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A technique for identifying congestion in Data Envelopment Analysis

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Abstract
Data Envelopment Analysis (DEA) is a non-parametric mathematical programming method used to assess performance and measure the efficiency of Decision-making Units (DMUs) that operate with multiple concurrent inputs and outputs. The performance of these units is influenced by the utilization of input resources. While an increase in input utilization typically leads to higher production levels, there are scenarios where increased input usage results in decreased outputs. This phenomenon is termed congestion. Given that alleviating congestion can reduce costs and enhance production, it holds significant importance in economics. This paper introduces a method for identifying congestion based on a defined modeling framework. A DMU is considered congested when reducing inputs in at least one component leads to increased outputs in at least one component, and increasing inputs in at least one component can be achieved by reducing outputs in at least one component, without improvement in other indicators. The paper explores congestion in DMUs with both increasing and decreasing inputs.

Keywords: DEA, Congestion, Efficiency, Inefficiency

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1. Introduction

Data Envelopment Analysis (DEA) is a scientific methodology used to assess organizational performance across various public and private sectors. Its inception can be traced back to the pioneering work of Charnes and Cooper [1]. Over time, several approaches have emerged to delve into the intricacies of congestion. The concept of congestion was initially articulated by Grosskopf and Far [2], while Cooper et al. [3] introduced a method leveraging slack variables that effectively identifies sources of congestion [4]. This method enables the quantification of congestion within input vectors. Further advancements include Cooper et al.'s development of the Additive model for studying congestion [5], and Brockett et al.'s development of the Chinese Industry Congestion Technology (CCT) model [6]. Jahanshahloo and Khodabakhshi [7] proposed a method centered on the flexible combination of inputs, with Khodabakhshi [8] extending this approach to stochastic models for congestion determination. Wi and Yan explored additive models to detect congestion existence, while Tone and Sahoo [9] introduced a novel method for calculating elastic scale amidst congestion, introducing the concepts of strong and weak congestion.

Nora et al. [10] contributed alternative methods for identifying input congestion in DEA, while Khoveyni et al. [11] adapted Tone and Sahoo's method to discern strong and weak congestion, particularly relevant in scenarios with multiple optimal solutions. They further extended this work to handle negative data [12]. Sueshy et al. [13] proposed a method to distinguish between undesirable congestion, typically beyond natural control, and desirable congestion manageable under operational control, with a focus on the US electricity power industry. Meng et al. [14] introduced a two-stage model for evaluating congestion in mixed energy systems. This methodology was applied to analyze inefficiency and congestion across 16 OPEC countries, revealing that fossil energy contributes significantly to congestion in these nations. Mehdiloozad et al. [15] demonstrated that all points within a given region exhibit similar congestion characteristics, even when dealing with negative data. Chen et al. [16], in their research, categorized energy congestion into two distinct types: Undesirable Energy Congestion (DEC) and Desirable Energy Congestion (DEC). They utilized DEC and UEC models to quantify energy congestion and assess inefficiencies in coal production in China from 2004 to 2013. In another work, Chen et al. [17] proposed a novel congestion measurement approach and delineated three congestion types aligned with political objectives.

Saati et al. [18] delved into supply chain congestion pertaining to inputs or intermediate products, exploring various scenarios that could lead to congestion in intermediate products to optimize supply chain efficiency. Their study focused on a two-stage serial supply chain, identifying units within the production possibilities set that exhibited strong or weak congestion in intermediate products through comparative analysis. Shadab et al. [19] examined the potential for congestion within DMUs, identifying units with efficiency scores below one as candidates for congestion assessment. Their method involved comparing each DMU's actual performance against the efficiency frontier to pinpoint inefficiencies caused by congestion. They also scrutinized inputs and output levels of these overall efficient DMUs to pinpoint areas of suboptimal resource utilization or existing inefficiencies, contributing valuable insights for enhancing overall efficiency. Table 1 provides a comprehensive scrutiny of a variety of methodologies put forth for identifying congestion within the paradigm of DEA. This table summarizes the key contributions and advancements in the field of congestion measurement in DEA, highlighting the development of methodologies to identify and address congestion in DMUs. Each entry provides insights into the methodological description, distinctive attributes, and reference citation for further exploration.

Table 1 literature review of some congestion methods based on DEA and NDEA

Author	Method Description	Structure	Characteristics
Grosskopf and Far (2002)	Introduced the concept of congestion.	Conceptual framework	Conceptual understanding of congestion
Cooper et al. (2000)	Introduced a method using slack variables to identify sources of congestion.	Method leveraging slack variables	Source identification
Cooper et al. (1999)	Developed the Additive model for studying congestion.	Additive model	Congestion analysis
Brockett et al. (2005)	Developed the Chinese Industry Congestion Technology (CCT) model.	CCT model	Industry-specific congestion analysis
Jahanshahloo(2007)	Proposed a method for flexible combination of inputs.	Flexible input combination	Input flexibility in DEA
Khodabakhshi (2009)	Extended the flexible input combination approach to stochastic models for congestion determination.	Stochastic models	Stochastic approach to congestion
Wi and Yan	Explored additive models to detect congestion existence.	Additive models	Detection of congestion existence
Tone, Sahoo (2003)	Introduced a novel method for calculating elastic scale amidst congestion.	Elastic scale calculation	Calculation of scale amidst congestion
Nora et al. (2004)	Contributed alternative methods for identifying input congestion in DEA.	Alternative congestion identification methods	Various methods for identifying congestion
Khoveyni et al. (2011)	Adapted Tone and Sahoo's method to discern strong and weak congestion.	Adaptation of congestion identification method	Identification of strong and weak congestion
Sueshy et al. (2013)	Proposed a method to distinguish between undesirable and desirable congestion in the US electricity power industry.	Distinction between types of congestion	Undesirable vs. desirable congestion
Meng et al. (2015)	Introduced a two-stage model for evaluating congestion in mixed energy systems.	Two-stage congestion evaluation model	Evaluation of congestion in mixed energy systems
Mehdiloozad et al. (2017)	Demonstrated similar congestion characteristics within given regions, even with negative data.	Regional congestion characteristics	Similarity of congestion characteristics
Chen et al. (2018)	Categorized energy congestion into Undesirable Energy Congestion (DEC) and Desirable Energy Congestion (DEC).	Categorization of energy congestion	Classification of energy congestion types

Chen et al. (2019)	Proposed a novel congestion measurement approach and delineated three congestion types aligned with political objectives.	Novel congestion measurement approach	Political alignment of congestion types
Saati et al. (2020)	Delved into supply chain congestion pertaining to inputs or intermediate products.	Supply chain congestion analysis	Analysis of congestion in supply chains
Shadab et al. (2021)	Examined the potential for congestion within DMUs, identifying units with efficiency scores below one as candidates for congestion assessment.	DMU congestion assessment	Identification of inefficient DMUs for congestion assessment

This paper aims to introduce a congestion-based model, structured as follows: the subsequent section presents the Cooper method and the Jahanshloo and Khodabakhshi method for comparative detection. Section 3 introduces the proposed models for congestion detection, differentiating between strong and weak congestion. A numerical example is provided in Section 4, with the final conclusions presented in Section 5.

2. Preliminaries

Definition 1 (Efficiency): A DMU denoted as DMUo is classified as efficient within the model (1) when it reaches an optimal solution.

- $\phi_o^* = 1$
- All slack variables equate to zero

Definition 2 (Technical Inefficiency): DMUo is considered technically inefficient when there exists the potential to enhance certain inputs or outputs without deteriorating any other inputs or outputs.

Definition 3 (Congestion): DMU_o experiences input congestion if reductions in one or more inputs can result in increases in one or more outputs without worsening any other inputs or outputs. Conversely, it also exhibits input congestion if increases in one or more inputs lead to reductions in one or more outputs without improving any other inputs or outputs.

Definition 4 (Technical Efficiency): DMUo achieves technical efficiency if and only if it is impossible to enhance any inputs or outputs without worsening other inputs or outputs.

In this section, we provide a brief overview of congestion methodologies in the DEA literature, particularly focusing on the CCT and Jahanshahloo and Khodabakhsi approaches. A DMU, functioning as a unit that receives an input vector and generates an output vector, serves as a metric for efficiency assessment. To elucidate the concept of congestion, we first introduce the methodology presented by Cooper et al. (2002), which addresses the output-oriented BCC approach represented as model number (1).

$$
\begin{aligned}
&\varphi_o^* = \max \varphi + \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) \\
&st: \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \quad i=1,...,m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \phi_o y_{ro}, \quad r=1,...,s \\
&\sum_{j=1}^{n} \lambda_j = 1 \qquad (1) \\
&(\lambda_j , s_i^-, s_r^+) \geq 0 , \ j=1,...,n , \ r=1,...,s , \ i=1,...,m
\end{aligned}
$$

In this context, $\varepsilon > 0$ represents a non-Archimedean element defined as smaller than any positive real number. Let

$(\lambda^*, \varphi^*, s^{-*}, s^{+*})$ denote the optimal solution of model (1). To ascertain congestion, the subsequent model must be resolved:

$$\max \ \sum_{i=1}^{m} \delta_i^+$$
$$st: \sum_{j=1}^{n} \lambda_j x_{ij} - \delta_i^+ = \hat{x}_{io} \qquad i=1,...,m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} = \hat{y}_{ro} \qquad r=1,...,s$$
$$\sum_{j=1}^{n} \lambda_j = 1 \qquad (2)$$
$$s_i^{-*} \geq \delta_i^+ \qquad i=1,...,m$$
$$\lambda_j \geq 0 \ , \ \delta_i^+ \geq 0 \ , \ j=1,...,n \ , \ i=1,...,m$$

Where $(\hat{x}_o, \hat{y}_o)$ represents the projection of model 1, with $(\hat{x}_{io}, \hat{y}_{ro}) = (x_{io} - s_i^{-*}, \phi^* y_{ro} + s_r^{+*}) \forall i, r$.

The congestion quantity can be computed as follows:

$$s_i^c = s_i^{-*} - \delta_i^{+*} \ , \ i=1,...,m$$

Consequently, s^{-c*} denotes the congestion magnitude in evaluating DMU_o. Jahanshahloo and Khodabakhshi proposed the subsequent models for identifying congestion.

$$\max \ \phi_o + \varepsilon \left(\sum_{i=1}^{m} s_{i1}^- + \sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_{i2}^+ \right)$$
$$s.t: \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - s_{i1}^- + s_{i2}^+ \ , \ i=1,...,m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} = \phi_o y_{ro} + s_r^+ \ , \ r=1,...,s$$
$$\sum_{j=1}^{n} \lambda_j = 1 \qquad (3)$$
$$s_{i1}^-, s_{i2}^+ \geq 0 \ , \qquad i=1,...,m$$
$$\lambda_j, s_r^+ \geq 0 \ , \ r=1,...,s \ , \quad j=1,...,n$$

s_{i1}^-, s_{i2}^+ are the slack variables for input and s_r^+ input variables for the outputs. $\varepsilon > 0$ is a positive real number and non-Archimedean. suppose that $(\varphi_o^*, \lambda^*, s_1^{-*}, s_2^{-*}, s_r^{+*})$ is the optimum solution of model (1). To determine congestion, the following model should be solved

$$\max \sum_{i=1}^{m} \delta_i^+$$
$$st: \ \sum_{j=1}^{n} \lambda_j x_{ij} - \delta_i^+ = x_{io} - s_{i1}^{-*} + s_{i2}^{+*} \ , \ i=1,...,m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} = \phi_o^* y_{ro} + s_r^{+*} \qquad , \ r=1,...,s$$
$$\sum_{j=1}^{n} \lambda_j = 1 \qquad (4)$$
$$\delta_i^+ \leq s_i^{-*} \ , \qquad i=1,...,m$$
$$\delta_i^+, \lambda_j \geq 0 \ , \quad i=1,...,m, \quad j=1,...,n$$

"The quantity of congestion can be computed using the following methodology:"

$$s_i^c = s_i^{-*} - \delta_i^{+*} \ , \quad i=1,...,m$$

3. Proposed Methodology

3.1 Method Proposed for Congestion Identification in DMUs

To identify congestion in DMUs, where DMUs are denoted by (x_j, y_j), j=1,....,n we consider the production possibility set with return-to-scale technology as follows:

$$T_v = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{array}{l} \exists \lambda, \sum_{j=1}^{n} \lambda_j x_j \leq x, \\ \sum_{j=1}^{n} \lambda_j y_j \geq y, \qquad \geq 0 \ \ j=1,...,n \\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \end{array} \right\}$$

Since the congestion is inconsistent with the principle of the input possibility, therefore, to determine the congestion of the units, first, the principle of the possibility of input is eliminated and the set of production possibilities is defined as follows.

$$T_{new} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{array}{l} x = \sum_{j=1}^{n} \lambda_j x_j, \\ \sum_{j=1}^{n} \lambda_j y_j \ge y, \qquad \ge 0 \ \ \text{j=1,...,n} \\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \end{array} \right\}$$

Given the definition of unit congestion (X), the increase in one or more components (X) reduces one or more output index(Y). Also, the reduction of one or more input components (X) increases one or more Output index (Y) provided that other indicators do not improve. To determine the unit congestion (x,y), the model number 5 is solved to determine if there is a possibility of increasing output (Y) with decreasing input (X).

$$\begin{aligned} &\alpha^* = \max \ 1s^+ && (5) \\ &\text{st:} \quad X\lambda = x_p - s^- \\ &\qquad Y\lambda = y_p + s^+ \\ &\qquad 1\lambda = 1 \\ &\lambda \ge 0 \ , s^+ \ge 0 \ , \ s^- \ge 0 \end{aligned}$$

Suppose $\left(\lambda^*, s^{-*}, s^{+*}\right)$ is the optimal solution for model 5. If $s^{+*} = 0$ then y_p cannot be increased as a result of the unit No congestion. In order to obtain the maximum increase(Y) with a minimum reduction of x and also to avoid alternative optimal solution (S) in the model (6), the following model is solved.

$$\begin{aligned} &\beta^* = \min 1t^- && (6) \\ &\text{st:} \ \ X\mu = x_p - t^- \\ &\qquad Y\mu = y_p + s^{+*} \\ &\qquad 1\mu = 1 \\ &\qquad \mu \ge 0 \quad , \ \ t^- \ge 0 \end{aligned}$$

In which s_r^{+*} the optimal solution is model (6). Let $\left(\mu^*, t^*\right)$ be the optimal solution in Model 6. When $t^{-*} = 0$ equals zero, there exists no possibility of reducing the input for the $\left(x_p, y_p\right)$ unit, thus indicating the absence of congestion within the unit. Conversely, if $t^{-*} \ne 0$, the reduction in input for DMU_p occurs in at least one component while there is an increase in output in at least one component. To assess whether the increase in input x corresponds to a low or high amount of output, Model 7 is employed. It is important to note that if there is an increase in input in at least one component along with an increase in output of the unit $\left(x_p, y_p\right)$ at least one component, then this unit is free from congestion. In summary, Model 7 investigates whether increasing input X_p leads to an increase in output Y_p.

$$\begin{aligned} &P3: \gamma^* = \max \sum_{i=1}^{m} s^1{}_i + \sum_{r=1}^{s} s_r{}^2 \\ &\text{st:} \ \ \sum_{j=1}^{n} \lambda_j x_{ij} = \hat{x}_{ip} + s_i^1 \ , \quad \text{i=1,...,m} \\ &\qquad \sum_{j=1}^{n} \lambda_j y_{rj} = \hat{y}_{rp} + s_r^2 \ , \quad \text{r=1,...,s} \\ &\qquad \sum_{j=1}^{n} \lambda_j = 1 && (7) \\ &\qquad \sum_{i=1}^{m} s_i^1 \ge \varepsilon \\ &\qquad \lambda_j \ge 0 \ , \qquad \text{j=1,...,n} \\ &\qquad s^1{}_i, s_r{}^2 \ge 0 \ , \ \text{r=1,...,s} \ , \ \text{i=1,...,m} \end{aligned}$$

In model 7 epsilon is $\varepsilon > 0$.In which $\left(\hat{x}_p, \hat{y}_p\right)$ is the projection of $\left(x_p, y_p\right)$ in model 6. In other words: $\left(\hat{x}_p = x_p - t^{-*}, \hat{y}_p = y_p + s^{+*}\right)$.Therefore, in order to identify congestion and the level of congestion in $\left(x_p, y_p\right)$ unit, models 5and 6 and 7 will be solved.

Theorem 1) If and Model 7 is infeasible, then the DMU_P has t^{-*} congestion from the model (7).

Proof: as $\alpha^* > 0, \beta^* > 0$ then by reducing x_p the y_p is increased. This shows that the first condition of congestion is true.

Regarding the infeasibility of model 7, as $\hat{x}_p$ increases, $\hat{y}_p$ decreases or remains unchanged.

This shows that the possibility of producing of $(\bar{x}, \bar{y}) \in T_v$ is found which

$\left(\bar{x} \geq \hat{x}_p, \bar{y} \leq \hat{y}_p\right) \Rightarrow \left(\bar{x} \geq x_p - t^{-*}, \bar{y} \leq y_p + s^{+*}\right)$

These results in $x_p \leq \bar{x} + t^{-*}$ & $y_p \geq \bar{y} - s^{+*}$ Regarding $(\bar{x} + t^{-*}, \bar{y} - s^{+*}) \in T_v$ it is concluded that by increasing x_p It is reduced in size of y_p it means that the second condition of congestion is true too.

Theorem 2) If $\alpha^* > 0, \beta^* = 0$ then no DMU_P has congestion.

Proof: According to the definition of the model, it is obvious because there is a point on the PPS which dominates DMU_P. In other words, in model (2)

$\beta^* = \sum_{i=1}^{m} t_i^{-*} = 0 \Rightarrow t_i^{-*} = 0, \forall \text{i} \ \& \ s^{+*} \geq 0, s^{+*} \neq 0$

And this indicates that the output can increase but the input cannot be decreased; therefore, based on congestion definition, DMU_P has no congestion.

Theorem3) if $\alpha^* + \beta^* > 0$ then DMU_P is inefficient

Proof by conjunction

Status 1: Assume that $\alpha^* > 0$, therefore the conditions of the problem are satisfied, thus

$$\alpha^* > 0 \Rightarrow \sum_{r=1}^{s} s_r^{+*} > 0 \Rightarrow \exists \text{ r}; s_r^{+*} > 0$$

$$\sum_{j=1}^{n} \lambda_j^* x_{ij} = x_{ip} - s_i^{-*} \leq x_{ip} \quad , \quad \forall i$$

$$\sum_{j=1}^{n} \lambda_j^* y_{rj} = y_{rp} + s_r^{+*} \geq y_{rp} \quad , \quad \forall \text{r}$$

$$\sum_{j=1}^{n} \lambda_j^* = 1$$

$$\begin{cases} \bar{x} = x_p - s^{-*} \\ \bar{y} = y_p + s^{+*} \end{cases} \Rightarrow \begin{bmatrix} -\bar{x} \\ \bar{y} \end{bmatrix} \underset{\neq}{\geq} \begin{bmatrix} -x_p \\ y_p \end{bmatrix}$$

hence DMU_P is inefficient.

Status 2: $\beta^* > 0$ is satisfied similarly in the conditions of the problem. Therefore,

$$\beta^* > 0 \Rightarrow \sum_{i=1}^{m} t_i^{-*} > 0 \Rightarrow \exists i; t_i^{-*} > 0$$

$$\begin{cases} \sum_{j=1}^{n} \lambda^*{}_j x_{ij} = x_{ip} - t_i^{-*} \leq x_{ip} \quad , \quad \forall i \\ \sum_{j=1}^{n} \lambda^*{}_j y_{rj} = y_{rp} + s_r^{+*} \geq y_{rp} \quad , \quad \forall r \\ \sum_{j=1}^{n} \lambda^*{}_j = 1 \end{cases}$$

$$\begin{pmatrix} \bar{x} = x_p - t^{-*} \\ \bar{y} = y_p + s^{+*} \end{pmatrix} \Rightarrow \begin{pmatrix} -\bar{x} \\ \bar{y} \end{pmatrix} \underset{\neq}{\geq} \begin{pmatrix} -x_p \\ y_p \end{pmatrix}$$

Therefor DMU_P is inefficient.

Assume condition that $\alpha^* > 0, \beta^* > 0$ and model 8 is feasible. Suppose (λ^*, t^{-*}) is the optimal solution of model 7, therefore $Y\lambda^* = y_p + s^{+*} \underset{\neq}{\geq} y_p$ and $X\lambda^* = x_p - t^{-*} \underset{\neq}{\leq} x_p$ This issue indicates that there is a production possibility set in T_V that by taking lower input in at least one component produces more output in one component. Hence the first condition in defining congestion is satisfied. Now the second condition is considered like this:

Based on the assumption, model 5 is feasible and $(\lambda^*, s^{1*}, s^{2*})$ is its optimal solution.

$$X\lambda^* = \hat{x}_p + s^{*1} = x_p + (s^{*1} - t^{*-})$$
$$Y\lambda^* = \hat{y}_p + s^{*2} = y_p + \left(s^{*+} + s^{*2}\right)$$

If $\forall i\ ; k_i^* = s_i^{*1} - t_i^{-*} \leq 0$, there is no chance of input increasing because $\alpha^*>0, \beta^* > 0$, so DMU_P has $t^{-*} \neq 0$ congestion and if $\exists i\ ; k_i^* = s_i^{*1} - t_i^{-*} > 0$, thus has no congestion in i^th^ component, because by increasing input in i^th^ component $k_i^* > 0$, the level of output is $y_p + \left(s^{*+} + s^{*2}\right)$ that contradicts the second condition of congestion.

Consider disjoint sets of E^-, E^+ as follow

$$E^- = \left\{i \middle| k_i^* \leq 0\right\}, E^+ = \left\{i \middle| k_i^* > 0\right\}$$
$$E^+ \cup E^- = \{1,...,m\} \quad E^+ \cap E^- = \varnothing$$

We have following conditions:

1) $E^- = \{1,...,m\}$ so DMU_P has $t^{-*} \neq 0$ congestion. If $t^{-*} > 0$, DMU_P has strong congestion otherwise, it has weak congestion.
2) $E^+ = \{1,...,m\}$, so DMU_P doesn't have congestion because by increasing input in all components, the output will increase which contradicts the second condition of congestion.
3) $E^+ \neq \{1,...,m\}$, in this condition DMU_P doesn't have congestion in $i \in E^+$ component.

Definition 5: Strong Congestion

DMU_P, characterized by the coordinates $\left(x_p, y_p\right)$ is considered to exhibit strong congestion if and only if there exists a scenario within T where $\left(\bar{x}, \bar{y}\right)$ can be identified, whereby reducing input across all components results in an increase of at least one output in one component. In simpler terms:

$$(\bar{x} < x_p) \Rightarrow \bar{y} \underset{\neq}{\geq} y_p$$

3.2. Method to identify strong congestion

Assume for DMUp, $\alpha^* > 0, \beta^* > 0$ and model 8 is not feasible or $\alpha^* > 0, \beta^* > 0, E^- \neq \varnothing$ *therefore* DMU_P has congestion. In order to recognize the strong congestion and the level of congestion, the following model should be solved:

$$\begin{aligned} T^* = \quad & \min\ 1q^- \\ st. \quad & X\lambda = x_p - q^- \\ & Y\lambda = y_p + s^{+*} \\ & 1\lambda = 1 \\ & q^- \geq 1\varepsilon \\ & \lambda \geq 0 \end{aligned} \tag{8}$$

If model 8 is infeasible therefore DMU_P has weak congestion with the value of *of* t^{-*} from model 6. Suppose there is a condition in which model 8 is feasible and has the optimum *solution* $\left(\lambda^*, q^{-*}\right)$. Since DMU_P has congestion therefore $X_p > 0$ on the other hand, $q^{-*} > 0$ therefore $x^* = x_p - q^{-*} < x_p$ and $y^* = y_p + s^{+*} \underset{\neq}{\geq} y_p$.

Therefore, there is a possibility set $\left(x^*, y^*\right)$ $in\ T_v$ which that taking lower input in all components produces more output at least for one component. Therefore, DMU_P has strong congestion with the value of $q^{-*} > 0$ from model 6. It should be noted that in the condition of one input and one output, the strong and weak congestion is the same.

4. Numerical Examples

4.1 Example:

Consider eight DMUs operating under the condition of having one input and one output.

The corresponding T_{new} values in the DMUs are as follows: Refer to Figure 2 for details.

Consequently, the optimal solutions pertaining to models 5, 6, and 7 as depicted in Table 3 are elucidated herein. According to the optimal solution it can be concluded that for DMU_A there is no chance of increasing output and also there is no chance of decreasing input as well because $\alpha^* = \beta^* = 0$. In addition, since $\gamma^* > 0$ therefore increasing input leads to increasing output, so DMU_A doesn't have congestion. For $DMU_B : \alpha^* = \beta^* = 0$, model 7 is feasible, thus DMU_B doesn't have congestion. For $DMU_C : \alpha^* = \beta^* = 0$, model 7 is infeasible, thus DMU_C doesn't have congestion. For DMU_D : $\alpha^* > 0, \beta^* = 0$, model 7 is infeasible, thus DMU_D doesn't have congestion. For $DMU_E : \alpha^* > 0, \beta^* = 0$, model 7 is infeasible, thus DMU_E has $t^{-*} = 4$ units' congestion in input. The value of $s^{+*} = 2$ units will be added to the output by eliminating congestion and the coordination of DMU_E the benchmark of model 6 is as follows:

$$\left(\hat{x}_E = x_E - t^{-*} = 12 - 4 = 8\right)$$

$$\left(\hat{y}_E = y_E + s^{+*} = 3 + 2 = 5\right)$$

For $DMU_F : \alpha^* > \beta,^* > 0$, model 7 is infeasible, thus DMU_F has $t^{-*} = 1$ unit congestion in input which by eliminating of that, therefore the value of $s^{+*} = 4$ will be added to the output and the coordination of DMU_F benchmark of model 6 is as follow:

$$\left(\hat{x}_F = x_F - t^{-*} = 9 - 1 = 8\right)$$

$$\left(\hat{y}_F = y_F + s^{+*} = 1 + 4 = 5\right).$$

For DMU_K, DMU_G $\alpha^* > 0, \beta^* = 0$, model 5 is infeasible, thus DMU_K, DMU_G doesn't have congestion.

Table 2: presents eight DMUs operating with a single input and a single output condition.

	A	B	C	D	E	F	K	G
I	3	4	8	8	12	9	3	6
O	3	5	5	2	3	1	1	4

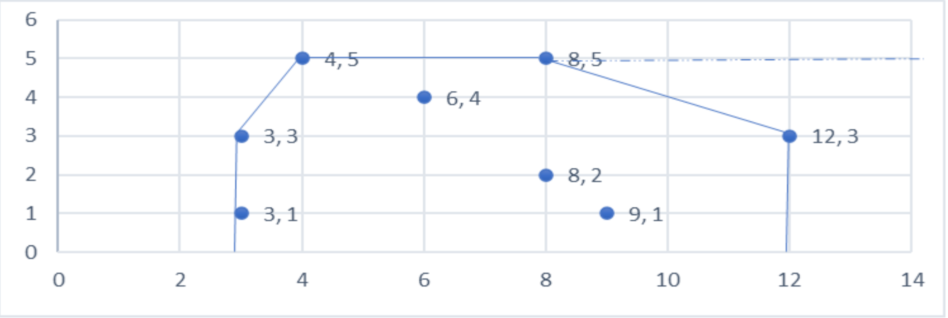

Figure 1 illustrates the Production Possibility Set (PPS) of T_{new} corresponding to the eight DMUs.

Table 3: Investigating Congestion in DMUs through Model Solutions 5, 6, and 7.

	α^*	β^*	γ^*	s_α^{-*}	s_α^{+*}	t_β^{-*}	s_γ^{*1}	s_γ^{*2}	result of Congestion
DMU_A	0	0	2	0	0	0	1	2	No
DMU_B	0	0	0	0	0	0	0.01	0	No
DMU_C	0	0	Inf	0	0	0	-----	-----	No
DMU_D	3	0	Inf	0	3	0	----	-----	No
DMU_E	2	4	Inf	8	2	4	----	----	Yes
DMU_F	4	1	Inf	1	4	1	----	-----	Yes
DMU_K	2	0	2	0	2	0	1	2	No
DMU_G	1	0	0	2	1	0	0	0	No

4.2 Example:

The data presented in Table 4 represents an extensive input-output dataset obtained from the textile industry in China, covering the time frame from 1981 to 1997. This dataset played a pivotal role in Cooper et al.'s landmark study in 2001, where it was utilized as a foundational component for computing density measures crucial to understanding the operational dynamics within the industry.

Figure 2 showcases the dataset pertinent to the textile industry, encompassing key quantitative variables essential for rigorous analysis within this sector. This dataset includes vital metrics such as labor, capital, and output, providing a comprehensive view of industry dynamics over time.

The tabulated outcomes in Table 5 unveil a distinctive categorization among the analyzed DMUs, highlighting the efficacious performance of DMU_{01}, DMU_{03}, DMU_{04}, DMU_{05}, DMU_{11}, DMU_{14}, DMU_{16}, and DMU_{17}, juxtaposed against the inefficiencies observed within DMU_{02}, DMU_{06}, DMU_{07}, DMU_{08}, DMU_{09}, DMU_{10}, DMU_{12}, DMU_{13}, and DMU_{15}. This categorization underscores a clear demarcation between DMUs exhibiting commendable efficiency and those characterized by suboptimal performance, thereby providing valuable insights into the efficacy of the evaluated units within the analyzed framework.

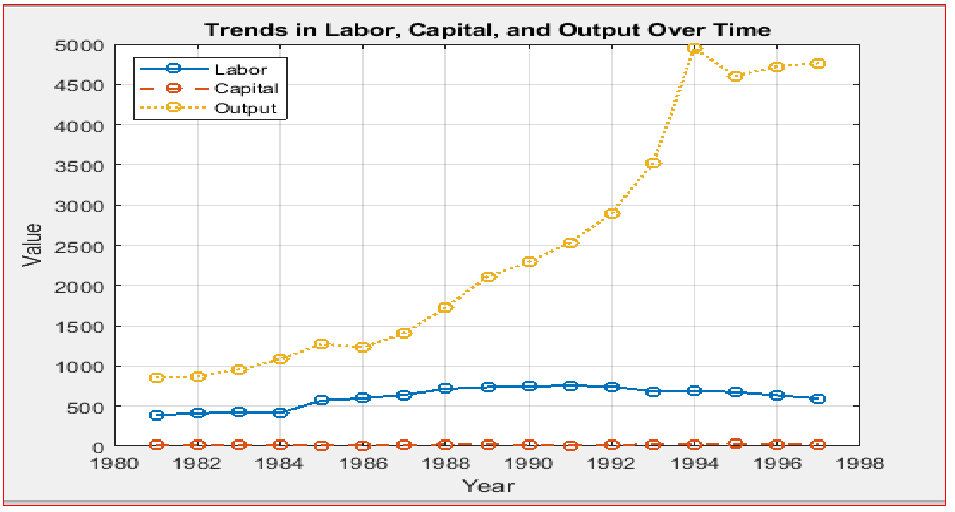

Figure 2. Data for the textile industry

Table 4: presents the input and output data derived from China's textile industry, encompassing two key input factors and one output measure.

Year	Labor	Capital	Output
DMU1=1981	389.00	19.86	856.02
DMU2=1982	412.30	21.16	866.85
DMU3=1983	423.50	17.08	956.04
DMU41984	417.30	18.10	1082.94
DMU5=1985	570.00	12.61	1273.20
DMU6=1986	600.50	13.45	1230.72
DMU7=1987	641.10	15.91	1410.66
DMU8=1988	715.30	23.72	1728.16
DMU9=1989	736.00	25.97	2109.57
DMU10=1990	745.00	18.24	2291.08
DMU11=1991	756.00	14.40	2533.27
DMU12=1992	743.00	17.50	2899.16
DMU13=1993	684.00	25.08	3520.74
DMU14=1994	691.00	25.45	4949.93
DMU15=1995	673.00	29.35	4604.00
DMU16=1996	634.00	23.05	4722.29
DMU17=1997	595.00	25.02	4760.28

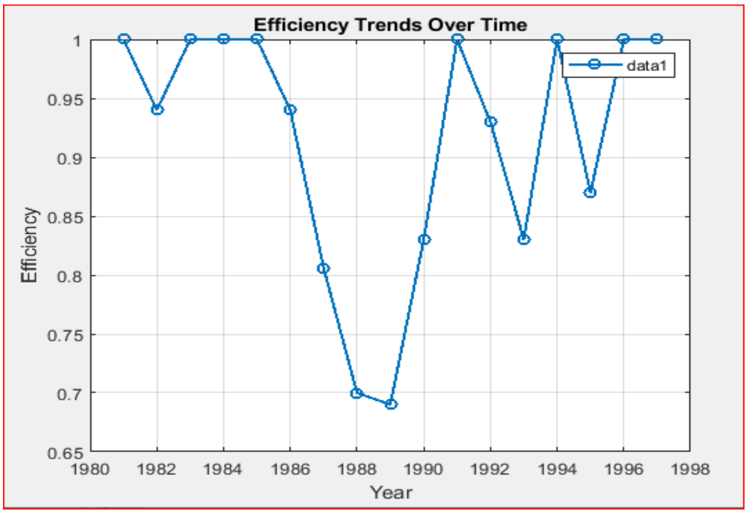

Figure 3. Efficiency Trends over Time

Table 5: The BCC model's optimal solution to the input oriented for the data

	θ^*	S_1^{-*}	S_2^{-*}	S^{+*}	Efficient unit
DMU1=1981	1	0	0	0	✓
DMU2=1982	0.94	0	0.13	0	------
DMU3=1983	1	0	0	0	✓
DMU41984	1	0	0	0	✓
DMU5=1985	1	0	0	0	✓
DMU6=1986	0.94	0	0	36	-------
DMU7=1987	0.806	0	0	0	------
DMU8=1988	0.7	0	0	0	------
DMU9=1989	0.69	0	0	0	------
DMU10=1990	0.83	0	0	0	------
DMU11=1991	1	0	0	0	✓
DMU12=1992	0.93	0	0	0	------
DMU13=1993	0.83	0	0	0	-----
DMU14=1994	1	0	0	0	✓
DMU15=1995	0.87	0	0.77	0	-----
DMU16=1996	1	0	0	0	✓
DMU17=1997	1	0	0	0	✓

In Figure 3, the efficiency curve delineates the varying efficiency levels among distinct decision-making units (DMUs) within the textile industry spanning the years from 1981 to 1997. Each data point on the curve corresponds to the efficiency value of a specific DMU in a given year, providing a detailed depiction of efficiency variability across DMUs and temporal epochs. The curve exhibits fluctuations in efficiency over time, reflecting periods of diverse performance within the textile industry. Notably, from 1981 to 1984, efficiency maintains a consistently high level (around 1), indicative of stable performance or optimization during this interval. However, a notable decline in efficiency is observed in 1988 (DMU8), where efficiency decreased to 0.7, highlighting potential areas for improvement or operational challenges. Subsequently, efficiency levels fluctuate between 0.69 and 1 in subsequent years, showcasing variability in performance across DMUs.

The findings depicted in Table 6 provide indications that DMUs 08, 10, 11, and 12 are characterized by a state of congestion categorized as "weak." In contrast, DMU 09 is identified as exhibiting a notably elevated level of congestion, characterized as "strong." This distinction underscores the nuanced variation in congestion levels among the DMUs under scrutiny.

Table 6 The result from Cooper's approach

	s_1^{-*}	s_2^{-*}	δ_1^{+*}	δ_2^{+*}	$s_1^c = s_1^{-*} - \delta_1^{+*}$	$s_1^c = s_2^{-*} - \delta_2^{+*}$	Labor congestion	Capital congestion
DMU1=1981	0	0	0	0	0	0	0	0
DMU2=1982	0	0.7164	0	0	0	0.7164	0	0
DMU3=1983	0	0	0	0	0	0	0	0
DMU41984	0	0	0	0	0	0	0	0
DMU5=1985	0	0	0	0	0	0	0	0
DMU6=1986	0	0	0	0	0	0	0	0
DMU7=1987	0	0	0	0	0	0	0	0
DMU8=1988	65	0	0	0	65	0	24.3	0
DMU9=1989	45	0.52	0	0	45	0.52	45	0.52
DMU10=1990	43	0	0	0	43	0	54	0
DMU11=1991	0	0	0	0	0	0	65	0
DMU12=1992	31	0	0	0	31	0	52	0
DMU13=1993	1.78	0	0	0	1.787	0	0	0
DMU14=1994	0	0	0	0	0	0	0	0
DMU15=1995	0	3.99	0	0	0	3.99	0	3.9
DMU16=1996	0	0	0	0	0	0	0	0
DMU17=1997	0	0	0	0	0	0	0	0

Table 7 Result of proposed Model

	MODEL3				MODEL4			MODEL 5				
	α^*	s_1^{-*}	s_2^{-*}	s^{+*}	β^*	t_1^{-*}	t_2^{-*}	t^*	s_1^{*1}	s_2^{*2}	s_2^{*2}	congestion
DMU1=1981	0	0	0	0	0	0	0	4405	200	5.5	4100	No
DMU2=1982	430	0	0.726	430	0.716	0	0.716	3900	280	5	3700	No
DMU3=1983	0	0	0	0	0	0	0	4300	270	8.3	4000	NO
DMU41984	0	0	0	0	0	0	0	4100	270	7.5	3900	No
DMU5=1985	0	0	0	0	0	0	0	3800	120	13	3700	No
DMU6=1986	420	0	0	420	0	0	0	3400	91	12	3300	No
DMU7=1987	1100	0	0	1100	0	0	0	2500	50	9.5	2400	No
DMU8=1988	3100	65	0	3100	65	65	0	210	41	1.7	160	No
DMU9=1989	2800	45	0.52	2800	46	45	0.53	inf	-----	----	-----	Yes
DMU10=1990	1200	33	0	1200	43	43	0	1000	0	5.3	4	No
DMU11=1991	0	0	0	0	0	0	0	Inf	-----	-----	-----	No
DMU12=1992	420	31	0	420	31	31	0	850	0	4.3	840	NO
DMU13=1993	1400	1.78	0	1400	1.78	1.78	0	400	8.7	0.37	350	NO
DMU14=1994	0	0	0	0	0	0	0	Inf	-----	-----	------	------
DMU15=1995	310	0	3.9	310	3.9	0	3.9	54	18	0.08	36	NO
DMU16=1996	0	0	0	0	0	0	0	290	57	2.4	230	NO
DMU17=1997	0	0	0	0	0	0	0	290	96	0.43	190	NO

Table 8: Results obtained from three different approaches

DMUs	Result of cooper approach	Result of Jahanshahloo & khodabakhshi approach	Result of proposed model
DMU1=1981	NO	No	NO
DMU2=1982	Weak congestion	No	NO
DMU3=1983	No	No	NO
DMU41984	No	No	NO
DMU5=1985	No	No	NO
DMU6=1986	No	No	NO
DMU7=1987	No	No	NO
DMU8=1988	Weak congestion	Weak congestion	NO
DMU9=1989	Strong congestion	Strong congestion	Strong congestion
DMU10=1990	Weak congestion	Weak congestion	NO
DMU11=1991	No	Weak congestion	NO
DMU12=1992	Weak congestion	Weak congestion	NO

DMU13=1993	Weak congestion	No	NO
DMU14=1994	No	No	NO
DMU15=1995	Weak congestion	No	NO
DMU16=1996	No	No	NO
DMU17=1997	No	No	NO

In table 8 presents the results obtained from three different approaches for assessing congestion among Decision-Making Units (DMUs). The Cooper approach, Jahanshahloo & Khodabakhshi approach, and the proposed model are evaluated across various DMUs and years, providing insights into the congestion levels observed in the dataset. The optimal solutions for models 5, 6 and 7 for table 3.

5. Conclusions

This paper presents a refined conceptualization of congestion within the context of data envelopment analysis (DEA), introducing two distinct conditions for its characterization. The first condition delineates congestion as occurring when a decrease in input results in an increase in output, while the second condition posits congestion when an increase in input leads to a decrease in output. Models 5, 6, and 7 are formulated based on these novel definitions, with Model 8 devised for discerning the severity of congestion. Through numerical illustrations, instances are demonstrated where certain DMUs exhibit a decrease in input accompanied by an increase in output, while simultaneously experiencing output growth with input augmentation. While conventional congestion detection methodologies, such as those advocated by Cooper, may identify these scenarios as indicative of congestion, our proposed framework distinguishes them as non-congestive. This differentiation arises from our method's unique capability to identify congestion regardless of the direction of input change, thus overcoming the limitations of existing approaches.Furthermore, our proposed methodology facilitates the precise localization and quantification of congestion within specific input units, as well as predicting the corresponding output units poised to expand upon congestion alleviation. Notably, the models presented herein operate within a non-radial framework, affording the capacity to distinguish between strong and weak manifestations of congestion.

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