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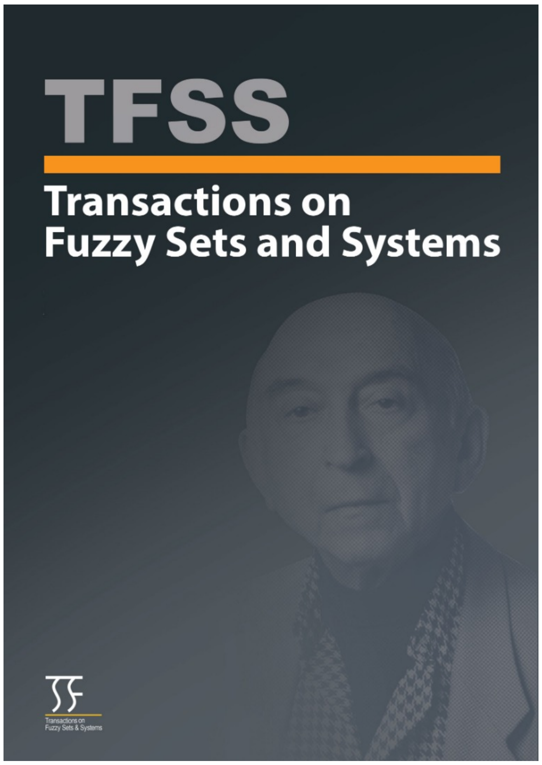

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New Elements in Hilbert Algebras

Ardavan Najafi*

Abstract. In this paper, the notion of the commutator of elements of a Hilbert algebra are introduced and some properties are given. The notions of involution element and Engel element in Hilbert algebras are introduced. Many different characterizations of them are given. Then, left (right) k-Engel elements as a natural generalization of commutators are introduced, and we discuss Engel elements, which are defined by left and right commutators. Finally, we will also study the relationships between these elements.

AMS Subject Classification 2020: MSC 06A11; MSC 03G25; MSC 03B22
Keywords and Phrases: Hilbert algebras, Commutator, Engel (involution) elements, Left (right) k-Engel element.

1 Introduction

Hilbert algebras are important tools for certain investigations in algebraic logic, since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value true. Following the introduction of Hilbert algebras by L. Henkin in the early 50-ties and A. Diego [4], the algebra and related concepts were developed by D. Busneag [1]. Y.B. Jun gave characterizations of deductive systems in Hilbert algebras [6], introduced the notion of commutative Hilbert algebras and gave some characterizations of a commutative Hilbert algebra.
A. Diego [4] proved that Hilbert algebras form a variety that is locally finite. They were studied from various points of view. Concerning congruence properties it is shown in [2] that Hilbert algebras form a congruence distributive variety the congruences which are in a one to one correspondence with ideals [5].
The present author introduced the commutator of two elements in a BCI-algebra, and used this notion to define a solvable BCI-algebra and considered solvable BCI-algebras using commutators. Then we gave a new definition for solvability, nilpotency, centralizer and pseudo-center in a BCI-algebra and considered their properties (see [7]).
In this paper, we present a definition for the notion of Engel elements in Hilbert algebras based on commutators. We define also the notions of left k-Engel elements and right k-Engel elements as a natural generalization of commutators in Hilbert algebra, give several characterizations of them and prove that a Hilbert algebra is commutative if and only if 1 is only commutator of it. So, the class of commutative Hilbert algebras and 1-Engel BCI-algebras are equal. One of the most important concepts in the study of groups is the notion of nilpotency. Finally, we present a definition for the involution elements in Hilbert algebras. We give several characterizations of them and we illustrate also these notions with some examples.

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2 Preliminaries

We include some elementary aspects of Hilbert algebras that are necessary for this paper, and for more details we refer to [1-6].
By a Hilbert algebra, we main an algebra $(H, \to, 1)$ of type $(2, 0)$, where H is a non-empty set, $\to$ is a binary operation on H, $1 \in H$ is an element such that the following three axioms are satisfied for every $x, y, z \in H$:
$(H1)$ $x \to (y \to x) = 1$,
$(H2)$ $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
(H3) if $x \to y = y \to x = 1$, then $x = y$.
In a Hilbert algebra H, the following properties hold:
$(P1)$ $x \to 1 = 1, 1 \to x = x$ and $x \to x = 1$,
$(P2)$ $x \to (y \to z) = (x \to y) \to (x \to z)$,
$(P3)$ $x \to (y \to z) = y \to (x \to z)$
If H is a Hilbert algebra, then the relation $x \leq y$ iff $x \to y = 1$ is a partial order on H, called the natural ordering on H. With respect to this ordering 1 is the greatest element of H and the following property is satisfied.
$(P4)$ $x \leq y$ implies $z \to x \leq z \to y$ and $y \to z \leq x \to z$.
For any x and y in a Hilbert algebra H, define $x \vee y$ as $(y \to x) \to x$. A Hilbert algebra H is said to be commutative, if for all $x, y \in H$,

$$(y \to x) \to x = (x \to y) \to y \quad i.e., \quad x \vee y = y \vee x.$$

Note that $x \vee y$ is the least upper bound of x and y, hence each commutative Hilbert algebra H is a semilattice with respect to $\vee$ (see [1]) and hence $\vee$ is commutative and associative. A non empety subset S of a Hilbert algebra H is called a subalgebra of H, if $x \to y \in S$, whenever $x, y \in S$. A bounded Hilbert algebra is a Hilbert algebra H with an element $0 \in H$ such that $0 \to x = 1$, for every $x \in H$. In a bounded Hilbert algebra H we define a unary operation $*$ as $x^* = x \to 0$, for each $x \in H$.
A Hilbert algebra is prelinear if $(x \to y) \vee (y \to x) = 1$, for all $x, y \in H$. We say that an element x of H is minimal if $y \leq x$ (i.e., $y \to x = 1$) implies $x = y$, for any $y \in H$.

Example 2.1. [4] It is of great importance that every partially ordered set $(P, \to, 1)$ with the greatest element 1 can be regarded as a Hilbert algebra, namely, if for any $x, y \in P$ we define:

$$x \to y = \begin{cases} 1 & \text{if x} \leq \text{y} \\ y & \text{otherwise.} \end{cases}$$

then (P, $\to$, 1) is a Hilbert algebra the natural ordering on which coincides with the relation $\leq$.

Lemma 2.2. [8] *If H is a bounded Hilbert algebra and $x, y \in H$, then*
i) $1^* = 0$ *and* $0^* = 1$,
ii) $x \leq x^{**}$,
iii) $x^{***} = x^*$,
iv) $x \to y^* = y \to x^*$,
v) if $x \leq y$, *then* $y^* \leq x^*$.

From now on, $(H, \to, 1)$ or simply H is a Hilbert algebra, unless otherwise specified.

3 Commutators of Two Elements

In this section, we introduce commutators of two elements of a Hilbert algebra and investigate some of their properties.

Definition 3.1. For elements x and y of a Hilbert algebra H, we define the commutator of x and y by $(y \vee x) \to (x \vee y)$ denoted by $[x, y]$, Namely,

$$[x, y] = (y \vee x) \to (x \vee y) = ((x \to y) \to y) \to ((y \to x) \to x)$$

For every $x \in H$, we obtain $[x, x] = (x \vee x) \to (x \vee x) = x \to x = 1$. Also, since 1 is the greatest element of H, $[1, x] = (x \vee 1) \to (1 \vee x) = 1 \to 1 = 1$. Similarity $[x, 1] = 1$. The set of all the commutators of elements of H is denoted by $Com(H)$ or $K(H)$. Obviously, $1 \in K(H)$.

Example 3.2. Let $H = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$, and a, b are incompatible, be a Hilbert algebra in which $\to$ operation is defined by the left table and the commutators of elements of H are calculation in the right table

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$[,]$	0	a	b	c	1
0	1	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	c	c	c	1	1
1	1	1	1	1	1

Lemma 3.3. *For any $x, y, z \in H$, $z \to [x, y] = [z \to x, z \to y]$.*

Proof.

$$\begin{aligned}
[z \to x, z \to y] =& (((z \to x) \to (z \to y)) \to (z \to y)) \\
&\to (((z \to y) \to (z \to x)) \to (z \to x)) \\
=& ((z \to (x \to y)) \to (z \to y)) \to ((z \to (y \to x)) \to (z \to x)) \\
=& (z \to ((x \to y) \to y)) \to (z \to ((y \to x) \to x)) \\
=& z \to (((x \to y) \to y) \to ((y \to x) \to x)) \\
=& z \to [x, y]
\end{aligned}$$

In this proof we use (P_2) in Lines 3, 4 , 5. $\square$

In the following theorem, we will show that $[x, y]$ is an upper bound for x and y, but in Example 4.5 we show that it is not a supremum of x, y, in general.

Theorem 3.4. *For all $x, y \in H$, we have*
i) $x \le [x, y]$ and $y \le [x, y]$,
ii) if $x \le y$, then $[x, y] = 1$,
iii) $[x \to y, x] = [x, y \to x] = 1$,
iv) $x \to y \le [x, y]$,
v) $[x, y] = y$ if and only if $y = 1$.

Proof. i) As $x \to [x, y] = [x \to x, x \to y] = [1, x \to y] = 1$, then $x \le [x, y]$. Also $y \to [x, y] = [y \to x, y \to y] = [y \to x, 1] = 1$, then $y \le [x, y]$.
ii) Let $x \le y$, then $x \to y = 1$. By use definition of $[x, y]$ and property P_3, we obtain

$$\begin{aligned}
[x, y] =& ((x \to y) \to y) \to ((y \to x) \to x) \\
=& (1 \to y) \to ((y \to x) \to x) \\
=& y \to ((y \to x) \to x) \\
=& (y \to x) \to (y \to x) = 1.
\end{aligned}$$

iii)

$$\begin{aligned}[x \to y, x] =&(((x \to y) \to x) \to x) \to ((x \to (x \to y)) \to (x \to y))\\ =&(((x \to y) \to x) \to x) \to (((x \to x) \to (x \to y)) \to (x \to y))\\ =&(((x \to y) \to x) \to x) \to (((1 \to (x \to y)) \to (x \to y))\\ =&(((x \to y) \to x) \to x) \to ((x \to y) \to (x \to y))\\ =&(((x \to y) \to x) \to x) \to 1 = 1.\end{aligned}$$

Since $x \leq y \to x$, by (ii), we have $[x, y \to x] = 1$.
iv)

$$\begin{aligned}(x \to y) \to [x, y] =&(x \to y) \to (((x \to y) \to y) \to ((y \to x) \to x))\\ =&((x \to y) \to y) \to ((x \to y) \to ((y \to x) \to x))\\ =&(x \to y) \to (y \to ((y \to x) \to x))\\ =&(x \to y) \to ((y \to x) \to (y \to x))\\ =&(x \to y) \to 1 = 1.\end{aligned}$$

v) Let $[x, y] = y$. Then $y = (y \vee x) \to (x \vee y) \geq (x \vee y) \geq y$. Hence $x \vee y = y$. Therefore $x \to y = x \to (x \vee y) = 1$, because $x \leq (x \vee y)$. So $x \to y = 1$ and hence $x \leq y$ and by (ii), we obtain $[x, y] = 1$. Thus $y = 1$.
Conversely, if $y = 1$, then $[x, y] = [x, 1] = 1$. □
By Example 3.2 we see that $[x, y] \neq [y, x]$, in general. Also, in this example we see $[c, b] = c$, so $[x, y] = x$ it cannot be said that $x = 1$, in general.

Theorem 3.5. *If H is a bounded Hilbert algebra and $x, y \in H$, then*
i) $[x, 0] = x^{**} \to x$,
ii) $[0, x] = 1$,
iii) $[x^*, x] = 1$.

Proof. i)

$$\begin{aligned}[x, 0] =&((x \to 0) \to 0) \to ((0 \to x) \to x)\\ =&(x^* \to 0) \to (1 \to x)\\ =&x^{**} \to x.\end{aligned}$$

ii) Since $0 \leq x$ for every $x \in H$, by Theorem 3.4 part (ii), we have $[0, x] = 1$.
iii)

$$\begin{aligned}[x^*, x] =&((x^* \to x) \to x) \to ((x \to x^*) \to x^*)\\ =&((x^* \to x) \to x) \to ((x \to (x \to 0)) \to x^*)\\ =&((x^* \to x) \to x) \to (((x \to x) \to (x \to 0)) \to x^*)\\ =&((x^* \to x) \to x) \to ((1 \to x^*) \to x^*)\\ =&((x^* \to x) \to x) \to (x^* \to x^*)\\ =&((x^* \to x) \to x) \to 1\\ =&1.\end{aligned}$$

□

Theorem 3.6. *H is commutative if and only if $[x,y]=1$, for every $x,y\in H$.*

Proof. Let H be a commutative Hilbert algebra. Then for every $x,y\in H$, $y\vee x=x\vee y$. Hence $[x,y]=(y\vee x)\to(x\vee y)=1$.
Conversely, let for every $x,y\in H$, we have $[x,y]=[y,x]=1$. Form $[x,y]=(y\vee x)\to(x\vee y)=1$, we deduce $y\vee x\leq x\vee y$. But $[y,x]=(x\vee y)\to(y\vee x)=1$, so $x\vee y\leq y\vee x$. Therefore $x\vee y=y\vee x$, for every $x,y\in H$. Hence H is a commutative Hilbert algebra. □

Theorem 3.7. *Let H be a finite Hilbert algebra of order n with $n\geq 2$ and $x,y\in H$. Then $[x,y]$ is not a minimal element of H.*

Proof. Suppose that there exist x and y in H such that $[x,y]$ is a minimal element of H. Then $[x,y]=y$, as $y\leq[x,y]$. Hence by Theorem 3.4 (v) we deduce $y=1$. Whence $[x,y]=[x,1]=1$. This is a contradiction. Because, if 1 is a minimal element of H, then from $x\leq 1$ we deduce $H=\{1\}$. □

4 Engel Elements in Hilbert Algebras

In this section, we introduce the notion of the Engel element of n elements of a Hilbert algebra.
Let $x_1,...,x_n$ be elements of H. For all positive integer n we define inductively $[x_1,...,x_n]$ as follows: $[x_1]=x_1$ and

$$[x_1,...,x_n]=([x_1,...,x_{n-1}]\vee x_n)\to(x_n\vee[x_1,...,x_{n-1}])$$

If $x_2=x_3=...=x_n$, then we denote $[x_1,...,x_n]$ by $[x_1,{}_nx_2]$. Note that $[x_1]=[x_1,{}_0x_2]=x_1$.

Definition 4.1. Suppose that $x,y\in H$. For a non-negative integer n we define inductively the *n-Engel left commutator* $[x,{}_ny]$ as follows:

$$[x,{}_0y]=x,...,[x,{}_ny]=[[x,{}_{n-1}y],y]$$

Also the *n-Engel right commutator* $[{}_nx,y]$ of the pair (x,y) is defined by induction as follows:

$$[{}_0x,y]=y,...,[{}_nx,y]=[x,[{}_{n-1}x,y]].$$

Especially, $[x,{}_1y]=[{}_1x,y]=[x,y]=(y\vee x)\to(x\vee y)$.

For a positive integer k, an element x of H is called a *right k-Engel element* of H whenever $[x,{}_ky]=1$, for all $y\in H$. An element x of H is called a *right Engel element* if it is right k-Engel for some non-negative integer k.
We denote by $R(H)$ and $R_k(H)$ the set of right Engel elements and right k-Engel elements, respectively. So

$$R_k(H)=\{x\in H:[x,{}_ky]=1,\forall y\in H\}.$$

and

$$R(H)=\bigcup_{k\in N}R_k(H).$$

Notice that the variable element y appears on the right of bracket and if n can be chosen independently of y, then x is a right n-Engel element of H. Left Engel elements are defined in a similar way.
For a positive integer k, an element x of H is called a *left k-Engel element* of H whenever $[y,{}_kx]=1$ for all

$y \in H$. Also x is said to be a *left Engel element* of H if it is left k-Engel for some non-negative integer k. We denote by $L(H)$ and $L_k(H)$ the set of left Engel elements and left k-Engel elements, respectively.

$$L_k(H) = \{x \in H : [y, {}_k x] = 1, \forall y \in H\}.$$

and

$$L(H) = \bigcup_{k \in N} L_k(H).$$

Where the variable y is on the left of bracket. Also, since $[x, 1] = [1, x] = 1$, for every $x \in H$, $1 \in R(H) \cap L(H)$. An element x of H that is both the left and right Engel element is said to be an *Engel element*.
The set of all Engel elements of H is denoted by $En(H)$. Obviously, 1 is an Engel element in any Hilbert algebra. Since $x, y \leq [x, y]$, for every $x, y \in H$,

$$[x, {}_0 y] = x \leq [x, {}_1 y] = [x, y] \leq [x, {}_2 y] = [[x, y], y] \leq [x, {}_3 y] = [[x, {}_2 y], y] \leq ...$$

Also

$$[{}_0 x, y] = y \leq [{}_1 x, y] = [x, y] \leq [{}_2 x, y] = [x, [x, y]] \leq [{}_3 x, y] = [x, [{}_2 x, y]] \leq ...$$

According to the above inequalities, we immediately have the following theorems.

Theorem 4.2. *Let $x, y \in H$ and m, n be non-negative integers. If $m \leq n$, then $[x,_m y] \leq [x,_n y]$ and $[_m x, y] \leq [_n x, y]$.*

Theorem 4.3.

$$R_1(H) \subseteq R_2(H) \subseteq R_3(H) \subseteq .... \subseteq R(H).$$
$$L_1(H) \subseteq L_2(H) \subseteq L_3(H) \subseteq .... \subseteq L(H).$$

Example 4.4. By simple calculation, for Hilbert algebra H in Example 3.2 we see that
$[c, {}_0 a] = c, [c, {}_1 a] = [c, a] = c, [c, {}_2 a] = [[c, a], a] = [c, a] = c$. So $[c, {}_n a] = c$ for $n \geq 2$, also $[{}_0 c, a] = a, [{}_1 c, a] = c, [{}_2 c, a] = [c, [c, a]] = [c, c] = 1$, so $[{}_n c, a] = 1$, for any $n \geq 2$.

$$R_1(H) = \{x \in H : [x, y] = 1, \forall y \in H\} = \{0, a, b, 1\}.$$

$$R_2(H) = \{x \in H : [x, {}_2 y] = 1, \forall y \in H\} = \{0, a, b, 1\}.$$

Also, for every $n \geq 2$ we have

$$R_n(H) = \{x \in H : [x, {}_n y] = 1, \forall y \in H\} = \{0, a, b, 1\}.$$

and

$$R(H) = \bigcup_{k \in N} R_k(H) = \{0, a, b, 1\}.$$

Also

$$L_1(H) = \{x \in H : [y, x] = 1, \forall y \in H\} = \{c, 1\}.$$

and for any $n \geq 2$ we obtain

$$L_n(H) = \{x \in H : [y, {}_n x] = 1, \forall y \in H\} = \{c, 1\}.$$

Therefore

$$L(H) = \bigcup_{k \in N} L_k(H) = \{c, 1\}.$$

Then $En(H) = R(H) \cap L(H) = \{1\}$.

Example 4.5. Let $H = \{a, b, c, 1\}$ be a Hilbert algebra in which $\rightarrow$ operation is defined by the left table and the commutators of elements of H are calculation in the right table

$\rightarrow$	a	b	c	1
a	1	a	a	1
b	1	1	a	1
c	1	1	1	1
1	a	b	c	1

$[,]$	a	b	c	1
a	1	1	1	1
b	1	1	a	1
c	1	1	1	1
1	1	1	1	1

By simple calculation, we see that $[a, {}_0 b] = a, [a, {}_1 b] = 1, [a, {}_2 b] = 1$ and so $[a, {}_n b] = 1$, also $[b, {}_0 c] = b, [b, {}_1 c] = a, [b, {}_2 c] = [[b, c], c] = [a, c] = 1$ and so $[b, {}_n c] = 1$, for any $n \geq 2$.
$[{}_0 c, a] = a, [{}_1 c, a] = 1, [{}_2 c, a] = [c, [c, a]] = [c, 1] = 1$, so $[{}_n c, a] = 1$, for any $n \geq 3$.

$$R_1(H) = \{a, c, 1\}.$$

for every $n \geq 2$

$$R_n(H) = \{a, b, c, 1\}.$$

$$L_1(H) = \{a, b, 1\}.$$

and for any $n \geq 2$ we obtain

$$L_n(H) = \{a, b, c, 1\}.$$

Then $En(H) = R(H) \cap L(H) = \{a, b, c, 1\}$.

In Example 4.5 we see that $c \leq b \leq a \leq 1$ and $[b, c] = a$. Hence $[b, c]$ is an upper bound for b and c, but it is not supremum.

Lemma 4.6. *Let $x, y \in H$. Then for each $n \in N$ the following assertions hold:*
i) $[{}_n x, y] = 1$, for every $n \geq 2$.
ii) if H is a Hilbert algebra with $|H| \geq 2$, then $[x, {}_n y]$ and $[{}_n x, y]$ are not minimal elements.

Proof. i) For any $x, y \in H$, since $x \leq [x, y]$, we obtain

$$[{}_2 x, y] = [x, [x, y]] = 1.$$

But

$$1 = [{}_2 x, y] \leq [{}_3 x, y] \leq [{}_4 x, y] \leq [{}_5 x, y] \leq ...$$

Hence $[{}_3 x, y] = [{}_4 x, y] = [{}_5 x, y] = ... = 1$.
ii) We proceed by induction on n. For $n = 1$, $[x, y]$ is not a minimal element of H, by Theorem 3.7.
Now assume that for $n \in N$, $[x, {}_n y]$ and $[{}_n x, y]$ are not minimal elements of H. Since $[x, {}_{n+1} y] = [[x, {}_n y], y] \geq [x, {}_n y]$, $[x, {}_{n+1} y]$ is not minimal. Also $[{}_{n+1} x, y] = [x, [{}_n x, y]] \geq [{}_n x, y]$. Since $[{}_n x, y]$ is not a minimal element of H, $[{}_{n+1} x, y]$ is not too. Hence the result holds for $n + 1$ in both cases. $\square$

5 Involution Elements

In this section, at first, we define the involution element in a bounded Hilbert algebra, and then we investigate the relationships of these elements with commutators.

Definition 5.1. For a bounded Hilbert algebra H, if an element x satisfies $x^{**} = x$, then x is called an involution of H.

The set of all involution elements of a bounded Hilbert algebra H is denoted by $S(H)$. The smallest element 0 and the greatest element 1 are two involutions of H, because $0^{**} = 1^* = 0$ and $1^{**} = 0^* = 1$. Since the elements 0 and 1 are contained in $S(H)$. Hence $S(H)$ is not empty.

Example 5.2. It is easy to see that $(H, \to, 1)$ in Example 3.2 is a bounded Hilbert algebra with unit 1. We obtain $S(H) = \{0, a, b, 1\}$. In this example $c \notin S(H)$, because $c^{**} = 0^* = 1 \neq c$. We saw that $K(H) = \{c, 1\}$ and $En(H) = \{1\}$. Hence $En(H), S(H)$ and $K(H)$ are the separate sets from each other, in general.
Also, if we choose $P = [0, 1]$ in Example 2.1, we obtain, $S(P) = \{0, 1\}$ and for any $0 < x \leq 1$, we get $[x, 0] = [x,_2 0] = [x,_3 0] = ... = x$. So $En(P) = K(P) = (0, 1]$.

Theorem 5.3. *In a bounded Hilbert algebra H, we have*

$$x^* \to y = y^* \to x$$

for all x and y in $S(H)$.

Proof. Let $x, y \in S(H)$. Then $x^* \to y = x^* \to y^{**} = y^* \to x^{**} = y^* \to x$. □

Theorem 5.4. *For any bounded Hilbert algebra H, $S(H)$ is a bounded subalgebra of H.*

Proof. Let $x, y \in S(H)$. Then by Lemma 2.2 and Theorem 5.3,

$$\begin{aligned}
(x \to y)^{**} \to (x \to y) &= (x \to y)^{**} \to (x \to y^{**}) \\
&= (x \to y)^{**} \to (y^* \to x^*) \\
&= y^* \to ((x \to y)^{**} \to x^*) \\
&= y^* \to (x^{**} \to (x \to y)^{***}) \\
&= y^* \to (x \to (x \to y)^*) \\
&= y^* \to ((x \to y) \to x^*) \\
&= (x \to y) \to (y^* \to x^*) \\
&= (x \to y) \to (x \to y^{**}) \\
&= (x \to y) \to (x \to y) \\
&= 1.
\end{aligned}$$

Therefore $(x \to y)^{**} \leq (x \to y)$. But $(x \to y) \to (x \to y)^{**} = (x \to y) \to (((x \to y) \to 0) \to 0) = ((x \to y) \to 0) \to ((x \to y) \to 0) = 1$. Hence $x \to y \leq (x \to y)^{**}$. We deduce $(x \to y)^{**} = (x \to y)$, which says $x \to y \in S(H)$, and consequently $S(H)$ is closed with respect to the binary operation $\to$. Also $0, 1 \in S(H)$, namely, $S(H)$ is a bounded subalgebra of H. □

Theorem 5.5. *Let $x, y \in S(H)$. Then for all $x, y \in H$ we have*
i) $x \to y = y^* \to x^*$,
ii) $x \leq y^*$ implies $y \leq x^*$.

Proof. i) Since $x, y \in S(H)$, we have $x^{**} = x$ and $y^{**} = y$. Hence by Lemma 2.2 part (iv), $x \to y = x \to y^{**} = y^* \to x^*$.
ii) Let $x \leq y^*$, we get $x \to y^* = 1$. Hence by Lemma 2.2 part (iv), $1 = x \to y^* = y \to x^*$. So, $y \leq x^*$. □
In the following theorem, we express the relationship between an involution element and the commutators.

Theorem 5.6. *Let H be a bounded Hilbert algebra, then for $x \in H$, $[x, 0] = 1$ if and only if $x \in S(H)$.*

Proof. Let H be a bounded Hilbert algebra. Let $x \in H$ and suppose that $[x, 0] = 1$. Since $[x, 0] = x^{**} \to x$, we get $x^{**} \to x = 1$, so $x^{**} \leq x$. But $x \leq x^{**}$ and therefore $x^{**} = x$. Thus $x \in S(H)$.
Conversely, let $x \in S(H)$, for bounded Hilbert algebra H. Then $x^{**} = x$. So, we obtain $[x, 0] = x^{**} \to x = x \to x = 1$. □

6 Conclusion

In the present paper, we have introduced the concepts of Engel elements in Hilbert algebras and investigated some of their properties. To develop the theory of Hilbert algebras, one of the most encouraging ideas could be investigating the Engel degree of Hilbert algebras and finding a relation diagram between subclasses of Hilbert algebras. For instance, 1-Engel Hilbert algebras are strictly commutative Hilbert algebras. It is hoped that this work contributes to further studies. Therefore, we think that the results presented in this paper and the forthcoming works can pave the way for a bright future for the theory of the Hilbert algebras. The major goal of Engelś theory in Hilbert algebras can be stated as follows: to find conditions on H which will ensure that $L(H)$ and $R(H)$ are subalgebras or ideals, if possible.

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