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Fixed point results of hybrid-type F-contractions

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Abstract. This manuscript studies new concepts of hybrid F-contractions on a complete metric space. It provides new conditions for the existence of fixed points for such mappings. The main idea of this paper unifies a few important results in the corresponding literature. Some of these consequences are highlighted and discussed as corollaries. In support of the assumptions forming the theorems presented herein, a comparative nontrivial example with a graphical illustration is provided.

Keywords: Hybrid contraction, Jaggi-type contraction, F-contraction, fixed point.

2010 AMS Subject Classification: 47H10; 54H25.

1. Introduction

In an attempt to extend the Banach contraction principle [4], Wardowski [20] introduced a new notion of contraction called F-contraction and proved a fixed point theorem. Shortly, Wardowski and Van Dung [22] initiated the concept of F-weak contraction on a metric space and obtained a generalization of F-contraction. In [18], Secelean noted that condition (F_2) in Wardowski's definition of F-contraction can be replaced with an equivalent and simpler one given by (F_2') : inf F$= -\infty$. In like manner, Piri and Kumam [16] established a variant of Wardowski's theorem by using (F_2'). Recently, Wardowski [21] proposed the replacement of the positive constant ξ in the original definition of F-contraction with a function φ fulfilling certain condition and established a new form of contraction on metric space under the name (φ, F)-contraction. Singh et al. [19] initiated the ideas of Boyd-Wong F-contractions of two types and proved new common fixed point theorems in partial b-metric spaces. Ehsan et al. [12] proved some fixed point theorems

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of set-valued F-contractions in the context of quasi-ordered metric spaces. Not long ago, Yahaya et al. [23] introduced the idea of a multi-valued contraction that unified Ćirić and Caristi contractions in the setting of a complete metric space. For some important advancements in F-contractions with related results, we refer to Karapınar et al. [11]. One of the important developments in fixed point theory of contractive-type inequalities is the introduction of hybrid contractions (for example, see Karapınar and Fulga [10], Alansari et al. [2], and some references therein). It is worthy of note that hybrid F-contractions are a powerful tool for solving fixed point problems in a variety of contexts. For example, hybrid F-contractions have been used to study the existence and uniqueness of solutions to differential equations, polynomial equations, the convergence of iterative algorithms, and so on. For some of these applications, we can consult Alansari et al. [2], Aydi et al. [3], Jiddah et al. [8], Ogbumba et al [14, 15] and some important applications of F-contractive operators presented in [1, 6, 13].

2. Preliminaries

In what follows, we recall specific preliminary concepts needed in the discussion of our main results. Throughout, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of natural numbers, non-negative reals and real numbers, respectively.

Let Ω_F be the set of all mappings $F : (0,\infty) \to \mathbb{R}$ satisfying the following conditions:

- $(F1)$ F is strictly increasing, that is, for $\alpha, \beta \in (0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;
- $(F2)$ for each sequence $\{a_n\}$ of positive numbers $\lim\limits_{n\to\infty} F(a_n)$=0 if and only if $\lim\limits_{n\to\infty} F(a_n) = -\infty$;
- $(F3)$ there exists $k \in (0,1)$ such that $\lim\limits_{a\to 0^+} a^k F(a) = 0$.

Definition 2.1 [20] Let (X,d) be a metric space and $T : X \longrightarrow X$ be a mapping. Then T is said to be an F-contraction if $F \in \Omega_F$ and there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx,Ty) > 0 \ \Rightarrow \tau + F(d(Tx,Ty)) \leqslant F(d(x,y)). \tag{1}$$

Considering some variants of the mapping $F \in \Omega_F$, we present the following examples:

***Example* 2.2** [20] Let $F : (0,\infty) \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that $F \in \Omega_F$. Then each self mappings T on a metric space (X,d) satisfying (1) is an F-contraction such that

$$d(Tx,Ty) \leqslant e^{-\tau} d(x,y), \text{for all } x,y \in X,\ Tx \neq Ty. \tag{2}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx,Ty) \leqslant e^{-\tau} d(x,y)$ also holds. Therefore, T satisfies

$$d(Tx,Ty) \leqslant Ld(x,y) \text{ for all x, y} \in X, \text{with } L = e^{-\tau}$$

and T is a contraction.

***Example* 2.3** [20] Let $F : (0,\infty) \to \mathbb{R}$ be given by the formula $F(\alpha) = \alpha + \ln \alpha$. It is clear that $F \in \Omega_F$. Then each self mapping T on a metric space (X,d) satisfying (1) is an F-contraction such that

$$\frac{d(Tx,Ty)}{d(x,y)}e^{d(Tx,Ty)-d(x,y)} \leqslant e^{-\tau}, \text{ for all } x,y \in X, \ Tx \neq Ty. \tag{3}$$

We can find some different examples for the function F belonging to Ω_F. In addition, Wardowski concluded that every F-contraction T is a contractive mapping; that is,

$$d(Tx,Ty) < d(x,y), \text{for all } x,y \in X, Tx \neq Ty.$$

Thus, every F-contraction is a continuous mapping. The following theorem, which was given by Wardowski, is a proper generalization of Banach contraction principle.

Theorem 2.4 [20] Let (X,d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.

The first two most embraced extensions of Banach-Cacciopoli principle involving rational inequalities were presented by Dass-Gupta [5] and Jaggi [7]. For the purpose of this paper, we recall the main result in [7] as follows:

Theorem 2.5 [7] Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a continuous mapping. Further, assume that T satisfies the condition:

$$d(Tx,Ty) \leqslant \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \eta d(x,y)$$

for all $x,y \in X$, $x \neq y$ and for some $\alpha, \eta \in [0,1)$ with $\alpha + \eta < 1$. Then T has a unique fixed point in X.

Definition 2.6 [9] Let (X,d) be a metric space and $T : X \longrightarrow X$ be a mapping. We shall call T a $(\lambda,\alpha,\beta)-\textit{interpolative}$ Kannan contraction if there exist $\lambda \in [0,1), \alpha,\beta \in (0,1)$ with $\alpha+\beta < 1$ such that $d(Tx,Ty) \leqslant \lambda d(x,Tx)^{\alpha} d(y,Ty)^{\beta}$ for all $x,y \in X$ with $x \neq Tx, y \neq Ty$.

We say that $\psi : [0,\infty) \to [0,\infty)$ is a (c)-comparison function (See, [17]) if it is nondecreasing and there exist $l_0 \in \mathbb{N}$, $k \in (0,1)$ and a convergent series $\sum_{l=1}^{\infty} z_l$ such that $z_l \geqslant 0$ and $\psi^{l+1}(t) \leqslant k\psi^l(t) + z_l$, (1) for $l \geqslant l_0$ and $t \geqslant 0$. The set of all function (c)-comparison function will be denoted by Ψ.

Lemma 2.7 [17] Every $\psi \in \Psi$ satisfies the following:

(a) $\psi(t) < t$, for any $t \in \mathbb{R}_+$;
(b) ψ is continuous at 0;
(c) the series $\sum_{l=1}^{\infty} \psi^l(t)$ is convergent for $t \geqslant 0$.

Karapınar and Fulga [10] introduced a new hybrid contraction in the following manner:

Definition 2.8 [10] A self-mapping T on (X,d) is called a Jaggi-type hybrid contraction if there is $\psi \in \Psi$ so that $d(Tx,Ty) \leqslant \psi(J_T^s(x,y))$ for all distinct $x,y \in X$, where $s \geqslant 0$ and

$$J_T^s(x,y) = \begin{cases} \left[\sigma_1 \left(\frac{d(x,Tx)\cdot d(y,Ty)}{d(x,y)}\right)^s + \sigma_2 (d(x,y))^s\right]^{\frac{1}{s}} & \text{for } s > 0, x,y \in X, x \neq y \\ (d(x,Tx))^{\sigma_1}(d(y,Ty))^{\sigma_2} & \text{for } s = 0, x,y \in X \setminus Fix(T) \end{cases}$$

for $\sigma_i \geqslant 0, i = 1, 2$, such that $\sigma_1 + \sigma_2 = 1$ and $Fix(T) = \{z \in X : Tz = z\}$.

The main result in [10] is given hereunder.

Theorem 2.9 [10] A continuous self-mapping T on (X, d) possesses a fixed point x provided that T is a Jaggi-type hybrid contraction. Moreover, for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x.

A lot of work has been done in the area of fixed point results of contraction mappings on metric spaces. Some of these results include fixed point theorems of hybrid contractions and F-contractions. However, a combination of the last two principles has not been adequately examined. With this background information, in this manuscript, a new type of hybrid F-contraction that is a unification of a Jaggi-type contraction, interpolative type contraction and F-contraction in the setting of a complete metric spaces is introduced. Sufficient conditions for the existence and uniqueness of fixed point of such contractions are investigated. A comparable example is presented to support the hypothesis of our result and its enhancements over earlier findings. With the help of some consequences presented, it has been established that the idea proposed herein is a generalization of some well-known invariant point results in the domain of F-contractive operators in metric space.

3. Main Results

Definition 3.1 A self-mapping T on a metric space (X, d) is called a Jaggi-type hybrid F-contraction if there exist $\tau > 0, \psi \in \Psi$ and $F \in \Omega_F$ such that
$d(Tx, Ty) > 0$ implies

$$\tau + F(d(Tx, Ty) \leqslant F(\psi(J_T^s(x, y))) \tag{4}$$

for all distinct $x, y \in X$, where $s \geqslant 0$,

$$J_T^s(x, y) = \begin{cases} \left[\sigma_1 \left(\dfrac{d(x, Tx)d(y, Ty)}{d(x, y)}\right)^s + \sigma_2 (d(x, y))^s\right]^{\frac{1}{s}} & \text{for } s > 0,\ x, y \in X, x \neq y \\ (d(x, Tx))^{\sigma_1} (d(y, Ty))^{\sigma_2} & \text{for } s = 0,\ x, y \in X \backslash Fix(T) \end{cases},$$

$\sigma_i \geqslant 0, i = 1, 2$, such that $\sigma_1 + \sigma_2 = 1$ and $Fix(T) = \{z \in X : Tz = z\}$.

Theorem 3.2 A continuous self-mapping T on a complete metric space (X, d) possesses a unique fixed point x, provided that T is a Jaggi-type hybrid F-contraction. Moreover, for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x.

Proof. Let $x_0 \in X$, be arbitrary but fixed. We define an iterative sequence $\{x_n\}$ by

$$x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0, \cdots,\ x_n = Tx_{n-1} = T^n x_0.$$

Suppose that $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. On the contrary, if the inequality above does not hold, that is if there exists $k_0 \in \mathbb{N}_0$ such that $d(x_{k_0}, x_{k_0+1}) = 0$, then we get $x_{k_0} = x_{k_0+1} = Tx_{k_0}$. This means x_{k_0} is a fixed point of T, and the proof ends. We shall prove the claim of the theorem by examining two distinct cases: $s = 0$ and $s > 0$.

Case (i): In the case where $s > 0$, letting $x = x_{n-1}$ and $y = Tx_{n-1}$, the expression 4 becomes

$$\tau + F(d(Tx_{n-1}, T^2x_{n-1})) \leqslant F(\psi(J_T^s(x_{n-1}, Tx_{n-1}))), \tag{5}$$

where

$$\begin{aligned} J_T^s(x_{n-1}, Tx_{n-1}) &= \left[\sigma_1 \left(\frac{d(x_{n-1}, Tx_{n-1}) \cdot d(Tx_{n-1}, T^2x_{n-1})}{d(x_{n-1}, Tx_{n-1})}\right)^s + \sigma_2(d(x_{n-1}, Tx_{n-1}))^s\right]^{\frac{1}{s}} \\ &= \left[\sigma_1(d(x_n, x_{n+1}))^s + \sigma_2(d(x_{n-1}, x_n))^s\right]^{\frac{1}{s}}. \end{aligned} \tag{6}$$

Therefore, from (5) and (6), we obtain

$$\tau + F(d(x_n, x_{n+1})) \leqslant F(\psi\left[\sigma_1(d(x_n, x_{n+1}))^s + \sigma_2(d(x_{n-1}, x_n))^s\right]^{\frac{1}{s}}). \tag{7}$$

Suppose that $d(x_n, x_{n+1}) \geqslant d(x_{n-1}, x_n)$ and using the non-decreasing property of ψ, we have from (7)

$$\begin{aligned} \tau + F(d(x_n, x_{n+1})) &\leqslant F(\psi\left[\sigma_1(d(x_n, x_{n+1}))^s + \sigma_2(d(x_{n-1}, x_n))^s\right]^{\frac{1}{s}}) \\ &\leqslant F(\psi\left[\sigma_1(d(x_n, x_{n+1}))^s + \sigma_2(d(x_n, x_{n+1}))^s\right]^{\frac{1}{s}}) \\ &= F(\psi\left[(\sigma_1 + \sigma_2)(d(x_n, x_{n+1}))^s\right]^{\frac{1}{s}}) \\ &= F(\psi(d(x_n, x_{n+1}))) < F(d(x_n, x_{n+1})). \end{aligned}$$

Therefore, $\tau + F(d(x_n, x_{n+1})) < F(d(x_n, x_{n+1}))$, which is a contradiction. Hence, we obtain that $d(x_n, x_{n+1}) \leqslant d(x_{n-1}, x_n)$. Substituting the above inequality in (6) yields

$$\tau + F(d(x_n, x_{n+1})) \leqslant F(\psi(d(x_{n-1}, x_n))). \tag{8}$$

Also, suppose that $d(x_{n-1}, x_n) \leqslant d(x_{n-2}, x_{n-1})$, then we obtain

$$\tau + F(d(x_{n-1}, x_n)) \leqslant F(\psi(d(x_{n-2}, x_{n-1}))). \tag{9}$$

The inequality (8) and (9) can be rewritten respectively as follows

$$F(d(x_n, x_{n+1})) \leqslant F(\psi(d(x_{n-1}, x_n))) - \tau. \tag{10}$$

In like manner, we have

$$F(d(x_{n-1}, x_n)) \leqslant F(\psi(d(x_{n-2}, x_{n-1}))) - \tau. \tag{11}$$

From (10) and (11), we obtain $F(d(x_n, x_{n+1})) \leqslant F(\psi^2(d(x_{n-2}, x_{n-1}))) - 2\tau$. Continuing in this way, we deduce inductively that

$$F(d(x_n, x_{n+1})) \leqslant F(\psi^n(d(x_0, x_1))) - n\tau. \tag{12}$$

Let $d(x_n, x_{n+1}) = \gamma_n$, then (12) can be rewritten as

$$F(\gamma_n) \leqslant F(\psi^n(\gamma_0)). \tag{13}$$

Letting $n \to \infty$ in (13), we obtain $\lim\limits_{n\to\infty} F(\gamma_n) = -\infty$, and together with (F_2), it gives

$$\lim_{n\to\infty} \gamma_n = 0. \tag{14}$$

From (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0. \tag{15}$$

By (13), the following holds

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\psi^n(\gamma_0)) \leqslant \gamma_n^k (F(\psi^n(\gamma_0)) - n\tau) - \gamma_n^k F(\psi^n(\gamma_0))$$

for all $n \in \mathbb{N}$. This implies that $\gamma_n^k F(\gamma_n) - \gamma_n^k F(\psi^n(\gamma_0)) \leqslant -\gamma_n^k n\tau$, from which we have $\frac{1}{\tau}\left[\gamma_n^k F(\psi^n(\gamma_0)) - \gamma_n^k F(\gamma_n)\right] \geqslant n\gamma_n^k$. Hence,

$$n\gamma_n^k \leqslant \frac{1}{\tau}\left[\gamma_n^k F(\psi^n(\gamma_0)) - \gamma_n^k F(\gamma_n)\right]. \tag{16}$$

Letting $n \longrightarrow$ in (16) and using (14) and (15) produce $\lim\limits_{n\to\infty} n\gamma_n^k = 0$. Now, let us observe that from (15) there exist $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \leqslant 1$ for all $n \geqslant n_1$. Consequently,

$$\gamma_n \leqslant \frac{1}{n^{\frac{1}{k}}}, \text{ for all } n \geqslant n_1. \tag{17}$$

In order to show that $\{x_n\}_n \in \mathbb{N}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geqslant n_1$. By triangle inequality and from (17),

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1} \\ &= \sum_{i=n}^{m-1} \gamma_i < \sum_{i=n}^{\infty} \gamma_i \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

From the completeness of (X, d), there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Also, by the continuity of T, we obtain

$$d(Tx^*, x^*) = \lim_{n\to\infty} d(Tx_n, x_n) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0,$$

which shows that x^* is a fixed point of T.

Case (ii): In the case where $s = 0$, using (2) for $x = x_{n-1}$ and $y = Tx_{n-1}$,

$$\tau + F(d(Tx_{n-1}, T^2x_{n-1})) \leqslant F(\psi(J_T^s(x_{n-1}, Tx_{n-1}))),$$

where

$$\begin{aligned} J_T^s(x_{n-1}, Tx_{n-1}) &= (d(x_{n-1}, Tx_{n-1}))^{\sigma_1}(d(Tx_{n-1}, T^2x_{n-1}))^{\sigma_2} \\ &\leqslant (d(x_{n-1}, x_n))^{\sigma_1}(d(x_n, x_{n+1}))^{\sigma_2}. \end{aligned}$$

This implies

$$\tau + F(d(x_n, x_{n+1})) \leqslant F(\psi(d(x_{n-1}, x_n)))^{\sigma_1}(d(x_n, x_{n+1}))^{\sigma_2}). \tag{18}$$

Suppose that $d(x_n, x_{n+1}) \geqslant d(x_{n-1}, x_n)$, then from (18) we have

$$\begin{aligned} \tau + F(d(x_n, x_{n+1})) &\leqslant F(\psi(d(x_n, x_{n+1})))^{\sigma_1}(d(x_n, x_{n+1}))^{\sigma_2}) \\ &\leqslant F(\psi(d(x_n, x_{n+1})))^{\sigma_1+\sigma_2} \\ &= F(\psi(d(x_n, x_{n+1}))) < F(d(x_n, x_{n+1})), \end{aligned}$$

which is a contradiction. Hence, $d(x_n, x_{n+1}) \leqslant d(x_{n-1}, x_n)$. Thus, it follows that

$$\tau + F(d(x_n, x_{n+1})) \leqslant F(\psi(d(x_{n-1}, x_n))). \tag{19}$$

Also, $d(x_{n-1}, x_n) \leqslant d(x_{n-2}, x_{n-1})$ and revisiting (18) give

$$\tau + F(d(x_{n-1}, x_n)) \leqslant F(\psi(d(x_{n-2}, x_{n-1}))). \tag{20}$$

From (19) and (20), we obtain $F(d(x_n, x_{n+1})) \leqslant F(\psi^2(d(x_{n-2}, x_{n-1}))) - 2\tau$. Continuing in this way, we deduce (inductively) that

$$F(d(x_n, x_{n+1})) \leqslant F(\psi^n(d(x_0, x_1))) - n\tau. \tag{21}$$

Using similar method as in Case (i) for $s > 0$, we obtain that $\{x_n\}$ forms a Cauchy sequence on a complete matric space. Hence, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$, and from the continuity of T, we get that the point x^* is a fixed point of the mapping T. Next, we show that T has a unique fixed point. To see this suppose that T has two distinct fixed points x^* and y^* such that $Tx^* = x^* \neq y^* = Ty^*$. Then, from the contractive inequality in (4),

$$\tau + F(d(Tx^*, Ty^*)) \leqslant F(\psi(J_T^s(x^*, y^*))). \tag{22}$$

Again we consider two cases, for $s > 0$ and $s = 0$.
Case (i): for $s > 0$,

$$\begin{aligned} J_T^s(x^*, y^*) &= \left[\sigma_1\left(\frac{d(x^*, Tx^*)d(y^*, Ty^*)}{d(x^*, y^*)}\right)^s + \sigma_2(d(x^*, y^*))^s\right]^{\frac{1}{s}} \\ &= \left[\sigma_1\left(\frac{d(x^*, x^*)d(y^*, y^*)}{d(x^*, y^*)}\right)^s + \sigma_2(d(x^*, y^*))^s\right]^{\frac{1}{s}} \\ &= \sigma^{\frac{1}{s}} d(x^*, y^*) \\ &\leqslant d(x^*, y^*). \end{aligned}$$

Now substituting the above inequality into (22) leads to

$$\tau + F(d(x^*, y^*)) \leqslant F(\psi(d(x^*, y^*))) < F(d(x^*, y^*)),$$

which is a contradiction, since $\tau > 0$.
Case (ii) for $s = 0$,

$$J_T^s(x^*, y^*) = (d(x^*, Tx^*))^{\sigma_1}(d(y^*, Ty^*))^{\sigma_2} = (d(x^*, x^*))^{\sigma_1}(d(y^*, y^*))^{\sigma_2} = 0.$$

Also, substituting the above inequality into (22) yields

$$\tau + F(d(x^*, y^*)) \leqslant F(\psi(0)) < F(0).$$

Since $d(x^*, y^*) > 0$ and F is strictly increasing by F_1, $F(0) < F(d(x^*, y^*))$. Thus,

$$\tau < F(0) - F(d(x^*, y^*)) < 0.$$

That is, $\tau < 0$, which is also a contradiction since $\tau > 0$. Hence, we conclude that T has a unique fixed point in X. ■

In what follows, we construct an example to support the assumptions of Theorem 3.2.

***Example* 3.3** Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space. Take $\tau > 0$ and consider the mapping $T : X \longrightarrow X$ defined by

$$Tx = \begin{cases} \frac{1}{4}xe^{-\tau}, & \text{if} \quad x \in [0, 1]; \\ \frac{1}{4}e^{-\tau}, & \text{if} \quad x > 1 \end{cases}$$

for all $x \in X$. Then T is continuous. Define the function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $\psi(t) = \frac{1}{4}t$, for all $t > 0$ and $F(\alpha) = \ln(\alpha^2 + \alpha), \alpha > 0$. It is clear that ψ is a (c)-comparison function and $F \in \Omega_F$.

Observe that for all $x, y \in (1, \infty)$, there is nothing to show. For $x, y \in [0, 1]$ such that $Tx \neq Ty$, let $\sigma_1 = 0$ and $\sigma_2 = 1$. Then, to show that the mapping T is a Jaggi-type hybrid F-contraction, we examine the following two cases:
Case 1: For $s > 0$, take $s = 1$. Then

$$d(Tx, Ty) = \frac{1}{4}e^{-\tau}|x - y| = e^{-\tau}\psi(d(x, y)) \leqslant e^{-\tau}\psi(J_T^s(x, y)). \tag{23}$$

From (23), we obtain

$$\begin{aligned} d(Tx, Ty)[d(Tx, Ty) + 1] &\leqslant e^{-\tau}\psi(d(x, y)) \cdot [e^{-\tau}\psi(d(x, y)) + 1] \\ &\leqslant e^{-\tau}[\psi(J_T^s(x, y)) \cdot (\psi(J_T^s(x, y)) + 1)]. \end{aligned}$$

This implies that

$$\begin{aligned}\tau + F(d(Tx,Ty)) &= \tau + \ln(d(Tx,Ty)^2 + d(Tx,Ty)) \\ &\leqslant \tau + \ln[e^{-\tau}(\psi(J_T^s(x,y))^2 + \psi(J_T^s(x,y)))] \\ &= \ln[(\psi(J_T^s(x,y)))^2 + \psi(J_T^s(x,y))] \\ &= F((\psi(J_T^s(x,y))).\end{aligned}$$

Case 2: Similarly, for $s = 0$, take $\sigma_1 = 0$, $\sigma_2 = 1$ and $x = Ty$, for all $x, y \in X$. Hence, we obtain

$$d(Tx,Ty) \leqslant e^{-\tau}\psi(J_T^s(x,y)). \tag{24}$$

As in Case 1, the inequality (24) yields $\tau + F(d(Tx,Ty)) \leqslant F((\psi(J_T^s(x,y)))$. In Figure 1, we illustrate the validity of the contractive inequality (4) using Example 3.3. Figure 1

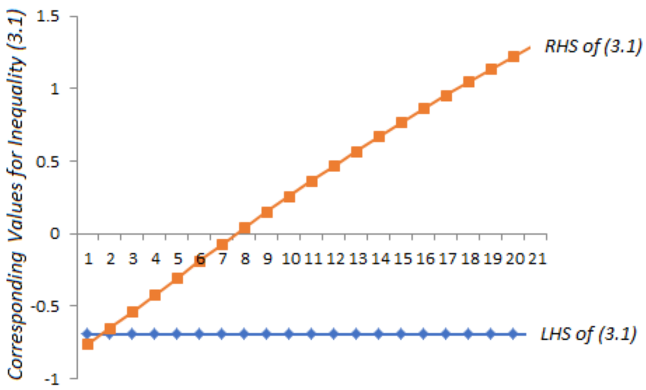

Figure 1. Illustration of the contractive inequality (4) using Example 3.3

illustrates that the right-hand side (RHS) of the contractive inequality (4) predominates the left-hand side (LHS) as defined in Example 3.3. Hence, all the hypotheses of Theorem 3.2 are satisfied. Consequently, we see that $x = 0$ is the unique fixed point of T.

On the other hand, it is easy to check that the main result of Karapınar and Fulga [10] is not applicable to this example. In fact, suppose that the mapping T is a Jaggi-type hybrid contraction; that is, for all $x, y \in X \backslash Fix(T)$,

$$d(Tx,Ty) \leqslant \psi(J_T^s(x,y)), \tag{25}$$

where $J_T^s(x,y)$ is as defined in (4). Then, for the chosen parameters $\sigma_1 = 0, \sigma_2 = 1, s = 1, \psi(t) = \frac{1}{4}t, t > 0$ we take $x = \frac{1}{2}e^{-\tau}$ and $y = \frac{3}{4}e^{-\tau}$, for $\tau > 0$. Clearly, $x, y \in [0,1]$. Then, by direct calculation, we have

$$d(Tx,Ty) = |Tx - Ty| = \left|\frac{1}{4}e^{-2\tau} - \frac{9}{16}e^{-2\tau}\right| = \frac{5}{16}e^{-2\tau}.$$

Also,

$$J_T^s(x,y) = d(x,y) = |x - y| = \left|\frac{1}{2}e^{-\tau} - \frac{3}{4}e^{-\tau}\right| = \frac{1}{4}e^{-\tau}.$$

It follows that $\psi(J_T^s(x,y)) = \frac{1}{4}[\frac{1}{4}e^{-\tau}] = \frac{1}{16}e^{-\tau}$. By (25), $\frac{d(Tx,Ty)}{\psi(J_T^s(x,y))} \leqslant 1$. Thus, we obtain

$$\frac{5e^{-2\tau}}{e^{-\tau}} \leqslant 1. \tag{26}$$

Letting $\tau \to \infty$ in (26), yields $\infty \leqslant 1$, which is a contradiction. Hence, the result in this article is a proper generalization of the main ideas of Karapınar and Fulga [10].

Corollary 3.4 Let (X, d) be a complete metric space and T be a continuous self-mapping on X. If there exist $\tau > 0$ and $F \in \Omega_F$ such that for all $x, y \in X$,
$\tau + F(d(Tx, Ty)) \leqslant F(\eta(A_T^s(x, y)))$, where

$$A_T^s(x,y) = \left[\sigma_1 \left(\frac{d(x,Tx)d(y,Ty)}{d(x,y)}\right)^s + \sigma_2(d(x,y))^s\right]^{\frac{1}{s}}, \tag{27}$$

$\sigma_1, \sigma_2 \geqslant 0$ with $\sum_{i=1}^{2} \sigma_i = 1$, $s > 0$ and $\eta \in (0, 1)$. Then T has a unique fixed point in X.

Proof. By defining the mapping $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as $\psi(t) = \eta t$, for all $t > 0$, $\eta \in (0, 1)$, Theorem 3.2 can be applied to find a unique $u \in X$ such that $u = Tu$. ■

Corollary 3.5 Let (X, d) be a complete metric space and T be a continuous self-mapping on X. If there exist $\tau > 0$ and $F \in \Omega_F$ such that for all $x, y \in X$,

$$\tau + F(d(Tx, Ty)) \leqslant F(\psi(R_T^s(x, y))),$$

where

$$R_T^s(x, y) = (d(x, Tx))^{\sigma_1}(d(y, Ty))^{\sigma_2},$$

$\sigma_1, \sigma_2 \geqslant 0$ with $\sum_{i=1}^{2} \sigma_i = 1$. Then T has a unique fixed point in X.

Proof. Considering the case where $s = 0$ in Theorem 3.2, the proof follows. ■

Corollary 3.6 Let (X, d) be a complete metric space and T be a continuous self-mapping on X. If there exist $\tau > 0$ and $F \in \Omega_F$ such that for all $x, y \in X$,
$\tau + F(d(Tx, Ty)) \leqslant F(\eta(R_T^s(x, y)))$, where

$$R_T^s(x, y) = \sqrt{d(x, Tx) \cdot d(y, Ty)},$$

$\sigma_1, \sigma_2 \geqslant 0$ with $\sum_{i=1}^{2} \sigma_i = 1$ and $\eta \in (0, 1)$. Then T has a unique fixed point in X.

Proof. Taking $\sigma_1 = \sigma_2 = \frac{1}{2}$ in Corollary 3.5, the result is obtained. ■

In line with Corollaries 3.4 and 3.5, more consequences of Theorem 3.2 can be obtained by particularizing the parameters.

4. Conclusion

Over the last few years, the study of fixed point theory has witnessed many notable contributions in the investigation of fixed points of self mappings defined on metric spaces. The study of fixed point results of hybrid F-contractions is a relatively new area of research, and there is still much work to be done. The presented hybrid fixed point theorem in this manuscript and the obtained corollaries showed that many existing fixed point concepts can be followed as special cases. In particular, by using members of Ω_F, various fixed point theorems of F-contractive type mappings can be discussed. In like manner, the established theorem in this work revealed that several linear and non-linear contractions can be obtained by specialising the constants used in the Jaggi-type hybrid F-contraction.

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