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A remark on some inequalities for positive linear maps

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Abstract. The objective of this paper is to reveal that an analogue of Jensen's inequality holds for positive unital linear maps and matrix s-convex functions. We prove that the restriction to the matrix s-convex functions is not necessary in the case of 2×2 matrices in some sense.

Keywords: Convex function, s-convex function, matrix s-convex function, P-class function, Jensen's inequality.

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1. Introduction

It is well known that a function $f : J \to \mathbb{R}$, $J \subseteq \mathbb{R}$ is called convex whenever $f(\lambda x+(1-\lambda)y) \leqslant \lambda f(x)+(1-\lambda)f(y)$ holds for all $x, y \in J$ and $\lambda \in [0,1]$, and the function $f : J \to \mathbb{R}$ is concave whenever $-f$ is convex. The convexity of functions plays a significant role in many fields, such as in economy, probability and optimization.

A function $f : [0,\infty) \to \mathbb{R}$ is s-convex (the second sense) whenever

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1}$$

holds for all $x, y \in [0,\infty)$ and $\lambda \in [0,1]$, and for some fixed $s \in (0,1]$.

The class of s-convex functions in the second sense was defined in [10] and it was proved that all s-convex functions in the second sense are nonnegative. There is an identity between the class of 1-convex functions and the class of nonnegative convex functions. Indeed, the s-convexity means just the convexity when $s = 1$. Moreover, when $s \to 0$

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we reach another rich class of functions the so-called P-class functions. We refer to see some inequalities for P-class functions in [13]. For many research results and applications related to the s-convex functions, see [3, 6, 7, 9, 10, 14, 15, 17] and references therein.

Jensen's inequality for convex functions is one of the most important results in the theory of inequalities and for appropriate choices of the function many other famous inequalities are particular cases of this inequality. In classical analysis, there is a general version of Jensen's inequality.

Let M_n and M_n^+ be the set of all $n \times n$ complex matrices and the set of all $n \times n$ positive matrices, respectively, and $\Phi : M_n \to M_k$ be a positive unital linear map. Let A be a self–adjoint matrix with spectrum in (a, b). Davis [4] showed that if f is a matrix convex function and Φ is a completely positive linear map, then

$$f(\Phi(A)) \leqslant \Phi(f(A)). \tag{2}$$

The restriction to completely positive linear maps was removed by Choi [2] who showed that (2) remains valid for all positive unital linear maps Φ. Indeed, when one restricts to matrix convex functions instead of ordinary convex functions an analogue of Jensen's inequality holds for positive unital linear maps on the noncommutative matrix algebras. Bhatia and Sharma [1] showed that the restriction to matrix convex functions is not necessary in the case of 2×2 matrices and one can use ordinary convex functions.

We denote the class of nonnegative matrix convex functions, the class of nonnegative convex functions, the class of matrix s-convex functions, and the class of s-convex functions by NMCF, NCF, MSCF, and SCF, respectively. Our interest is to extend the inequality (2) to s-convex functions. This extension has an impact because of the following inclusions:

$$\text{SCF} \supseteq \text{MSCF} \supseteq \text{NMCF} \subseteq \text{NCF} \subseteq \text{SCF}.$$

In the present paper, we give a matrix interpretation for what is called s-convex functions, and present some inequalities including a variant of Hansen-Pedesen inequality. Using some known results, we demonstrate that Jensen's type inequality (2) holds for matrix s-convex functions, see (4), but these are not our main goals. Our main purpose is to show that the restriction to matrix s-convex functions is not necessary in the case of 2×2 matrices and one can use ordinary s-convex functions, see (8).

The arrangement of this paper is as follows. In Section 2, we extend some well-known results for s-convex functions for the convenience of the readers. The main results are included in Section 3.

2. Matrix s-convex functions

In this section, we show that some Jensen's type inequalities hold for matrix s-convex functions. We define matrix s-convex functions in the second sense as follows.

Definition 2.1 [8] Let f be a real–valued continuous function defined on $[0, \infty)$. The function f is matrix s-convex on $[0, \infty)$ if $f(\lambda A + (1 - \lambda)B) \leqslant \lambda^s f(A) + (1 - \lambda)^s f(B)$ for all $A, B \in M_n^+$, $\lambda \in [0, 1]$ and $s \in (0, 1]$.

We now give some Jensen's type inequalities. In [5], it has been shown that

$$f\left([C^* A^p C + D^* B^p D]^{1/p}\right) \leqslant 2h\left(\frac{1}{2}\right)(C^* f(A)C + D^* f(B)D)$$

in which h is a super-multiplicative function, f is operator (p,h)-convex and $C^*C + D^*D = I$. Taking $h(t) = t^s$, $D = 0$, and $p = 1$, this concludes Theorem 2.2.

Theorem 2.2 Let f be a matrix s-convex function on $[0,\infty)$. If $C \in M_n$ is an isometry and $A \in M_n^+$, then

$$f(C^*AC) \leqslant 2^{1-s}C^*f(A)C. \tag{3}$$

The isometry in Theorem 2.2 can be replaced by a contraction.

Corollary 2.3 Let f be matrix s-convex on $[0,\infty)$ such that $f(0) = 0$. If $C \in M_n$ is a contraction and $A \in M_n^+$, then (3) holds.

Proof. For every contraction $C \in M_n$, we put $D = \sqrt{I - C^*C}$. Let $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\tilde{C} = \begin{pmatrix} C \\ D \end{pmatrix}$. It follows that $\tilde{C}^*\tilde{C} = C^*C + D^*D = I$, and by Theorem 2.2, we have

$$f(C^*AC) = f(\tilde{C}^*\tilde{A}\tilde{C}) \leqslant 2^{1-s}\tilde{C}^*f(\tilde{A})\tilde{C} = 2^{1-s}C^*f(A)C.$$

■

The following theorem is also a standard reforming.

Theorem 2.4 Let $\Phi : M_n \to M_k$ be a unital positive linear map. If f is matrix s-convex on $[0,\infty)$ and $A \in M_n^+$, then

$$f(\Phi(A)) \leqslant 2^{1-s}\Phi(f(A)). \tag{4}$$

Proof. Let Ψ be the restriction of Φ to the C^*-algebra $\mathcal{C}^*(A, I)$ generated by I and A. So, Ψ is a unital completely positive map on $\mathcal{C}^*(A, I)$. By the Stinespring dilation theorem [16], there exist an isometry $V : \mathbb{C}^n \to \mathbb{C}^n$ and a unital $*$-homomorphism $h : \mathcal{C}^*(A, I) \to M_n$ such that $\Psi(A) = V^*h(A)V$. Consequently, by Theorem 2.2, we have

$$\begin{aligned} f(\Phi(A)) &= f(\Psi(A)) \\ &= f(V^*h(A)V) \\ &\leqslant 2^{1-s}V^*f(h(A))V \\ &= 2^{1-s}V^*h(f(A))V \\ &= 2^{1-s}\Psi(f(A)) \\ &= 2^{1-s}\Phi(f(A)). \end{aligned}$$

■

3. Jensen's inequality for 2 × 2 matrices and s-convex functions

In this section, we address the main purpose of this paper. We show how we can remove the restriction to matrix s-convexity and use the ordinary s-convex functions in Theorem 2.4. We obtain an analogue of Jensen's type inequality for ordinary s-convex functions. We begin with some lemmas for s-convex functions in the second sense and $s \in (0, 1]$.

Lemma 3.1 [11] Let $0 < s \leqslant r < 1$. Then the function $f(t) = t^r$ for $t \geqslant 0$ is s-convex.

Lemma 3.2 [11] Let X be a normed space and $0 < s \leqslant r < 1$. Then the function $f : X \to \mathbb{R}$ defined by $f(t) = ||t||^r$ is s-convex.

Lemma 3.3 Let f be a real valued s-convex function on $[0, \infty)$ containing (a, b). Then

$$f(x) \leqslant L(x) \tag{5}$$

for all $a \leqslant x \leqslant b$, where $L(x) = \frac{f(b)-f(a)}{b-a} sx - \frac{f(b)a-f(a)b}{b-a} + (1-s)f(b)$.

Proof. Consider the function g defined by $g(x) = f(x) - L(x)$. Then $g(a) = -(1 - s)f(b) \leqslant 0$ and $g(b) = -(1 - s)f(a) \leqslant 0$. Define $u(t) = t^s$ and $v(t) = (1 - t)^s$ for $0 \leqslant t \leqslant 1$. The functions u and v are concave on $[0, 1]$. The line passing through the point $(1, 1)$ is tangent to the function u and this line is above the function u. Indeed,

$$t^s \leqslant st + 1 - s \tag{6}$$

for $0 \leqslant t \leqslant 1$. Moreover, The line passing through the point $(0, 1)$ is tangent to the function v and this line is above the function v which means

$$(1 - t)^s \leqslant -st + 1 \tag{7}$$

for $0 \leqslant t \leqslant 1$. Let $x = (1 - t)a + tb$ for $0 \leqslant t \leqslant 1$. Since f is s-convex, in view of (6) and (7), we have

$$\begin{aligned} g(x) &= g((1-t)a + tb) \\ &= f((1-t)a + tb) - L((1-t)a + tb) \\ &\leqslant (1-t)^s f(a) + t^s f(b) - L((1-t)a + tb) \\ &\leqslant (-st + 1)f(a) + (st + 1 - s)f(b) - L((1-t)a + tb). \end{aligned}$$

An algebraic manipulation indicates that the right hand side of the last inequality is equal to zero and this entails the desired inequality (5). ■

We now demonstrate that ordinary s-convexity can be replaced by matrix s-convexity in the case of 2×2 matrices. We distinguish the situation when the matrix A has one eigenvalue. In this situation, we have $A = \lambda I$ and so

$$f(\Phi(A)) = f(\Phi(\lambda I)) = f(\lambda)I = f(\lambda)\Phi(I) = \Phi(f(\lambda I)) = \Phi(f(A)).$$

Indeed, the equality holds when A has one eigenvalue.

Theorem 3.4 Let $\Phi : M_2 \to M_k$ be a positive unital linear map. If f is s-convex on $[0, \infty)$, then

$$f(\Phi(A)) \leqslant \Phi(f(A)) + (1 - s)K(\lambda_1, \lambda_2), \tag{8}$$

where $K(\lambda_1, \lambda_2) = \frac{\lambda_1(f(\lambda_1)+f(\lambda_2))-2\lambda_2 f(\lambda_1)}{\lambda_1-\lambda_2}$ and λ_1, λ_2 are two distinct eigenvalues of $A \in M_2^+$ with $\lambda_1 > \lambda_2 \geqslant 0$.

Proof. Assume that A has distinct eigenvalues $\lambda_1 > \lambda_2$. Choose orthogonal projections P_1 and P_2 such that $A = \lambda_1 P_1 + \lambda_2 P_2$. Since Φ is a unital linear map, it is proved in [1] that

$$\Phi(f(A)) = \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2}\Phi(A) - \frac{\lambda_2 f(\lambda_1) - \lambda_1 f(\lambda_2)}{\lambda_1 - \lambda_2}. \tag{9}$$

According to (9) and Lemma 3.3, we see

$$\begin{aligned}
&\Phi(f(A)) + (1-s)f(\lambda_1) \\
&\qquad \geqslant s\Phi(f(A)) + (1-s)f(\lambda_1) \\
&\qquad = \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2}s\Phi(A) - \frac{\lambda_2 f(\lambda_1) - \lambda_1 f(\lambda_2)}{\lambda_1 - \lambda_2}s + (1-s)f(\lambda_1) \\
&\qquad = \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2}s\Phi(A) - \frac{\lambda_2 f(\lambda_1) - \lambda_1 f(\lambda_2)}{\lambda_1 - \lambda_2} + (1-s)f(\lambda_1) \\
&\qquad\quad + (1-s)\frac{\lambda_2 f(\lambda_1) - \lambda_1 f(\lambda_2)}{\lambda_1 - \lambda_2} \\
&\qquad = L(\Phi(A)) + (1-s)\frac{\lambda_2 f(\lambda_1) - \lambda_1 f(\lambda_2)}{\lambda_1 - \lambda_2} \\
&\qquad \geqslant f(\Phi(A)) + (1-s)\frac{\lambda_2 f(\lambda_1) - \lambda_1 f(\lambda_2)}{\lambda_1 - \lambda_2}.
\end{aligned}$$

This entails that

$$\Phi(f(A)) + (1-s)\left(f(\lambda_1) - \frac{\lambda_2 f(\lambda_1) - \lambda_1 f(\lambda_2)}{\lambda_1 - \lambda_2}\right) \geqslant f(\Phi(A)),$$

whence we can deduce the result. ■

We can reach a result for P-class functions.

Corollary 3.5 Let $\Phi : M_2 \to M_k$ be a positive unital linear map and f be a P-class function on $[0, \infty)$. Then

$$f(\Phi(A)) \leqslant \Phi(f(A)) + \frac{\lambda_1(f(\lambda_1) + f(\lambda_2)) - 2\lambda_2 f(\lambda_1)}{\lambda_1 - \lambda_2},$$

where λ_1, λ_2 are two distinct eigenvalues of $A \in M_2^+$ with $\lambda_1 > \lambda_2 \geqslant 0$.

Proof. It is enough to consider $s \to 0$ in Theorem 3.4. ■

The following corollary recovers the result presented by Bhatia and Sharma [1] in part. Indeed, the restriction to matrix convex functions in (2) is not necessary in the case of 2×2 matrices and one can use ordinary nonnegative convex functions.

Corollary 3.6 Let $\Phi : M_2 \to M_k$ be a positive unital linear map and f be a nonnegative convex function on $[0, \infty)$ containing the eigenvalues of $A \in M_2^+$. Then $f(\Phi(A)) \leqslant \Phi(f(A))$.

Proof. It is enough to consider $s = 1$ in Theorem 3.4. ■

Some useful results can also be deduced.

Corollary 3.7 Let $\Phi : M_2 \to M_k$ be a positive unital linear map. Then

$$\Phi(A^r) \leqslant (\Phi(A))^r \leqslant \Phi(A^r) + (1-s)\frac{\lambda_1(\lambda_1^r + \lambda_2^r) - 2\lambda_2\lambda_1^r}{\lambda_1 - \lambda_2},$$

where $0 < s \leqslant r < 1$ and λ_1, λ_2 are two distinct eigenvalues of $A \in M_2^+$ with $\lambda_1 > \lambda_2 \geqslant 0$.

Proof. Consider the function $f(t) = t^r$, $t \geqslant 0$ and $0 < s \leqslant r < 1$. The first inequality follows from (2) and the fact that the function f is matrix concave on $[0, \infty)$ for $0 < s \leqslant r < 1$. Lemma 3.1 infers that the function f is s-convex and so one can get the second inequality from Theorem 3.4. ∎

Corollary 3.8 Let $\Phi : M_2 \to M_k$ be a positive unital linear map. Then

$$\Phi(\log A) \leqslant \log(\Phi(A)) \leqslant \Phi(\log A) + \log\left(\frac{(\lambda_1\lambda_2)^{\lambda_1}}{\lambda_1^{2\lambda_2}}\right)^{\frac{1-s}{\lambda_1-\lambda_2}},$$

where $0 < s < 1$ and λ_1, λ_2 are two distinct eigenvalues of $A \in M_2^+$ with $\lambda_1 > \lambda_2 \geqslant 1$.

Proof. Consider the function $f(t) = \log t$ for $t \geqslant 1$. The first inequality follows from (2) and the fact that the function f is matrix concave on $[1, \infty)$. Regarding Lemma [12, Lemma 5] the function f is s-convex and so, one can get the second inequality from Theorem 3.4. ∎

Corollary 3.9 Let $\Phi : M_2 \to M_k$ be a positive unital linear map. Then

$$\|\Phi(A)\|^r \leqslant \Phi(\|A\|^r) + (1-s)\frac{\lambda_1(\lambda_1^r + \lambda_2^r) - 2\lambda_2\lambda_1^r}{\lambda_1 - \lambda_2},$$

where $0 < s \leqslant r < 1$ and λ_1, λ_2 are two distinct eigenvalues of $A \in M_2^+$ with $\lambda_1 > \lambda_2 \geqslant 0$.

Proof. Consider the function $f(t) = \|t\|^r$, $t \geqslant 0$ and $0 < s \leqslant r < 1$. The function f is s-convex by Lemma 3.2 and so the inequality follows from Theorem 3.4 and the fact that Φ is positive and unital. ∎

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