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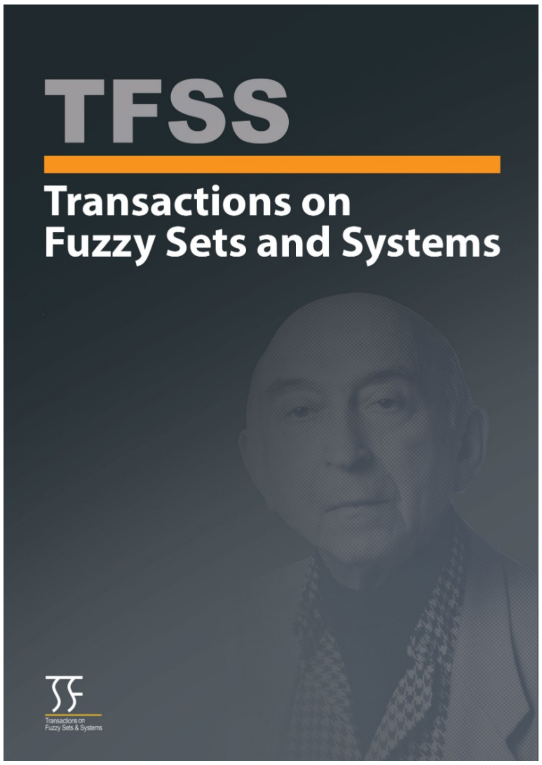

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Stable Topology on Ideals for Residuated Lattices

Ariane Gabriel Tallee Kakeu*, **Luc Emery Diekouam Fotso**, **Blaise Bleriot Koguep Njionou**, **Lele Celestin**, **Daniel Akume**

Abstract. Residuated lattices are the major algebraic counterpart of logics without contraction rule, as they are more generalized logic systems including important classes of algebras such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and De Morgan residuated lattices among others, on which filters and ideals are sets of provable formulas. This paper presents a meaningful exploration of the topological properties of prime ideals of residuated lattices. Our primary objective is to endow the set of prime ideals with the stable topology, a topological framework that proves to be more refined than the well-known Zariski topology. To achieve this, we introduce and investigate the concept of pure ideals in the general framework of residuated lattices. These pure ideals are intimately connected to the notion of annihilator in residuated lattices, representing precisely the pure elements of quantales. In addition, we establish a relation between pure ideals and pure filters within a residuated lattice, even though these concepts are not dual notions. Furthermore, thanks to the concept of pure ideals, we provide a rigorous description of the open sets within the stable topology. We introduce the i-local residuated lattices along with their properties, demonstrating that they coincide with local residuated lattices. The findings presented in this study represent an extension beyond previous work conducted in the framework of lattices, and classes of residuated lattices.

AMS Subject Classification 2020: MSC 03G10; MSC 06B10; MSC 06F30; MSC 06B75
Keywords and Phrases: Residuated lattice, Prime ideal, Pure ideal, Zariski topology, Stable topology.

1 Introduction

It is well known that non-classical logic is a formal and useful technique for computer science to deal with fuzzy and uncertain information in classification problems, artificial intelligence, data organization, and formal concept analysis. In this way, several algebraic structures such as MV-algebras, BL-algebras, Gödel algebras, MTL-algebras, De Morgan residuated lattices, and residuated lattices (see [1, 14, 17, 29]) have been introduced, and provide an algebraic framework to fuzzy logic and fuzzy reasoning. Among these structures, Pavelka showed in [22] that residuated lattices are more generalized logic systems on which filters and ideals are sets of provable formulas. The study of their algebraic properties is therefore deciphered through the notions of ideal and filter.

In the framework of residuated lattices, previous works, such as [4, 10, 17, 24, 32], were more focused on filters. In [4], Busneag et al. endowed the set of prime filters with the spectral topology and used the concepts of co-annihilator as well as pure filter to study the stable topology. On the other hand, the notion of ideal was recently introduced in residuated lattices by Busneag et al. in [6], generalizing the one in

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BL-algebras. A year later, Luo ([20]) pursued this by bringing in another definition for ideals of residuated lattices, with which he introduced a congruence relation associated with ideals. That congruence relation was later on revised by Liu et al. ([19]), who set forth the concept of fuzzy ideals. In [14], Holdon established the equivalence between Luo's and Busneag's definition of an ideal of a residuated lattice, providing additional properties on ideals of residuated lattices. Motivated by the fact that the Zariski topology allows tools from topology to be used to interpret algebraic varieties, Dana Piciu ([23]) introduced the Zariski topology on ideals of residuated lattices. One can observe that in that topology, clopen sets are *stable*, that is, they are simultaneously stable under ascent and descent. The question that arises: does it exist stable sets other than clopen sets? If so, how can we describe them?

To answer this question, we introduce and study pure ideals in residuated lattices, based on the notion of annihilator in residuated lattices which generalize the one done in De Morgan residuated lattices [21] and MV-algebras ([2, 8]).

The paper is organized as follows: in Section 2, we recall basic notions of residuated lattices and describe some properties that will be needed in the sequel. In Section 3, we introduce the concept of pure ideal in residuated lattices and provide some of its properties. Moreover, we discuss the relationship between pure ideals and pure filters of a residuated lattice. Section 4 is devoted to the characterization of open stable sets by the means of pure ideals of a residuated lattice, setting up the stable topology.

2 Preliminaries

A *residuated lattice* ([4, 19, 20, 29]) is an algebraic structure $(L; \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$, where:

(L1) $(L; \vee, \wedge, 0, 1)$ is a bounded lattice;

(L2) $(L; \odot, 1)$ is a commutative ordered monoid;

(L3) For every $x, y, z \in L, x \leq y \rightarrow z$ iff $x \odot y \leq z$.

In what follows, unless otherwise specified, by $\mathcal{L}$ we denote a residuated lattice $(L;\ \vee,\ \wedge,\ \odot,\ \rightarrow,\ 0,\ 1)$. A subset X of L is *proper* if $X \neq L$. Every residuated lattice $\mathcal{L}$ has the *negation* operation defined by $x' := x \rightarrow 0$, for all $x \in L$.

We will use the notations

$$x^n := \underbrace{x \odot \cdots \odot x}_{n \ \ times}, \text{ for any } x \in L \text{ and } n \geq 1;$$

$$X' := \{x' : x \in X\}, \text{ for any } X \subseteq L.$$

Recall from [14, 24] that a residuated lattice $\mathcal{L}$ is called:

(i) a *De Morgan* residuated lattice if the De Morgan law $(x \wedge y)' = x' \vee y'$, for all $x, y \in L$ holds;

(ii) an *MTL-algebra* if it satisfies $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y \in L$ *(prelinearity)*;

(iii) a *BL-algebra* if it is an MTL-algebra where $x \wedge y = x \odot (x \rightarrow y)$, for all $x, y \in L$ (*divisibility*);

(iv) an *MV-algebra* if it is a BL-algebra that verifies $x'' = x$, for all $x \in L$ (*double negation*).

When $x'' = x$ for all x in L, we say that $\mathcal{L}$ is *regular*.

The following rules of calculus in residuated lattices shall be needed in the sequel.

Proposition 2.1. *[4, 6, 14, 24, 29] Let L be a residuated lattice. Then, for all $x, y, z \in L$, we have:*

(P1) $x \leq y$ *iff* $x \to y = 1,\ x \odot y \leq x \wedge y,\ x \odot y \leq x \to y$;

(P2) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(P3) $x \to y \leq (x \odot y')',\ \ x \odot (y \to z) \leq y \to (x \odot z) \leq (x \odot y) \to (x \odot z)$;

(P4) *If* $x \leq y$, *then* $y \to z \leq x \to z\ ,\ z \to x \leq z \to y,\ x \odot z \leq y \odot z,\ y' \leq x'$;

(P5) $x \odot (x \to y) \leq y,\ \ x \leq (x \to y) \to y$ *and* $((x \to y) \to y) \to y = x \to y$;

(P6) $1 \to x = x,\ \ x \to x = 1,\ \ x \to 1 = 1,\ \ x \leq y \to x,$
$x \to y \leq y' \to x',\ \ x \leq x'',\ \ x''' = x'$;

(P7) $x \odot x' = 0,\ x \odot y = 0$ *iff* $x \leq y'$;

(P8) $(x \odot y)' = x \to y' = y \to x' = x'' \to y',\ \ (x \wedge y)' \geq x' \vee y',$
$(x \vee y)' = x' \wedge y',\ \ 0' = 1$ *and* $1' = 0$;

(P9) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z), x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z),\ x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$ *and hence* $(x \vee y)^{mn} \leq x^m \vee y^n$, *for every* $n, m \geq 1$;

(P10) $x \to (y \wedge z) = (x \to y) \wedge (x \to z),\ (x \vee y) \to z = (x \to z) \wedge (y \to z),$
$(x \wedge y) \to z \geq (x \to z) \vee (y \to z),\ x \to (y \vee z) \geq (x \to y) \vee (x \to z)$;

(P11) $x' \odot y' \leq (x \odot y)',\ x'' \odot y'' \leq (x \odot y)'',\ x' \odot y' \leq (x' \to y)'$ *and* $x, y \leq (x' \odot y')'$;

(P12) $x \vee y = 1$ *implies* $x \odot y = x \wedge y$ *and* $x^n \vee y^n = 1$, *for every* $n \geq 1$.

The operation $\oplus$ defined on L by $x \oplus y = (x' \odot y')' = x' \to y''$, for all $x, y \in L$ is commutative, associative, and compatible with the order [6].

For any $x \in L$, $nx := \underbrace{x \oplus \cdots \oplus x}_{\text{n times}}$, $n \geq 1$.

Recall from [5, 23] that

(P13) For every $m, n \geq 2$,

$$[(x')^n]' = nx, x \wedge (ny) \leq n(x'' \wedge y'') \text{ and } (mx) \wedge (ny) = (mn)(x'' \wedge y'').$$

The operation $\oslash$ defined for every $x, y \in L$ by $x \oslash y := x' \to y$ is neither associative nor commutative and is called the pseudo-addition (see [20]). We shall have in mind that it is compatible with the order.

Remark 2.2. We easily see that the operation $\oslash$ verifies $x \oslash (y \wedge z) = (x \oslash y) \wedge (x \oslash z)$ and $x \oslash (y \oslash z) = y \oslash (x \oslash z)$, for every $x, y, z \in L$.

We recall that a nonempty subset F of $\mathcal{L}$ is called a *filter* ([4]) if it verifies:
(F1) For every $x, y \in L$, if $x \leq y$ and $x \in F$, then $y \in F$;
(F2) For every $x, y \in F$, $x \odot y \in F$.
A filter F of $\mathcal{L}$ is *proper* if $F \neq L$ (i.e., $0 \notin F$).

A *deductive system* of a residuated lattice $\mathcal{L}$ is a nonempty subset F of $\mathcal{L}$ containing 1 such that for all $x, y \in L$, $x \to y \in F$ and $x \in F$ imply $y \in F$.

It is known that in a residuated lattice, filters and deductive systems coincide.

A filter M of $\mathcal{L}$ is called a *maximal filter* if it is a maximal element of the set of all proper filters of $\mathcal{L}$. A residuated lattice $\mathcal{L}$ is called *local* if it has a unique maximal filter ([4]).
From [20], a nonempty subset I of a residuated lattice $\mathcal{L}$ is said to be an *ideal* of $\mathcal{L}$ if the following properties hold:

(I1) For every $x, y \in I$, $x \oslash y \in I$;

(I2) For every $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$.

An ideal I of $\mathcal{L}$ is *proper* if $I \neq L$ (i.e., $1 \notin I$). I is a *maximal ideal* of $\mathcal{L}$ if it is not contained in any other proper ideal of $\mathcal{L}$ ([23]). A residuated lattice is called *i-local* if it has a unique maximal ideal.
Holdon proved the following proposition.

Proposition 2.3. *[14] A nonempty subset I of a residuated lattice $\mathcal{L}$ is an ideal of $\mathcal{L}$ if and only if:*

(I'1) For every $x, y \in I$, $x \oplus y \in I$;

(I2) For every $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$.

We denote by $\mathcal{I}(\mathcal{L})$ the set of ideals of $\mathcal{L}$. We shall notice that $\{0\}$ and L are trivial ideals of $\mathcal{L}$, and each ideal of $\mathcal{L}$ contains 0.

Proposition 2.4. *[14] Let L be a residuated lattice, and I an ideal of $\mathcal{L}$. Then, $x \in I$ iff $x'' \in I$, for every $x \in L$.*

We recall that an algebraic structure $(L; \wedge, \vee, \odot, 0, 1)$ is a *quantale* if $(L; \wedge, \vee, 0, 1)$ is a complete lattice and $(L; \odot)$ a semigroup such that the operator $\odot$ verifies the infinite distributive laws: $a \odot \bigvee X = \bigvee\{a \odot x : x \in X\}$, for all $a \in L$ and $X \subseteq L$ ([11, 26]). An element $a \in L$ is said to be *compact* if for every $X \subseteq L$ such that $a \leq \bigvee X$, there is a finite subset $X_1 \subseteq X$ such that $a \leq \bigvee X_1$.

Recall also from [3] that a *Heyting algebra* is a lattice $(L; \wedge, \vee)$ with 0 in which for every $x, y \in L$, there is an element $x \to y := \bigvee\{a : a \wedge x \leq y\} \in L$, called the *pseudocomplement* of x with respect to y. We say that L is *pseudocomplemented* if every element of L is pseudocomplemented with respect to 0. For any $x \in L$, we will denote by x^* the pseudocomplement of x with respect to 0. Note that a complete Heyting algebra is a quantale in which the operators $\odot$ and $\wedge$ coincide [16]. It is also known that a residuated lattice $\mathcal{L}$ is a Heyting algebra iff $x \odot y = x \wedge y$, for every $x, y \in L$ (see [14]).

An element x of a quantale L is called *pure* if for every compact element a of L, $a \leq x$ implies $x \vee a^* = 1$ (see [11, 13]).

Given a nonempty subset X of L, the least ideal of $\mathcal{L}$ containing X (called the *ideal generated* by X) will be denoted $\langle X \rangle$, and for all $x \in L$, $\langle\{x\}\rangle$ will be denoted $\langle x \rangle$.

Proposition 2.5. *[23] Let $\mathcal{L}$ be a residuated lattice and $x \in L$. Then,*

(i) $\langle X \rangle := \{a \in L : a \leq x_1 \oplus ... \oplus x_n$, for some $n \geq 1$ and $x_1, x_2, ..., x_n \in X\}$. Particularly, $\langle x \rangle = \{a \in L : a \leq nx$, for some $n \geq 1\}$.

(ii) For any $I \in \mathcal{I}(L)$, if $x \notin I$, then
$\langle I \cup \{x\} \rangle = \{a \in L : a \leq i \oplus nx$, for some $i \in I$ and $n \geq 1\}$.

(iii) $(\mathcal{I}(L), \wedge, \vee, \to)$ is a complete Heyting algebra
where $I \wedge J := I \cap J$,
$I \vee J = \langle I \cup J \rangle := \{x \in L : x \leq i \oplus j, i \in I, j \in J\}$ and
$I \to J := \{x \in L : \langle x \rangle \cap I \subseteq J\}$, for $I, J \in \mathcal{I}(L)$.

Definition 2.6. [19, 23] Let P be a proper ideal of a residuated lattice $\mathcal{L}$. Then,

(i) P is called a *prime ideal* of $\mathcal{L}$ if P is a prime element of $(\mathcal{I}(\mathcal{L}), \wedge, \vee, \to)$, that is, if I, J are ideals of $\mathcal{L}$ and $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

(ii) P is a *prime ideal of the second kind* of $\mathcal{L}$ if for every $x, y \in A$, $x \wedge y \in P$ implies $x \in I$ or $y \in P$.

(iii) We say that P is a *prime ideal of third kind* of $\mathcal{L}$ if for all $x, y \in L$, $(x \to y)' \in P$ or $(y \to x)' \in P$.

(iv) A prime ideal P which is minimal in the poset of prime ideals containing an ideal I is called a *minimal prime ideal belonging to I*. A minimal prime ideal belonging to $\{0\}$ is called *minimal prime ideal*. In other words, P is a *minimal prime ideal* if P is prime, and for every prime ideal Q, if $Q \subseteq P$, then $P = Q$.

We denote by $Max_{Id}(\mathcal{L})$, $Spec_{Id}(\mathcal{L})$, and by $Min_{Id}(\mathcal{L})$ the set of maximal ideals of $\mathcal{L}$, the set of all prime ideals of $\mathcal{L}$, and the set of minimal prime ideals of $\mathcal{L}$, respectively. Note that $Max_{Id}(\mathcal{L}) \subseteq Spec_{Id}(\mathcal{L})$, and $Min_{Id}(\mathcal{L}) \subseteq Spec_{Id}(\mathcal{L})$ (see [23]).

Proposition 2.7. *[14, 23] Let M be a proper ideal of $\mathcal{L}$. Then, the following are equivalent:*

(i) $M \in Max_{Id}(\mathcal{L})$

(ii) For any $x \in L$, $x \notin M$ iff $(nx)' \in M$, for some natural number $n \geq 1$

(iii) For all $x \notin M$ there is $y \in M$ and $n \geq 1$ such that $y \oplus (nx) = 1$

It follows from Zorn's lemma that every proper ideal of a residuated lattice is contained in a maximal ideal.

The next proposition characterizes prime ideals of residuated lattices.

Proposition 2.8. *[23] Let P be a proper ideal of $\mathcal{L}$. Then, the following are equivalent:*

(i) P is prime.

(ii) $x'' \wedge y'' \in P$ implies $x \in P$ or $y \in P$, for all $x, y \in A$

(iii) If $I, J \in \mathcal{I}(\mathcal{L})$ and $I \cap J = P$, then $I = P$ or $J = P$.

Obviously, every prime ideal of third kind of $\mathcal{L}$ is a prime ideal of second kind of $\mathcal{L}$. The converse, however, is not true, see [19].

Moreover, every prime ideal of second kind of $\mathcal{L}$ is also a prime ideal of $\mathcal{L}$. But the converse is not always guaranteed. Nevertheless, these three types of prime ideals coincide in a De Morgan residuated lattice (see [23]).

Consequently, the results presented in this study, using prime ideals of residuated lattices, constitute an extension of what was done with prime ideals of the second kind in [14, 15].

Before stating the prime ideal theorem, recall that a nonempty subset F of L is a *lattice filter* (or *ℓ-filter*) of $\mathcal{L}$ if:

(i) $\forall x, y \in F, x \wedge y \in F$;

(ii) $\forall x \in F, \forall y \in L, x \leq y \Rightarrow y \in F$.

If F is a filter of $\mathcal{L}$, then F is also a lattice filter of $\mathcal{L}$. But the converse is not always true ([4]).

Theorem 2.9. *(Prime ideal theorem) [5] Let $\mathcal{L}$ be a residuated lattice. If I is an ideal and F is a lattice filter of $\mathcal{L}$ such that $I \cap F = \emptyset$, then there exists a prime ideal P of $\mathcal{L}$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

As a direct consequence of the prime ideal theorem, for any proper ideal I of $\mathcal{L}$, we have $I = \cap\{P \in spec_{Id}(\mathcal{L}) : I \subseteq P\}$.

Proposition 2.10. *[23] For every ideal I of $\mathcal{L}$ and $x \in A \setminus I$, there is a minimal prime ideal P such that $I \subseteq P$ and $x \notin P$. Singularly, for every $x \in L$, there exists a minimal prime ideal P such that $x \notin P$, whenever $x \neq 0$.*

For any nonempty subset X of L, the ideal

$$X^{\perp} := \{a \in L : x'' \wedge a'' = 0, \text{ for all } x \in X\}$$

is called the *annihilator* of X in $\mathcal{L}$ (see [27, 28]). For all $x \in L$, $\{x\}^{\perp}$ will simply be denoted $x^{\perp}$.

Recall that for any ideal I of $\mathcal{L}$, $I^{\perp}$ is the pseudocomplement of I in $(\mathcal{I}(\mathcal{L}); \wedge, \vee, \rightarrow)$, that is $I^{\perp} = I^*$. Recall also that the set $\{a \in L : x \wedge a = 0, \text{ for all } x \in X\}$ is not always an ideal of $\mathcal{L}$ as shown in [27]. However, the above definition of annihilator in residuated lattices has the benefit of generalizing the existing one in subclasses of residuated lattices such as De Morgan residuated lattices, MTL-algebras, BL-algebras, MV-algebras.

Below are some properties of annihilators in residuated lattices.

Lemma 2.11. *[27] Let $x, a, b \in L$. Then, we have:*

(i) $1^{\perp} = \{0\}$, $0^{\perp} = L$;

(ii) If $a \leq b$, then $b^{\perp} \subseteq a^{\perp}$;

(iii) $a^{\perp} \cap b^{\perp} = (a \vee b)^{\perp}$;

(iv) $a^{\perp} \cup b^{\perp} \subseteq (a \wedge b)^{\perp}$;

(v) If $x \in a^{\perp}$, then $a \leq x'$ and $x \leq a'$.

In order to make the paper self-contained, we recall the following result.

Theorem 2.12. *[27] Let X, Y be nonempty subsets of L. Then,*

(i) $X \subseteq Y$ implies $Y^{\perp} \subseteq X^{\perp}$;

(ii) $X^{\perp}$ is an ideal. In addition if $X \neq \{0\}$, then $X^{\perp}$ is a proper ideal;

(ii) $L^{\perp} = \{0\}$;

(iii) $X \subseteq X^{\perp\perp}$;

(iv) $X^{\perp} = X^{\perp\perp\perp}$;

(v) $X \cap X^{\perp} \subseteq \{0\}$;

(vi) $X^{\perp} \cup Y^{\perp} \subseteq (X \cap Y)^{\perp}$;

(vii) $(X \cup Y)^{\perp} = X^{\perp} \cap Y^{\perp}$;

(viii) $\langle X \rangle^{\perp} = X^{\perp}$. Particularly, $\emptyset^{\perp} = L$;

(ix) If $X \subseteq L$, then $\langle X \rangle \cap X^{\perp} = \{0\}$;

(x) $X^{\perp} = \bigcap_{x \in X} x^{\perp}$.

Let $Ann(\mathcal{L}) = \{X^{\perp}, X \subseteq L\}$ be the set of annihilators of $\mathcal{L}$. Since $X^{\perp} = \langle X \rangle^{\perp}$, we have $Ann(\mathcal{L}) = \{I^{\perp}, I \in \mathcal{I}(\mathcal{L})\}$. Then, $(Ann(\mathcal{L}), \wedge, \vee_{Ann(\mathcal{L})}, ^{\perp}, \{0\}, L)$ is a complete boolean algebra where $I \wedge J := I \cap J$ and $I \vee_{Ann(\mathcal{L})} J := (I \cup J)^{\perp\perp}$, for all $I, J \in Ann(\mathcal{L})$ (see [28]).

3 Pure Ideals of Residuated Lattices

The notion of pure ideal has been studied in rings by De Marco ([9]), in distributive lattices by Georgescu and Voiculescu ([12]), as well as in MV-algebras by Cavaccini et al. ([8]), and in De Morgan residuated lattices by Holdon ([15]) and Mihaela ([21]). In this section, we introduce the notion of pure ideal in residuated lattices using the concept of annihilator and explore some of its properties.

For any ideal I of a residuated lattice $\mathcal{L} = (L; \vee, \wedge, \odot, \rightarrow, 0, 1)$, we define

$$\sigma(I) := \{x \in L : \text{ there are } a \in I \text{ and } b \in x^{\perp} \text{ such that } a \oplus b = 1\}.$$

Proposition 3.1. *Let I be an ideal of $\mathcal{L}$, $\sigma(I)$ is an ideal of $\mathcal{L}$ and $\sigma(I) \subseteq I$.*

Proof. Since $0 \in I, 1 \in 0^{\perp}$ and $0 \oplus 1 = 1$, we obtain $0 \in \sigma(I)$. Thus $\sigma(I) \neq \emptyset$.
Let $x_1, x_2 \in L$, such that $x_1 \leq x_2$ and $x_2 \in \sigma(I)$. Then, there are $a_2 \in I$, $b_2 \in x_2^{\perp}$ such that $a_2 \oplus b_2 = 1$. From Lemma 2.11 (ii), $x_1 \leq x_2$ implies that $x_2^{\perp} \subseteq x_1^{\perp}$. Then, $b_2 \in x_1^{\perp}$ and $x_1 \in \sigma(I)$.

In addition, if $x_1, x_2 \in \sigma(I)$, then, there are $a_1, a_2 \in I$, $b_1 \in x_1^{\perp}, b_2 \in x_2^{\perp}$ such that $a_1 \oplus b_1 = 1 = a_2 \oplus b_2$.

Consider $a = a_1 \oplus a_2$ and $b = b_1 \wedge b_2$. Then, $a \in I$ from Proposition 2.3 (I'1).
Let us show that $b \in (x_1 \oslash x_2)^{\perp}$ and $a \oplus b = 1$.

$$\begin{aligned}
(x_1 \oslash x_2)'' \wedge b'' &= [(x_1' \rightarrow x_2)' \vee b']', && \text{from } (P8)\\
&= [(x_1' \rightarrow x_2)' \vee (b_1 \wedge b_2)']', \\
&\leq [(x_1' \rightarrow x_2)' \vee (b_1' \vee b_2')]', && \text{from } (P4) \text{ and } (P8)\\
&\leq [(x_1' \odot x_2') \vee (b_1' \vee b_2')]', && \text{from } (P4) \text{ and } (P11)\\
&\leq [((b_1' \vee b_2') \vee x_1') \odot ((b_1' \vee b_2') \vee x_2')]', && \text{from } (P4) \text{ and } (P9)\\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow ((b_1' \vee b_2') \vee x_2')', && \text{from } (P8)\\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow ((b_1' \vee b_2')' \wedge x_2''), && \text{from } (P8)\\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow ((b_1'' \wedge b_2'') \wedge x_2''), && \text{from } (P8)\\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow ((x_2'' \wedge b_2'') \wedge b_1''), \\
&= ((b_1' \vee b_2') \vee x_1') \rightarrow 0, && \text{since } b_2 \in x_2^{\perp}\\
&= ((b_1' \vee b_2') \vee x_1')', \\
&= (b_1' \vee b_2')' \wedge x_1'', && \text{from } (P8)\\
&= (b_1'' \wedge b_2'') \wedge x_1'', && \text{from } (P8)\\
&= (b_1'' \wedge x_1'') \wedge b_2'', \\
&= 0.
\end{aligned}$$

Therefore, $(x_1 \oslash x_2)'' \wedge b'' = 0$, which means that $b \in (x_1 \oslash x_2)^{\perp}$.

Also,

$$
\begin{aligned}
a \oplus b &= (a_1 \oplus a_2) \oplus (b_1 \wedge b_2), \\
&= a_1 \oplus (a_2 \oplus (b_1 \wedge b_2)), && \text{since } \oplus \text{ is associative} \\
&= a_1 \oplus (a_2' \odot (b_1 \wedge b_2)')', \\
&= a_1 \oplus (a_2' \to (b_1 \wedge b_2)''), && \text{from } (P8) \\
&\geq a_1 \oplus (a_2' \to (b_1 \odot b_2)''), && \text{from } (P1) \text{ and } (P4) \\
&\geq a_1 \oplus (a_2' \to (b_1'' \odot b_2'')), && \text{from } (P11) \text{ and } (P4) \\
&\geq a_1 \oplus (b_1'' \odot (a_2' \to b_2'')), && \text{from } (P3) \\
&= a_1 \oplus (b_1'' \odot (a_2 \oplus b_2)), \\
&= a_1 \oplus (b_1'' \odot 1), && \text{from hypothesis} \\
&= a_1 \oplus b_1''
\end{aligned}
$$

and we obtain $a \oplus b \geq a_1 \oplus b_1 = 1$ because $\oplus$ is compatible with the lattice order and from the hypothesis. Thus, $a \oplus b = 1$, and it follows that $(x_1 \oslash x_2) \in \sigma(I)$. Hence, $\sigma(I)$ is an ideal of $\mathcal{L}$.

Now, let us show that $\sigma(I) \subseteq I$.
Let $x \in \sigma(I)$. Then, there are $a \in I$, $b \in x^{\perp}$ (i.e., $x'' \wedge b'' = 0$) such that $a \oplus b = 1$. We have: $x'' = x'' \wedge 1 = x'' \wedge (a \oplus b) = x'' \wedge (a' \odot b')' \overset{(P8)}{=} [x' \vee (a' \odot b')]' \overset{(P4),\,(P9)}{\leq} [(x' \vee a') \odot (x' \vee b')]' \overset{(P8)}{=} (x' \vee a') \to (x' \vee b')' \overset{(P8)}{=} (x' \vee a') \to (x'' \wedge b'') = (x' \vee a') \to 0 = (x' \vee a')' \overset{(P8)}{=} (x'' \wedge a'')$, i.e., $x'' \leq x'' \wedge a''$. Thus, $x'' = x'' \wedge a''$, which implies that $x'' \leq a''$. Since $a'' \in I$, then $x'' \in I$. Therefore, $x \in I$. $\square$

The following lemma highlights some properties of the ideal $\sigma(I)$.

Lemma 3.2. *Let L be a residuated lattice and $I,\ J$ two ideals of L. Then,*

(i) $I \subseteq J$ *implies* $\sigma(I) \subseteq \sigma(J)$;

(ii) $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$;

(iii) $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$;

(iv) $\sigma(\sigma(I)) = \sigma(I)$.

Proof.

(i) Straightforward.

(ii) From (i) we obtain $\sigma(I \cap J) \subseteq \sigma(I) \cap \sigma(J)$.
On the other hand, let $x \in \sigma(I) \cap \sigma(J)$. Then, there are $a_1 \in I$, $a_2 \in J$, and $b_1, b_2 \in x^{\perp}$ such that $a_1 \oplus b_1 = 1 = a_2 \oplus b_2$. Set $a = a_1 \wedge a_2$, and $b = b_1 \oplus b_2$. Since $x^{\perp}$, I and J are ideals, it follows that

$b \in x^{\perp}$, $a \in I \cap J$. Moreover, we have:

$$
\begin{aligned}
a \oplus b &= b \oplus a, \text{ since } \oplus \text{ is commutative} \\
&= (b_1 \oplus b_2) \oplus (a_1 \wedge a_2), \\
&= b_1 \oplus (b_2 \oplus (a_1 \wedge a_2)), \text{ since } \oplus \text{ is associative} \\
&= b_1 \oplus (b_2' \odot (a_1 \wedge a_2)')', \\
&= b_1 \oplus (b_2' \rightarrow (a_1 \wedge a_2)''), \text{ from } (P8) \\
&\geq b_1 \oplus (b_2' \rightarrow (a_1 \odot a_2)''), \text{ from } (P1) \text{ and } (P4) \\
&\geq b_1 \oplus (b_2' \rightarrow (a_1'' \odot a_2'')), \text{ from } P(11) \text{ and } (P4) \\
&\geq b_1 \oplus (a_1'' \odot (b_2' \rightarrow a_2'')), \text{ from } (P3) \\
&= b_1 \oplus (a_1'' \odot (b_2 \oplus a_2)), \text{ from the definition of } \oplus \\
&= b_1 \oplus (a_1'' \odot 1), \text{ since } b_2 \oplus a_2 = 1 \\
&= b_1 \oplus a_1'', \\
&\geq b_1 \oplus a_1, \text{ since } \oplus \text{ is compatible with the lattice order} \\
&= 1, \text{ from hypothesis}
\end{aligned}
$$

Then, $a \oplus b = 1$. Therefore, there are $b \in x^{\perp}$, $a \in I \cap J$ such that $a \oplus b = 1$. It follows that $x \in \sigma(I \cap J)$. Hence, $\sigma(I) \cap \sigma(J) \subseteq \sigma(I \cap J)$.

(iii) Straightforward from (i) and the fact that $\sigma(I) \vee \sigma(J) = \langle \sigma(I) \cup \sigma(J) \rangle$.

(iv) It follows from Proposition 3.1 that $\sigma(\sigma(I)) \subseteq \sigma(I)$.
Conversely, let $x \in \sigma(I)$. Then, there are $a \in I$ and $b \in x^{\perp}$ such that $a \oplus b = 1$. We have
$x \oplus b = b' \rightarrow x'' \overset{(P4)}{\geq} b' \rightarrow (x \odot a)'' \overset{(P4),\,(P11)}{\geq} b' \rightarrow (x'' \odot a'') \overset{(P3)}{\geq} x'' \odot (b' \rightarrow a'') = x'' \odot (b \oplus a) = x'' \odot 1 = 1$.
Therefore, $x \oplus b = 1$ with $x \in \sigma(I)$ and $b \in x^{\perp}$. Hence, $x \in \sigma(\sigma(I))$.

□

In light of Proposition 3.1, we define the concept of pure ideal in residuated lattices.

Definition 3.3. Let $\mathcal{L}$ be a residuated lattice. Then, I is a *pure ideal* of $\mathcal{L}$ if I is an ideal of $\mathcal{L}$ such that $\sigma(I) = I$.

Remark 3.4. We observe that $\{0\}$ and L are trivial pure ideals of $\mathcal{L}$.

We denote by $\mathcal{I}_{\sigma}(\mathcal{L})$ the set of pure ideals of $\mathcal{L}$. As an illustration of Definition 3.3, we have this example.

Example 3.5. Consider the residuated lattice $\mathcal{L}_1 = (L; \vee, \wedge, \odot, \rightarrow, 0, 1)$ where the underlying poset is depicted in Figure 1, and the operations $\rightarrow$ and $\odot$ are given in Table 1 ([18])

The only proper ideals of $\mathcal{L}_1$ are $I = \{0, d\}$ and $J = \{0, a, b, c\}$. We have: $0^{\perp} = L$, $a^{\perp} = b^{\perp} = c^{\perp} = \{0, d\}$, $d^{\perp} = \{0, a, b, c\}$, $e^{\perp} = \{0\}$, $f^{\perp} = \{0\}$, $1^{\perp} = \{0\}$. Thus, $\sigma(I) = \{0, d\} = I$, and $\sigma(J) = \{0, a, b, c\} = J$. Hence I and J are pure ideals of $\mathcal{L}_1$.

The next example shows that not all ideals of residuated lattices are pure ideals.

Example 3.6. Let $\mathcal{L}_2 = (L; \vee, \wedge, \odot, \rightarrow, 0, 1)$ be the residuated lattice ([30]) whose associated Hasse diagram is depicted in Figure 2, and the operations $\rightarrow$ and $\odot$ given in Table 2.

For $I = \{0, d, e, f\}$, we have $\sigma(I) = \{0\} \neq I$, meaning that I is not a pure ideal of $\mathcal{L}_2$.

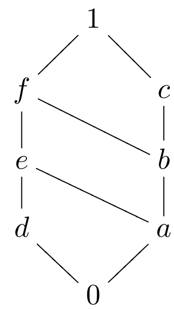

Figure 1: Hasse diagram of $\mathcal{L}_1$

Table 1: Operation tables of $\rightarrow$ and $\odot$ for $\mathcal{L}_1$ in Example 3.5

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

The set of pure ideals is closed under the infimum and supremum as shown below.

Proposition 3.7. *If I and J are pure ideals of $\mathcal{L}$, then $I \cap J$ and $I \vee J$ are pure ideals of $\mathcal{L}$.*

Proof. Let I and J be pure ideals of $\mathcal{L}$. Then $\sigma(I) = I$ and $\sigma(J) = J$.
According to Proposition 3.1, we have $\sigma(I \cap J) \subseteq I \cap J$.
Now $I \subseteq \sigma(I)$ and $J \subseteq \sigma(J)$ imply that $I \cap J \subseteq \sigma(I) \cap \sigma(J)$. Moreover, $\sigma(I) \cap \sigma(J) = \sigma(I \cap J)$ from Lemma 3.2. It follows that $\sigma(I \cap J) = I \cap J$. Hence, $I \cap J$ is a pure ideal of $\mathcal{L}$.

In addition, applying Proposition 3.1 and Lemma 3.2, we have $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J) \subseteq I \vee J$. This implies that $I \vee J \subseteq \sigma(I \vee J) \subseteq I \vee J$. Therefore, $\sigma(I \vee J) = I \vee J$.
□

The next result is a characterization of pure ideals of residuated lattices.

Proposition 3.8. *An ideal I of $\mathcal{L}$ is pure if and only if $\langle I \cup x^{\perp} \rangle = L$, for all $x \in I$.*

Proof. Assume I is pure, that is $I = \sigma(I)$. Let $x \in I = \sigma(I)$. Then, there are $a \in I$ and $b \in x^{\perp}$ such that $a \oplus b = 1$. This means that $1 = a \oplus b \in \langle I \cup x^{\perp} \rangle$, and it follows that $\langle I \cup x^{\perp} \rangle = L$. Hence, for all $x \in I$, $\langle I \cup x^{\perp} \rangle = L$.
Conversely, suppose $\langle I \cup x^{\perp} \rangle = L$, for every $x \in I$. It suffices to show that $I \subseteq \sigma(I)$.
Let $x \in I$. From $\langle I \cup x^{\perp} \rangle = L$, we have $1 \in \langle I \cup x^{\perp} \rangle$; then, by Proposition 2.5, there are $i \in I$ and $j \in x^{\perp}$ such that $1 \leq i \oplus j$. This means that there are $i \in I$ and $j \in x^{\perp}$ such that $i \oplus j = 1$; i.e., $x \in \sigma(I)$. Then, $I \subseteq \sigma(I)$. □

The characterization obtained in 3.8 clearly shows that a pure ideal I of $\mathcal{L}$ is exactly a pure element of the quantale $\mathcal{I}(\mathcal{L})$.

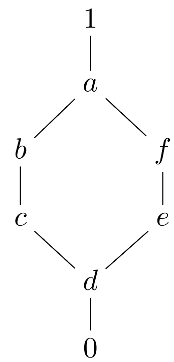

Figure 2: Hasse diagram of $\mathcal{L}_2$

Table 2: Operation tables of $\rightarrow$ and $\odot$ for $\mathcal{L}_2$ in Example 3.6

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

Recall from [7] that a mapping $g : L \longrightarrow L$ on a bounded lattice that associates to any element a from L its image $g(a) \in L$ is an *interior operator* of L if it verifies the following properties for all $a, b \in L$:

(i) $a \leq b$ implies $g(a) \leq g(b)$;

(ii) $g(a) \leq a$;

(iii) $g^2(a) = g(a)$;

(iv) $g(1) = 1$.

The set $\mathcal{O} := \{a \in L : g(a) = a\}$ is the set of fixed elements of L by g.

As a direct consequence of Proposition 3.1, Lemma 3.2 (i), (iv), and Remark 3.4, we have the following proposition.

Proposition 3.9. *Let $\mathcal{L}$ be a residuated lattice. Then, the operator σ is an interior operator on $(\mathcal{I}(\mathcal{L}), \subseteq)$.*

We easily observe that $\mathcal{I}_\sigma(\mathcal{L})$ is the set of fixed elements of $\mathcal{I}(\mathcal{L})$ by σ.

Now, since the notions of ideal and filter in (non-regular) residuated lattices are not perfectly dual, we analyze the relation between pure ideals and pure filters studied in [4, 25]. We first of all recall some useful properties.

Let $(L; \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice and X a nonempty subset of L. The set of elements of L having their negation in X is denoted and defined by:

$$N(X) := \{x \in L : x' \in X\}.$$

The following properties of the operator N shall be needed.

Remark 3.10. [30] Let F be a filter and I an ideal of $\mathcal{L}$. Then,

(i) $N(I)$ is a filter of $\mathcal{L}$, and $I = N(N(I))$;

(ii) $N(F)$ is an ideal of $\mathcal{L}$, and $F \subseteq N(N(F))$.

Proposition 3.11. *[31] Let $\mathcal{L}$ be a residuated lattice.*

(i) If I is a maximal ideal of $\mathcal{L}$, then $N(I)$ is a maximal filter of $\mathcal{L}$;

(ii) If F is a maximal filter of $\mathcal{L}$, then $N(F)$ is a maximal ideal of $\mathcal{L}$.

For all $x \in L$, the set

$$^{\perp}x := \{y \in L : x \vee y = 1\}$$

which is called the co-annihilator of x is a filter. Also, for any filter F of $\mathcal{L}$, the set

$$\delta(F) := \{x \in L : \text{ there are } f \in F \text{ and } z \in {}^{\perp}x \text{ such that } f \odot z = 0\}$$

is a filter of $\mathcal{L}$ and $\delta(F) \subseteq F$ (see [4]). Moreover, a filter F of $\mathcal{L}$ is called *pure filter* of $\mathcal{L}$ if $\delta(F) = F$ (i.e., $F \subseteq \delta(F)$).

Example 3.12. [4] Consider the residuated lattice $\mathcal{L}_3$ with the Hasse diagram of the underlying poset pictured in Figure 3, and the operations $\rightarrow$ and $\odot$ defined in Table 3. Then, $\{1\}$, $\{1, d\}$, $\{1, a, c\}$ and $\{0, a, b, c, d, 1\}$ are pure filters.

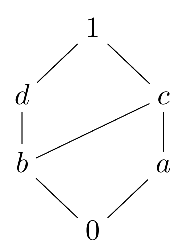

Figure 3: Hasse diagram of $\mathcal{L}_3$

Table 3: Operation tables of $\rightarrow$ and $\odot$ for $\mathcal{L}_3$ in Example 3.12

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

The following proposition establishes a relation between pure ideals and pure filters of a residuated lattice.

Proposition 3.13. *Let $\mathcal{L}$ be a residuated lattice.*

(i) If F is a pure filter of $\mathcal{L}$, then $N(F)$ is a pure ideal of $\mathcal{L}$;

(ii) If I is a pure ideal of $\mathcal{L}$, then $N(I)$ is not necessarily a pure filter of $\mathcal{L}$.

Proof.

(i) Assume that F is a pure filter of $\mathcal{L}$; it suffices to show that $N(F) \subseteq \sigma(N(F))$.
For all $x \in N(F)$, we have $x' \in F = \delta(F)$, since F is pure. This implies that, there are $f \in F$ and $y \in {}^{\perp}(x')$ such that $f \odot y = 0$. We deduce that $f' \in F' \subseteq N(F)$.
In addition, since $x' \vee y = 1$, we obtain $1 = x' \vee y = x''' \vee y \leq x''' \vee y''$. Then, $x''' \vee y'' = 1$, which implies that $(x''' \vee y'')' = 0$. Thus, $x'''' \wedge y''' = 0$, and hence $y' \in (x'')^{\perp}$.
We also have $f \odot y = 0$, which implies that $f'' \odot y'' \overset{(P11)}{\leq} (f \odot y)'' = 0$. Therefore, $f'' \odot y'' = 0$, and then $(f'' \odot y'')' = 1$, i.e., $f' \oplus y' = 1$.
Thus, there are $f' \in N(F)$ and $y' \in (x'')^{\perp}$ such that $f' \oplus y' = 1$, which means that $x'' \in \sigma(N(F))$. Since $\sigma(N(F))$ is an ideal, it becomes clear that $x \in \sigma(N(F))$. Hence, $\sigma(N(F)) = N(F)$.

(ii) From Example 3.5, we have $I_1 = \{0, d\}$ is a pure ideal. But $N(I_1) = \{a, b, c, e, f, 1\}$ is not a pure filter, since $\delta(N(I_1)) = \{1, c\} \neq N(I_1)$.

□

Remark 3.14. If I is a pure ideal of a regular residuated lattice $\mathcal{L}$, then $N(I)$ is a pure filter of $\mathcal{L}$.

Indeed, assume that I is a pure ideal of a regular residuated lattice $\mathcal{L}$. It suffices to show that $N(I) \subseteq \delta(N(I))$.
For all $x \in N(I)$, $x' \in I = \sigma(I)$ since I is pure. Then, there are $a \in I$ and $b \in (x')^{\perp}$ such that $a \oplus b = 1$. We obtain that $a' \in I' \subseteq N(I)$. Since $x''' \wedge b'' = 0$, we have $(x''' \wedge b'')' = 1$, i.e., $(x'' \vee b')'' = 1$, from (P8). It follows from the regularity of $\mathcal{L}$ that $x'' \vee b' = 1$, and then $x \vee b' = 1$. Hence, $b' \in {}^{\perp}x$.

Moreover, since $a \oplus b = 1$, i.e., $(a' \odot b')' = 1$, we deduce that $a' \odot b' = 0$. Thus there are $a' \in N(I)$ and $b' \in {}^{\perp}x$ such that $a' \odot b' = 0$. Hence, $x \in \delta(N(I))$, as required.

The symbol $ord(x)$ which stands for the *order of nilpotence* or simply *order* of an element $x \in L$ is the smallest number $n \in \mathbb{N}^{\star}$ such that $x^n = 0$, i.e., $\underbrace{x \odot \cdots \odot x}_{n \ \ times} = 0$. If there is no such n, then the order of x is infinite, i.e., $ord(x) = \infty$. Obviously, we always have $ord(1) = \infty$. For every $x, y \in L$, if $x \leq y$ and $ord(y) < \infty$, then $ord(x) < \infty$. Similarly, if $x \leq y$ and $ord(x) = \infty$ then $ord(y) = \infty$.

Proposition 3.15. *[4] A residuated lattice $\mathcal{L}$ is local if and only if $ord(x) < \infty$ or $ord(x') < \infty$, for every $x \in L$.*

We say that a residuated lattice $\mathcal{L}$ is *locally finite* if $ord(x) < \infty$ for all $x \neq 1$ in L.

Proposition 3.16. *[21] For any $x \in L$,*

(i) There exists a proper ideal I of $\mathcal{L}$ such that $x \in I$ iff $ord(x') = \infty$;

(ii) $\langle x \rangle$ is proper iff $ord(x') = \infty$;

(iii) $ord(x') < \infty$ iff $x \notin P$ for every prime ideal P.

For any residuated lattice $\mathcal{L}$, we consider the set $\Re := \{x \in L : ord(x') = \infty\} = \{x \in L : (x')^n \neq 0, \text{ for all } n \geq 1\}$.

Proposition 3.17. *Let $\mathcal{L}$ be a residuated lattice. In case $\Re$ is an ideal of $\mathcal{L}$, if $x, y \in \Re$, then $(x')^n \oplus (y')^n \neq 0$, for all $n \geq 1$.*

Proof. Let $x^n, y^n \neq 0$ for all $n \geq 1$, as $x, y \in \Re$. Then, $x \oplus y \in \Re$, since $\Re$ is an ideal by hypothesis. This implies that $[(x \oplus y)']^n \neq 0$. But, $0 \neq [(x \oplus y)']^n = [(x' \odot y')'']^n \overset{P(11)}{\leq} [(x' \odot y')^n]'' = [(x')^n \odot (y')^n]'' \overset{P(11)}{\leq} [((x')^n)' \odot ((y')^n)']' = (x')^n \oplus (y')^n$. Hence, $(x')^n \oplus (y')^n \neq 0$, for all $n \geq 1$. □

A residuated lattice is called *i-local* if it has a unique maximal ideal.
We recall some characterizations of i-local residuated lattices.

Proposition 3.18. *[21] The following statements are equivalent:*

(i) $\Re$ is an ideal of $\mathcal{L}$.

(ii) $\langle \Re \rangle$ is a proper ideal of $\mathcal{L}$.

(iii) $\mathcal{L}$ is i-local.

(iv) $\Re$ is the only maximal ideal of $\mathcal{L}$.

The next proposition shows that the notions of local and i-local residuated lattices are equivalent.

Proposition 3.19. *A residuated lattice $\mathcal{L}$ is local if and only if it is i-local.*

Proof. Assume that $\mathcal{L}$ is local, that is, $\mathcal{L}$ has only one maximal filter F. From Proposition 3.11 (ii), $N(F)$ is a maximal ideal of $\mathcal{L}$.
Let I be a maximal ideal of $\mathcal{L}$. Then, from Proposition 3.11 (i), $N(I)$ is a maximal filter of $\mathcal{L}$. This implies that $N(I) = F$, by the uniqueness of the maximal filter. By applying Remark 3.10 (i), it yields that $I = N(N(I)) = N(F)$. Therefore, $N(F)$ is the unique maximal ideal of $\mathcal{L}$, that is, $\mathcal{L}$ is i-local.

Conversely, if $\mathcal{L}$ is i-local, then it has a unique maximal ideal I. Applying Proposition 3.11 (i), $N(I)$ is a maximal filter of $\mathcal{L}$.
Consider a maximal filter F of $\mathcal{L}$. We deduce from Proposition 3.11 (ii) that $N(F)$ is a maximal ideal of $\mathcal{L}$. Since $\mathcal{L}$ is i-local, then $N(F) = I$. Thus, from the maximality of F and applying Remark 3.10 (ii), yields we obtain $F = N(N(F)) = N(I)$. Hence, $\mathcal{L}$ has only one maximal filter $N(I)$, that is, $\mathcal{L}$ is local. □

Corollary 3.20. *A residuated lattice $\mathcal{L}$ is i-local if and only if $ord(x) < \infty$ or $ord(x') < \infty$, for every $x \in L$.*

Since the notion of ideal of residuated lattices is also defined from the commutative and associative operation $\oplus$, we now introduce the concept of *$\oplus$-order* of an element, from which we will provide a new characterization of i-local residuated lattices.

Definition 3.21. Let $\mathcal{L}$ be a residuated lattice, and $x \in L$. Then, the *$\oplus$-order* of x denoted $ord_{\oplus}(x)$ is the smallest number $n \in \mathbb{N}^{\star}$ such that $nx = 1$. When there is no such n, we say that the $\oplus$-order of x is infinite, that is, $ord_{\oplus}(x) = \infty$.

In the example below, we compute the $\oplus$-order of some elements of the residuated lattice $\mathcal{L}_2$ from Example 3.6.

Example 3.22. In the residuated lattice $\mathcal{L}_2$ of Example 3.6, we have:

- $ord_{\oplus}(a) = 1 < \infty = ord(a)$;
- $ord(d) = 1 < \infty = ord_{\oplus}(d)$;
- $ord(e) = 2 < \infty = ord_{\oplus}(e)$.

Proposition 3.23. *A residuated lattice $\mathcal{L}$ is i-local if and only if $ord_{\oplus}(x) < \infty$ or $ord_{\oplus}(x') < \infty$, for every $x \in L$.*

Proof. Assume that $\mathcal{L}$ is i-local, that is $\mathcal{L}$ has a unique maximal ideal I. Suppose by contrary that there is $x \in L$ such that $ord_{\oplus}(x) = \infty = ord_{\oplus}(x')$. Then, $\langle x\rangle$ is proper; otherwise, if $\langle x\rangle = L$, then from Proposition 2.5 (i) there exists $n \in \mathbb{N}^{\star}$ such that $nx = 1$, which is a contradiction.

Similarly, $\langle x'\rangle$ is proper. Since I is the unique maximal ideal of $\mathcal{L}$, we deduce that $\langle x\rangle$, $\langle x'\rangle \subseteq I$, which implies that $x, x' \in I$. Thus, $1 = x \oplus x' \in I$, a contradiction. Hence, $ord_{\oplus}(x) < \infty$ or $ord_{\oplus}(x') < \infty$.

Conversely, assume that $ord_{\oplus}(x) < \infty$ or $ord_{\oplus}(x') < \infty$, for every $x \in L$. Suppose by contrary that there are two distinct maximal ideals I and J of $\mathcal{L}$. Then, for any $y \in I \setminus J$, there is $n \in \mathbb{N}^{\star}$ such that $(ny)' \in J$, from Proposition 2.7 (ii). Set $a = ny$; then, $a' \in J$, which implies that $ma' \in J$ for all $m \in \mathbb{N}$. Thus, $ord_{\oplus}(a') = \infty$, implying from hypothesis that $ord_{\oplus}(a) < \infty$. This means that there is $k \in \mathbb{N}^{\star}$ such that $ka = 1$, that is, $kny = 1$. Since $y \in I$, we have $kny \in I$, that is, $1 = kny \in I$ which contradicts the fact that I is maximal. Therefore, $\mathcal{L}$ has only one maximal ideal, and hence is i-local. □

4 Stable Topology for Ideals of Residuated Lattices

Piciu in [23] endowed the set of prime ideals of a residuated lattice $\mathcal{L}$ with the Zariski topology. Let X be a nonempty subset of L and $D(X) := \{P \in Spec_{Id}(\mathcal{L}) : X \nsubseteq P\}$. The following proposition presents some properties of $D(X)$.

Proposition 4.1. *[23] Let $x, y \in L$, and $X, X_1, X_2, \{X_\gamma\}_{\gamma\in\Gamma} \subseteq L$. Then,*

(i) $X_1 \subseteq X_2$ *implies* $D(X_1) \subseteq D(X_2)$;

(ii) $D(X) = Spec_{Id}(\mathcal{L})$ *if and only if* $\langle X\rangle = L$. *Particularly,* $D(x) = Spec_{Id}(\mathcal{L})$ *if and only if* $\langle x\rangle = L$;

(iii) $D(X) = \emptyset$ *if and only if* $X = \{0\}$ *or* $X = \emptyset$. *In particular,* $D(x) = \emptyset$ *if and only if* $x = 0$;

(iv) $D(1) = D(L) = Spec_{Id}(\mathcal{L})$ *and* $D(\{0\}) = D(\emptyset) = \emptyset$;

(v) $\bigcup_{\gamma\in\Gamma} D(X_\gamma) = D(\bigcup_{\gamma\in\Gamma} X_\gamma)$;

(vi) $D(X) = D(\langle X\rangle)$;

(vii) $D(X_1) \cup D(X_2) = D(\langle X_2\rangle \cup \langle X_1\rangle)$ *and* $D(X_1) \cap D(X_2) = D(\langle X_2\rangle \cap \langle X_1\rangle)$;

(viii) $\langle X_1\rangle = \langle X_2\rangle$ *if and only if* $D(X_1) = D(X_2)$;

(ix) $D(x) \cup D(y) = D(x \vee y) = D(x \oplus y)$; $D(x) \cap D(y) = D(x'' \wedge y'')$.

The family $\{D(\{x\})\}_{x\in L}$ where $D(x) = \{P \in Spec_{Id}(\mathcal{L}) : x \notin P\}$, for all $x \in L$, is a basis for a topology $\tau_{\mathcal{L}} := \{D(X) : X \subseteq L\}$ on $Spec_{Id}(\mathcal{L})$. The topological space $(Spec_{Id}(\mathcal{L}), \tau_{\mathcal{L}})$ is called the *prime ideals space* of $\mathcal{L}$.

One can observe that for any ideal I of $\mathcal{L}$, $D(I)$ is an open set and $V(I) := \{P \in Spec_{Id}(\mathcal{L}) : I \subseteq P\}$ is a closed set for $(Spec_{Id}(\mathcal{L}), \tau_{\mathcal{L}})$. The set $D(I)$ is *stable under descent*, that is, if $P \in D(I)$, $Q \in Spec_{Id}(\mathcal{L})$ and $Q \subseteq P$, then $Q \in D(I)$. Moreover, $V(I)$ is *stable under ascent*, that is, if $P \in V(I)$, $Q \in Spec_{Id}(\mathcal{L})$ and $P \subseteq Q$, then $Q \in V(I)$. Therefore, the sets that are simultaneously open and closed (called clopen) are stable, that is, they are stable under ascent and descent.
The *stable topology* for $\mathcal{L}$ is the collection $\mathcal{S}_L$ of open stable subsets $D(I)$ of $Spec_{Id}(\mathcal{L})$ defined by $\mathcal{S}_L := \{D(I) : I \in \mathcal{I}(\mathcal{L}) \text{ and } D(I) \text{ is stable under ascent}\}$.

In what follows, we characterize open stable sets by means of pure ideals.

Theorem 4.2. *Let $\mathcal{L}$ be a residuated lattice and I an ideal of $\mathcal{L}$. Then, I is pure iff $D(I)$ is stable in $Spec_{Id}(\mathcal{L})$.*

Proof. For the first implication, let us show that $D(I)$ is stable under ascent. To this end, assume that I is pure and let P, Q be prime ideals of $\mathcal{L}$ such that $P \subseteq Q$ and $P \in D(I)$. Then, $I \nsubseteq P$, which implies that there is $x \in I \setminus P$. From the fact that $x \in I = \sigma(I)$, there are $a \in I$ and $b \in x^{\perp}$ such that $a \oplus b = 1$. But $b \in x^{\perp}$ implies that $b'' \wedge x'' = 0 \in P$. Since P is prime and $x \notin P$, we deduce that $b \in P \subseteq Q$. It yields that $Q \in D(I)$, otherwise we will have $I \subseteq Q$, implying that $a \in Q$ and $1 = a \oplus b \in Q$, contradicting the assumption that Q is proper.

On the other hand, assume that $D(I)$ is stable in $Spec_{Id}(\mathcal{L})$ and suppose by contrary that I is not a pure ideal of $\mathcal{L}$, that is, $\sigma(I) \subsetneq I$. Then, there exists $x \in I \setminus \sigma(I)$. Applying Proposition 2.10, there exists a minimal prime ideal P of $\mathcal{L}$ such that $\sigma(I) \subseteq P$ and $x \notin P$. This implies that $I \nsubseteq P$, that is, $P \in D(I)$.

Applying Proposition 3.8, I is not pure iff $x^{\perp} \vee I \neq L$, which implies that $x^{\perp} \vee I$ is proper. Thus, from the prime ideal theorem (see Theorem 2.9), there exists a prime ideal Q such that $x^{\perp} \vee I \subseteq Q$. This implies that $I \subseteq Q$, and therefore $Q \notin D(I)$. But $\sigma(I) \subseteq I \subseteq Q$, and by the minimality of P, we have $P \subseteq Q$. Since $D(I)$ is stable and $P \in D(I)$, it follows that $Q \in D(I)$, which is a contradiction. Hence, I is a pure ideal of $\mathcal{L}$.
□

Corollary 4.3. *For a residuated lattice $\mathcal{L}$, the assignment $I \rightsquigarrow D(I)$ is a bijection between the set of pure ideals of $\mathcal{L}$ and the set of open stable subsets of $Spec_{Id}(\mathcal{L})$.*

Theorem 4.2 yields the following separation property.

Theorem 4.4. *Let I be a pure ideal of $\mathcal{L}$, let P_1, P_2 be minimal ideals and P a prime ideal of $\mathcal{L}$ such that $P_1, P_2 \subseteq P$. Then, $I \subseteq P_1$ iff $I \subseteq P_2$.*

Proof. Suppose by contrary that $I \subseteq P_1$ and $I \nsubseteq P_2$. Then, $P_2 \in D(I)$. Since I is pure, we deduce from Theorem 4.2 that $D(I)$ is stable. Thus, from $P_2 \subseteq P$ and $P_2 \in D(I)$ it follows that $P \in D(I)$, which means that $I \nsubseteq P$. But, $I \subseteq P_1$ and $P_1 \nsubseteq P$ imply that $I \subseteq P$, which is a contradiction. □

For any maximal ideal M of $\mathcal{L}$, we set $\widehat{M} := \{P \in Spec_{Id}(\mathcal{L}) : P \subseteq M\}$.

Corollary 4.5. *For any pure ideal I and any maximal ideal M of $\mathcal{L}$, either $I \subseteq P$ for every $P \in \widehat{M}$, or $I \nsubseteq P$ for every $P \in \widehat{M}$.*

Proof. Assuming by contrary that there are $P_1, P_2 \in \widehat{M}$ such that $I \subseteq P_1$ and $I \nsubseteq P_2$, is in contradiction with Theorem 4.4.
□

To investigate the stable topology on i-local residuated lattices, we need the following results.

Proposition 4.6. *Let I be an ideal of $\mathcal{L}$. If $\sigma(I) \neq \{0\}$, then there is an element $a \in I$ such that $ord(a'') = \infty$.*

Proof. If $x \in \sigma(I)$ with $x \neq 0$, then there are $a \in I$ and $b \in x^{\perp}$ such that $a \oplus b = 1$. It follows that $a' \to b'' = 1$, which means that $a' \leq b''$. This implies that $(b')^n \leq (a'')^n$, for every $n \geq 1$. It is sufficient to show that $ord(b') = \infty$. We have $x'' \wedge b'' = 0$ (since $b \in x^{\perp}$) , which implies that $n.n(x'' \wedge b'') = 0$, for every $n \geq 2$. From (P13), $(nx) \wedge (nb) \leq n.n(x'' \wedge b'') = 0$, i.e., $(nx) \wedge (nb) = 0$, which is equivalent to $[(x')^n]' \wedge [(b')^n]' = 0$, for every $n \geq 2$. If by contrary $[(b')^n]' = 1$ for some $n \geq 2$, then $[(x')^n]' = 0$, i.e., $(nx) = 0$, implying that $x = 0$ (as $x \leq nx = 0$), contradicting the hypothesis. Thus, $[(b')^n]' \neq 1$. We deduce that $(b')^n \neq 0$, for every $n \geq 2$.
It is worth noticing that if $n = 1$, then $b' \neq 0$. Otherwise, we will have $b'' = 1$, implying from $x'' \wedge b'' = 0$ that $x'' = 0$, which is equivalent to $x = 0$, a contradiction to the hypothesis. Therefore, $ord(b') = \infty$ and hence $ord(a'') = \infty$.
□

Corollary 4.7. *Let I be a proper ideal of $\mathcal{L}$. If $\mathcal{L}$ is i-local, then $\sigma(I) = \{0\}$, that is, the unique pure ideals of $\mathcal{L}$ are $\{0\}$ and L.*

Proof. Let I be a proper ideal of the i-local residuated lattice $\mathcal{L}$. Assume by contrary that $\sigma(I) \neq 0$. Then, there exists $a \in I$ such that $ord(a'') = \infty$, from Proposition 4.6. Since $\mathcal{L}$ is i-local, we deduce from Corollary 3.20 that $ord(a') < \infty$, i.e., $(a')^n = 0$ for some $n \geq 1$. Therefore, $1 = [(a')^n]' = na \in I$, i.e., $I = L$, contradicting the hypothesis that I is proper. Hence the only pure ideals of $\mathcal{L}$ are 0 and L.
□

The theorem below states that the stable topology for an i-local residuated lattice is trivial.

Theorem 4.8. *If $\mathcal{L}$ is i-local, then the stable topology S_L on $\mathcal{L}$ is trivial.*

Proof. From Theorem 4.2, $D(I)$ is stable in $Spec_{Id}(\mathcal{L})$ iff I is a pure ideal of $\mathcal{L}$. But if $\mathcal{L}$ is i-local, then it follows from Corollary 4.7 that the only pure ideals of $\mathcal{L}$ are $\{0\}$ and L . Thus, either $I = \{0\}$ or $I = L$, and therefore $D(I) = \emptyset$, or $D(I) = Spec_{Id}(\mathcal{L})$. Hence, $S_L = \{\emptyset, Spec_{Id}(\mathcal{L})\}$. □

5 Conclusion

This work aimed to equip the set of prime ideals of a residuated lattice with the stable topology, a topology coarser than Zariski topology. To achieve, based on the notion of annihilator, we have introduced the concept of pure ideal in residuated lattices, along with its properties. After establishing a relation between pure ideals and pure filters of a residuated lattice, we have characterized open stable sets relative to the stable topology on prime ideals of a residuated lattice.

In our forthcoming research, following the approach in [12], we will explore some sheaf representations of i-normal residuated lattices described in [23]. Also, we plan to construct the Belluce lattice using the prime ideals of a residuated lattice to offer some additional characterizations of pure ideals and a better understanding of the prime ideals space of a residuated lattice. Recognizing the limitations of existing models such as those used for De Morgan residuated lattices [21], or MV algebras [2], we acknowledge the need for a novel approach.

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