left-zero semigroup, we have $\mu(x*y) = \mu(x) \geq \min\{\mu(x), \mu(y)\}$, proving that μ is a fuzzy subgroupoid of X.
(ii) Assume that μ is contractive. Then $\mu(x*y) \leq \min\{\mu(x),\, u(y)\}$ for all $x, y \in X$. Since $(X,*)$ is a left-zero semigroup, we obtain that $\mu(x) \leq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. This shows that $\mu(x) \leq \mu(y)$ for all $x, y \in X$, i.e., μ is a constant function. □

Let $(X,*) \in Bin(X)$. A mapping $\mu : X \to [0,1]$ is said to be a *contained fuzzy subgroupoid* of X if, for all $x, y \in X$,

$$\mu(x*y) \leq \max\{\mu(x), \mu(y)\}.$$

Proposition 5.2. *Let $(X,*)$ be a left-zero semigroup. Then every mapping $\mu : X \to [0,1]$ is a contained fuzzy subgroupoid of X.*

Proof. The proof is straightforward. □

Let $(X,*) \in Bin(X)$ and let $\mu : X \to [0,1]$ be a mapping. A mapping $\mu^c : X \to [0,1]$ is said to be a *complement* of μ if, for all $x \in X$, $\mu^c(x) := 1 - \mu(x)$.

Proposition 5.3. *Let $(X,*)$ be a groupoid. If $\mu : X \to [0,1]$ is a contained fuzzy subgroupoid of X, then μ^c is a fuzzy subgroupoid of X.*

Proof. It follows from that $1 - \max\{\mu(x), \mu(y)\} = \min\{1-\mu(x), 1-\mu(y)\}$ for all $x, y \in X$. □

Let $(X,*) \in Bin(X)$. A mapping $\mu : X \to [0,1]$ is said to be a *expansive fuzzy subgroupoid* of X if, for all $x, y \in X$,

$$\mu(x*y) \geq \max\{\mu(x), \mu(y)\}.$$

Note that every expansive fuzzy subgroupoid of X is also a fuzzy subgroupoid of X. Moreover, a fuzzy subset μ is an expansive fuzzy subgroupoid of X if and only if μ^c is a contractive fuzzy subgroupoid of X.

Example 5.4. Let $X := [0, \infty)$. Define a binary operation $x * y := x + y$ for all $x, y \in X$ where "+" is the usual addition of real numbers. Then every order-preserving mapping μ is expansive, since $\mu(x+y) \geq \mu(x)$ and $\mu(x+y) \geq \mu(y)$ for all $x, y \in X$.

Theorem 5.5. *Let $(X,*), (X,\bullet) \in Bin(X)$ and let $(X,\square) := (X,*)\square(X,\bullet)$. Then the following are hold:*

(i) *if μ is contractive fuzzy subgroupoid on $(X,*)$ and $(X,\bullet)$, then it is also contractive fuzzy subgroupoid on $(X,\square)$,*

(ii) *if μ is contained fuzzy subgroupoid on $(X, *)$ and $(X, \bullet)$, then it is also contained fuzzy subgroupoid on $(X, \Box)$,*

(iii) *if μ is expansive fuzzy subgroupoid on $(X, *)$ and $(X, \bullet)$, then it is also expansive fuzzy subgroupoid on $(X, \Box)$.*

Proof. (i) Given $x, y \in X$, we have

$$\begin{aligned} \mu(x \Box y) &= \mu((x*y) \bullet (y*x)) \\ &\leq \min\{\mu(x*y), \mu(y*x)\} \\ &\leq \min\{\mu(x), \mu(y)\}, \end{aligned}$$

proving that μ is a contractive fuzzy subgroupoid on $(X, \Box)$. Others are similar to (i) and we omit the proofs. □

Proposition 5.6. *Let $(X, *) \in Bin(X)$. If $\mu : (X, *) \to [0, 1]$ is both a contractive fuzzy subgroupoid of X and a fuzzy subgroupoid of X, then $\mu(x * y) = \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Proof. It follows from that $\min\{\mu(x), \mu(y)\} \leq \mu(x * y) \leq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. □

Let $(X, *) \in Bin(X)$. A mapping $\mu : X \to [0, 1]$ is said to be *β-order-preserving* if $x\beta y$ implies $\mu(x) \leq \mu(y)$, and a mapping $\mu : X \to [0, 1]$ is said to be *β-order-reversing* if $x\beta y$ implies $\mu(x) \geq \mu(y)$. A mapping $\mu : X \to [0, 1]$ is said to be *expanding* if $\mu(x) \leq \mu(x * y)$ for all $x, y \in X$, and a mapping $\mu : X \to [0, 1]$ is said to be *contracting* if $\mu(x) \geq \mu(x * y)$ for all $x, y \in X$.

Proposition 5.7. *Let $(X, *) \in Bin(X)$. Then every expanding (resp., contractive) fuzzy subset $\mu : X \to [0, 1]$ is β-order-preserving (resp., reversing).*

Proof. Assume that $x\beta y$. Then $x * y = y$. Since μ is expanding, we obtain $\mu(x) \leq \mu(x * y) = \mu(y)$, proving that μ is β-order-preserving. The other part is similar, and we omit it. □

Theorem 5.8. *Let $(X, *) \in Bin(X)$ and let $a, b, a + b \in [0, 1]$. Then the following conditions hold:*

(i) *if μ and ν are β-order-preserving, then $a\mu + b\nu$ is also β-order-preserving,*

(ii) *if μ, ν are expanding, then $a\mu + b\nu$ is also expanding,*

(iii) *if μ is β-order-preserving, then μ^c is β-order-reversing,*

(iv) *if μ is expanding, then μ^c is contracting.*

Proof. Let $\mu, \nu : X \to [0, 1]$ be fuzzy subsets of X. Given $x \in X$, we have $(a\mu + b\nu)(x) = a\mu(x) + b\nu(x) \leq (a + b) \max\{\mu(x), \nu(x)\} \leq a + b \leq 1$.

We consider (i). If μ and ν are β-order-preserving and $x\beta y$, then $\mu(x) \leq \mu(y), \nu(x) \leq \nu(y)$. It follows that $(a\mu + b\nu)(x) = a\mu(x) + b\nu(x) \leq a\mu(y) + b\mu(y) = (a\mu + b\nu)(y)$. Other proofs can be shown easily, and we omit the proofs. □

Let $(X, *) \in Bin(X)$. A map $\mu : X \to [0, 1]$ is said to be *γ_β-order-preserving* if $x\beta z, z\beta y$ implies $\mu(x) \leq \mu(z) \leq \mu(y)$.

Note that every β-order-preserving mapping μ of a groupoid $(X, *)$ is γ_β-order-preserving.

Proposition 5.9. *Let $(X, *)$ be a groupoid with the following property (P):*

$$x\beta z \implies \exists\, y \in X \text{ such that } z\beta y. \tag{P}$$

If μ is a γ_β-order-preserving mapping on $(X, *)$, then it is a β-order-preserving mapping on $(X, *)$.

Proof. Let $x\beta z$. Since $(X,*)$ has the property (P), there exists $y \in X$ such that $z\beta y$. Since μ is a γ_β-order-preserving mapping, we obtain $\mu(x) \le \mu(z) \le \mu(y)$, which shows that μ is a β-order-preserving mapping. □

Example 5.10. Let $X := [0,\infty)$. Define a binary operation $*$ on X by $x * y := \max\{x,y\}$ for all $x,y \in X$. Assume $x\beta y$. Then $y = x * y = \max\{x,y\}$ and hence $x \le y$. If we put $z := y+1$, then $y * z = y * (y+1) = \max\{y, y+1\} = y+1 = z$. Hence $(X,*)$ has the property (P).

Example 5.11. Let $X := [0,\infty)$. Define a binary operation $*$ on X by $x * y := x + y$ for all $x,y \in X$, where "+" is the usual addition of real numbers. Assume $x\beta 1$. Then $1 = x * 1 = x + 1$, and hence $x = 0$, i.e., $0\beta 1$. If we assume that there is $y \in X$ such that $1\beta y$, then $y = 1 * y = 1 + y$, which shows that $1 = 0$, a contradiction. Hence $(X,*)$ does not have the property (P).

Let $(X,*) \in Bin(X)$. A mapping $\mu : X \to [0,1]$ is said to be *α-order-preserving* if $x\alpha y$ implies $\mu(x) \ge \mu(y)$. A map $\mu : X \to [0,1]$ is said to be *γ_α-order-preserving* if $x\alpha z, z\alpha y$ implies $\mu(x) \ge \mu(z) \ge \mu(y)$.

Proposition 5.12. *Let $(X,*)$ be a groupoid with the following property (Q):*

$$x\alpha z \implies \exists\, y \in X \text{ such that } z\alpha y. \tag{Q}$$

If μ is a γ_α-order-preserving mapping on $(X,*)$, then it is a α-order-preserving mapping on $(X,*)$.

Proof. The proof is similar to the proof of Proposition 5.9. □

Let $(\mathbf{R},*)$ be a leftoid for φ, where $\varphi(x) := x^2$ for all $x \in \mathbf{R}$. Then $1 * 2 = \varphi(1) = 1$ and hence $a\alpha 2$. If we assume $(\mathbf{R},*)$ satisfies the condition (Q), then there exists $y \in \mathbf{R}$ such that $2\alpha y$. It follows that $2 = 2 * y = \varphi(2) = 4$, a contradiction. Hence such a groupoid $(\mathbf{R},*)$ does not satisfy the condition (Q).

Given a groupoid $(X,*)$, a map $\mu : X \to [0,1]$ is said to be a *super-symmetric fuzzy subset* of $(X,*)$ if $\mu((x*y)*(y*z)) \ge \mu(x*z)$ for all $x,y,z \in X$.

Thus, if $(X,*)$ is a left-zero semigroup, then every mapping $\mu : (X,*) \to [0,1]$ is a super-symmetric fuzzy subset of $(X,*)$, since $\mu((x*y)*(y*z)) = \mu(x) \ge \mu(x) = \mu(x*z)$ for all $x,y,z \in X$. Similarly, for any right zero semigroup, every mapping $\mu : (X,*) \to [0,1]$ is also a super-symmetric fuzzy subset of $(X,*)$.

Proposition 5.13. *Let $(X,*)$ be a leftoid for φ. If $\mu : X \to [0,1]$ is a map with $\mu(\varphi(x)) \ge \mu(x)$ for all $x \in X$, then μ is a super-symmetric fuzzy subset of $(X,*)$.*

Proof. Given $x,y,z \in X$, since $(X,*)$ is a leftoid for φ, we have $\mu((x*y)*(y*z)) = \mu(\varphi(x)*\varphi(y)) = \mu(\varphi(\varphi(x)) \ge \mu(\varphi(x)) = \mu(x*z)$, proving the proposition. □

It is a question of some interest to determine a super-symmetric fuzzy subset of a groupoid $(X,*)$ to be a fuzzy subgroupoid of $(X,*)$, i.e., $\mu(x*y) \ge \min\{\mu(x),\mu(y)\}$ for all $x,y \in X$. Clearly, every map $\mu : X \to [0,1]$ of a left-zero semigroup $(X,*)$ is also a fuzzy subgroupoid of $(X,*)$.

Proposition 5.14. *Let $(X,*)$ be a leftoid for φ. If $\mu : X \to [0,1]$ is a map with $\mu(\varphi(x)) \ge \mu(x)$ for all $x \in X$, then μ is a fuzzy subgroupoid of $(X,*)$.*

Proof. If $\mu(\varphi(x)) \ge \mu(x)$ for all $x \in X$, then $\mu(\varphi(\varphi(x)) \ge \mu(\varphi(x)) \ge \mu(x)$ and hence $\mu(x*y) = \mu(\varphi(x)) \ge \mu(x) \ge \min\{\mu(x),\mu(y)\}$ for all $x,y \in X$, proving that μ is a fuzzy subgroupoid of $(X,*)$. □

Clearly, there is much more information waiting to be obtained here as well.

6 Conclusion

In this paper, we have continued the investigation started in [13] of what we may discover in the theory of groupoids (binary systems) by separating the concepts of below ($x\beta y$) and above ($x\alpha y$) in general groupoids, and then recombining them to obtain what looks to be a candidate for the best relation $\leq$ available in general. After doing so, we introduce the idea of super-transitivity in groupoids as a generalization of the notions β and α in identity form, $(x*y)*(y*z)=x*z$ which allows us to make claims about the class of groupoids for which this identity holds, i.e., that this yields a variety. Having done so we may then concern ourselves with introducing fuzzy subsets μ on groupoids $(X,*)$ which have certain properties of interest, e.g., being contracting on expanding which defined in the natural way provides new but not unexpected information. A bit stickier is the class of $\mu((x*y)*(y*z))\geq\mu(x*z)$ for super-symmetric fuzzy subsets of $(X,*)$ introduced with a standard looking inequality and the problem being the determination of fuzzy subsets of this type which are also fuzzy subgroups and conversely. Certain problems look innocent enough but may yet prove not to be trivial as they are solved.

Conflict of Interest: The authors declare no conflict of interest.

References

[1] Ahn SS, Kim YH, Neggers J. Fuzzy upper bounds in groupoids. *The Scientific World Journal*. 2014; 2014(1): 697012. DOI: https://doi.org/10.1155/2014/697012

[2] Ahsan J, Mordeson JN, Shabir M. *Fuzzy Semirings with Applications to Automata Theory*. New York: Springer; 2012.

[3] Allen PJ, Kim HS, Neggers J. Companion d-algebras. *Mathematica Slovaca*. 2007; 57(2): 93-106. DOI: hppts://doi.org/10.2478/s12175-007-0001-z

[4] Allen PJ, Kim HS, Neggers J. Several types of groupoids induced by two-variable functions. *Springer Plus*. 2016; 5: 1715-1725. DOI: https://doi.org/10.1186/s40064-016-3411-y

[5] Clark J, Holton DA. *A First Look at Graph Theory*. Singapore: World Scientic; 1991.

[6] Han JS, Kim HS, Neggers J. On linear fuzzifications of groupoids with special emphasis on BCK-algebras. *Journal of Intelligent & Fuzzy Systems*. 2013; 24(1): 105-110. DOI: https://doi.org/10.3233/IFS-2012-0534

[7] Huang Y. *BCI-Algebra*. Science Press: Beijing; 2006.

[8] Hwang IH, Kim HS, Neggers J. Some implicativities for groupoids and BCK-algebras. *Mathematics*. 2019; 7(10): 973-980. DOI: https://doi.org/10.3390/math7100973

[9] Kim YH, Kim HS, Neggers J. Selective groupoids and frameworks induced by fuzzy setsets. *Iranian Journal of Fuzzy Systems*. 2017; 14(3): 151-160. DOI: https://doi.org/10.22111/IJFS.2017.3261

[10] Kim HS, Neggers J. Fuzzy pogroupoids. *Information Sciences*. 2005; 175(1-2): 108-119. DOI: https://doi.org/10.1016/j.ins.2004.10.003

[11] Kim HS, Neggers J. The semigroups of binary systems and some perspectives. *Bulletin of Korean Mathematical Society*. 2008; 45(4): 651-661. DOI: https://doi.org/10.4134/BKMS.2008.45.4.651

[12] Kim HS, Neggers J, Ahn SS. On pre-commutative algebras. *Mathematics*. 2019; 7(4): Paper 336-342. DOI: https://doi.org/10.3390/math7040336

[13] Kim HS, Neggers J, So KS. Order related concepts for arbitrary groupoids. *Bulletin of Korean Mathematical Society*. 2017; 54(4): 1373-1386. DOI: https://doi.org/10.4134/BKMS.b160593

[14] Liu YL, Kim HS, Neggers J. Hyperfuzzy subsets and subgroupoids. *Journal of Intelligent & Fuzzy Systems*. 2017; 33(3): 1553-1562. DOI: https://doi.org/10.3233/JIFS-17104

[15] Meng J, Jun YB. *BCK-Algebras*. Kyung Moon Sa: Seoul; 1994.

[16] Mordeson JN, Malik DS. *Fuzzy Commutative Algebra*. Singapore: World Scientific; 1998.

[17] Neggers J. Partially ordered sets and groupoids. *Kyungpook Mathematical Journal*. 1976; 16(1): 7-20. https://koreascience.kr/article/JAKO197625748111702.pdf

[18] Rosenfeld A. Fuzzy groups. *Journal of Mathematical Analysis and Applications*. 1971; 35(3): 512-517. DOI: https://doi.org/10.1016/0022-247X(71)90199-5

[19] Shin SJ, Kim HS, Neggers J. On abelian and related fuzzy subsets of groupoids. *The Scientific World Journal*. 2013; 2013(1): 476057. DOI: https://doi.org/10.1155/2013/476057

[20] Zadeh LA. Fuzzy sets. *Information and Control*. 1965; 8(3): 338-353. DOI: https://doi.org/10.1016/S0019-9958(65)90241-X

Akbar Rezaei
Department of Mathematics
Payame Noor University
Tehran, Iran
E-mail: rezaei@pnu.ac.ir

Choonkil Park
Department of Mathematics
Hanyang University
Seoul, Korea
E-mail: baak@hanyang.ac.kr

Hee Sik Kim
Department of Mathematics
Hanyang University
Seoul, Korea
E-mail: heekim@hanyang.ac.kr